

EVERY SET OF FINITE HAUSDORFF MEASURE
IS A COUNTABLE UNION OF SETS
WHOSE HAUSDORFF MEASURE AND CONTENT COINCIDE

RICHARD DELAWARE

(Communicated by David Preiss)

ABSTRACT. A set $E \subseteq \mathbb{R}^n$ is h -straight if E has finite Hausdorff h -measure equal to its Hausdorff h -content, where $h : [0, \infty) \rightarrow [0, \infty)$ is continuous and non-decreasing with $h(0) = 0$. Here, if h satisfies the standard doubling condition, then every set of finite Hausdorff h -measure in \mathbb{R}^n is shown to be a countable union of h -straight sets. This also settles a conjecture of Foran that when $h(t) = t^s$, every set of finite s -measure is a countable union of s -straight sets.

1. INTRODUCTION

In [7], Foran introduced the notion of an s -straight set, that is, a set whose Hausdorff s -measure and Hausdorff s -content are equal (see Definitions 1 and 2 below). In [1] and [2], we continued the first analysis of such sets, among other results proving that in \mathbb{R}^2 a quarter circle is the countable union of 1-straight sets, verifying a conjecture of Foran. In [3] and [4], by different detailed arguments we extended that result, proving that in \mathbb{R}^2 the graphs of any convex, continuously differentiable, absolutely continuous, or increasing continuous functions, as well as regular sets of finite 1-measure, all consist of such countable unions.

Here we prove the more general result that if a function h satisfies the standard doubling condition (Condition 1), then every set of finite h -measure in \mathbb{R}^n is a countable union of h -straight sets (Definition 2, Theorem 4, and Theorem 5). This also settles the conjecture of Foran [7] that every set of finite s -measure is a countable union of s -straight sets.

Let d be the standard distance function on \mathbb{R}^n where $n \geq 1$. The diameter of an arbitrary nonempty set $U \subseteq \mathbb{R}^n$ is defined by $|U| = \sup\{d(x, y) : x, y \in U\}$, with $|\emptyset| = 0$. Given $0 < \delta \leq \infty$, let C_δ^n represent the collection of subsets of \mathbb{R}^n with diameter less than δ . Let $h : [0, \infty) \rightarrow [0, \infty)$ be continuous and non-decreasing, with $h(0) = 0$. (In Condition 1 we restrict $h(t)$ further.)

Definition 1 ([6, 2.10.1] or [8, p. 60]). For $E \subseteq \mathbb{R}^n$, let

$$\mathcal{H}_\delta^h(E) = \inf \left\{ \sum h(|E_i|) : E \subseteq \bigcup E_i \text{ where } E_i \in C_\delta^n \text{ for } i = 1, 2, \dots \right\}.$$

Received by the editors August 17, 2001 and, in revised form, March 27, 2002.
2000 *Mathematics Subject Classification*. Primary 28A78, 28A05, 28A12.
Key words and phrases. h -straight, Hausdorff measure, Hausdorff content.

Then $\mathcal{H}_\infty^h(E)$ is called the Hausdorff h -content of E , and $\mathcal{H}^h(E) = \sup_{\delta>0} \mathcal{H}_\delta^h(E)$ is called the Hausdorff h -measure, or just the h -measure of E .

Note that Hausdorff h -measure is Borel regular [6, 2.10.2 (1)]. Also [8, p. 54], since for any $0 < \alpha < \beta \leq \infty$ it follows that $\mathcal{H}_\beta^h(E) \leq \mathcal{H}_\alpha^h(E)$, we always have the inequality

$$\mathcal{H}_\infty^h(E) \leq \mathcal{H}^h(E).$$

Definition 2. Define a set $E \subseteq \mathbb{R}^n$ to be h -straight if

$$\mathcal{H}_\infty^h(E) = \mathcal{H}^h(E) < \infty.$$

A set which is the countable union of h -straight sets is called σh -straight. (When $h(t) = t^s$, the terms are s -straight and σs -straight, [1], [2].)

In [7], Foran proves the following theorem for the case of Hausdorff s -measure providing a useful equivalent definition of an s -straight set. We make the easy extension of his proof to h -straight sets.

Theorem 1. *Let $E \subseteq \mathbb{R}^n$ have finite h -measure. Then, E is h -straight if and only if for each $A \subseteq E$*

$$\mathcal{H}^h(A) \leq h(|A|).$$

In particular, sets of zero h -measure are h -straight.

Proof. (Based on [7, p. 733].) On the one hand, suppose for each $A \subseteq E$ that $\mathcal{H}^h(A) \leq h(|A|)$. Then

$$\begin{aligned} \mathcal{H}^h(E) &\geq \mathcal{H}_\infty^h(E) = \inf \left\{ \sum h(|E_i|) : E = \bigcup E_i \right\} \\ &\geq \inf \left\{ \sum \mathcal{H}^h(E_i) : E = \bigcup E_i \right\} \\ &\geq \mathcal{H}^h(E), \end{aligned}$$

where the infima are over countable covers $\{E_i\}$ of E . Since E has finite h -measure, it then follows that $\mathcal{H}_\infty^h(E) = \mathcal{H}^h(E) < \infty$. Hence E is h -straight. Conversely, suppose $\mathcal{H}_\infty^h(E) = \mathcal{H}^h(E) < \infty$. If there were a subset $A \subseteq E$ such that $\mathcal{H}^h(A) > h(|A|)$, then, since h is non-decreasing and $\mathcal{H}_\infty^h(E \setminus A) \leq \mathcal{H}^h(E \setminus A) \leq \mathcal{H}^h(E) < \infty$, we have

$$\begin{aligned} \mathcal{H}^h(E) &= \mathcal{H}^h(A) + \mathcal{H}^h(E \setminus A) \\ &> h(|A|) + \mathcal{H}_\infty^h(E \setminus A) \\ &\geq \mathcal{H}_\infty^h(A) + \mathcal{H}_\infty^h(E \setminus A) \\ &\geq \mathcal{H}_\infty^h(E), \end{aligned}$$

contradicting the assumption that $\mathcal{H}_\infty^h(E) = \mathcal{H}^h(E)$. □

Theorem 2 is proved in [1], [2], for s -straight sets using a standard exhaustion argument. The proof of the generalization here to h -straight sets is omitted.

Theorem 2. *Let $E \subseteq \mathbb{R}^n$ have finite h -measure. Then, every \mathcal{H}^h -measurable subset of positive h -measure of E contains an h -straight set of positive h -measure if and only if E is σh -straight.*

2. MAIN RESULT

To prepare for the proof of the main result, Theorem 4, we begin with a well-known definition.

Definition 3 ([5, p. 21]). Let $E \subseteq \mathbb{R}^n$ have finite h -measure, and $x \in E$. The upper convex density of E at x is defined to be

$$\overline{D}_c^h(E, x) = \limsup_{\eta \rightarrow 0} \left\{ \frac{\mathcal{H}^h(E \cap S)}{h(|S|)} : x \in S \text{ and } 0 < |S| \leq \eta \right\}$$

where the supremum is over all convex sets $S \subseteq \mathbb{R}^n$.

Note that since any set is contained in a convex set of the same diameter, Definition 3 may be interpreted as taking the supremum over *all* sets $S \subseteq \mathbb{R}^n$. We also assume h satisfies the following standard doubling condition.

Condition 1. For some $c < \infty$, $h(2t) \leq c \cdot h(t)$ for all $t > 0$.

Theorem 3 is the restriction we need here of a result of Federer, [6, 2.10.18 (3)].

Theorem 3 (Corollary to [6, 2.10.18 (3)]). Let $E \subseteq \mathbb{R}^n$ have finite h -measure, where h satisfies Condition 1. Then for \mathcal{H}^h -almost all $x \in E$ it follows that

$$\overline{D}_c^h(E, x) \leq 1.$$

Lemma 1 is then a straightforward consequence of Theorem 3. The proof, though a standard argument, is included for completeness.

Lemma 1. Let $E \subseteq \mathbb{R}^n$ have finite h -measure, where h satisfies Condition 1. Let $\{\varepsilon_j\}_{j=0}^\infty$ be a positive decreasing sequence of real numbers such that $\lim_{j \rightarrow \infty} \varepsilon_j = 0$. Then there exists $A \subseteq E$ with positive h -measure satisfying the condition that for each $j \geq 0$ there is $\rho_j > 0$ so that for all $S \subseteq \mathbb{R}^n$ with $0 < |S| \leq \rho_j$ we have

$$\mathcal{H}^h(A \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|).$$

Proof. Suppose h is such a function, $E \subseteq \mathbb{R}^n$ satisfies $\mathcal{H}^h(E) < \infty$, and $\{\varepsilon_j\}_{j=0}^\infty$ is a positive decreasing sequence of real numbers such that $\lim_{j \rightarrow \infty} \varepsilon_j = 0$. By Theorem 3, for \mathcal{H}^h -almost all $x \in E$ it follows that

$$\overline{D}_c^h(E, x) = \limsup_{\eta \rightarrow 0} \left\{ \frac{\mathcal{H}^h(E \cap S)}{h(|S|)} : x \in S \text{ and } 0 < |S| \leq \eta \right\} \leq 1.$$

So given $\varepsilon_j > 0$ there exists a corresponding $\eta_j(x) > 0$ such that

$$\sup \left\{ \frac{\mathcal{H}^h(E \cap S)}{h(|S|)} : x \in S \text{ and } 0 < |S| \leq \eta_j(x) \right\} \leq 1 + \varepsilon_j.$$

Then for each $S \subseteq \mathbb{R}^n$ with $x \in E \cap S$ and $0 < |S| \leq \eta_j(x)$ we have

$$\mathcal{H}^h(E \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|).$$

Now let $\{\rho_j\}_{j=0}^\infty$ be a positive decreasing sequence of real numbers such that $\lim_{j \rightarrow \infty} \rho_j = 0$. Define

$$A_j = \{x \in \mathbb{R}^n : \text{If } x \in S \text{ and } 0 < |S| \leq \rho_j, \text{ then } \mathcal{H}^h(E \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|)\}.$$

Then A_j is a Borel (in fact, closed) set, hence \mathcal{H}^h -measurable. Since $\overline{D}_c^h(E, x) \leq 1$ for \mathcal{H}^h -almost all $x \in E$, we may further choose ρ_j small enough so that given δ satisfying $0 < \delta < \mathcal{H}^h(E)$, we have $\mathcal{H}^h(E \setminus A_j) < \frac{\delta}{2^{j+1}}$. Hence, $\sum_{j=0}^\infty \mathcal{H}^h(E \setminus A_j) <$

$\sum_{j=0}^{\infty} \frac{\delta}{2^{j+1}} = \delta$. For such choices of ρ_j , define $A = \bigcap_{j=0}^{\infty} A_j$. By the definition of A as an intersection, for each $j \geq 0$ there is $\rho_j > 0$, so for all $S \subseteq \mathbb{R}^n$ with $0 < |S| \leq \rho_j$ we have that A satisfies

$$\mathcal{H}^h(A \cap S) \leq \mathcal{H}^h(E \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|).$$

Also, A has positive h -measure since

$$\begin{aligned} \mathcal{H}^h(A) &= \mathcal{H}^h\left(\bigcap_{j=0}^{\infty} A_j\right) = \mathcal{H}^h(E) - \mathcal{H}^h\left(\bigcup_{j=0}^{\infty} (E \setminus A_j)\right) \\ &\geq \mathcal{H}^h(E) - \sum_{j=0}^{\infty} \mathcal{H}^h(E \setminus A_j) > \mathcal{H}^h(E) - \delta > 0. \end{aligned}$$

So, $A \subseteq E$ is the desired set. □

The argument used in the proof of Theorem 4 below was developed from a suggestion of Preiss [10] for sets of finite s -measure made after reading a preprint of [3].

Theorem 4. *If $E \subseteq \mathbb{R}^n$ has finite h -measure where h satisfies Condition 1, then there exists $B \subseteq E$ with positive h -measure such that B is h -straight.*

Proof. Let $E \subseteq \mathbb{R}^n$ have finite h -measure, where h is such a function. Let $\{\varepsilon_j\}_{j=0}^{\infty}$ be a positive, decreasing sequence of real numbers such that $\sum_{j=0}^{\infty} \varepsilon_j < 1$. Hence, $\lim_{j \rightarrow \infty} \varepsilon_j = 0$. So, by Lemma 1 there exists $A \subseteq E$ with positive h -measure and corresponding $\rho_j > 0$ such that for all $S \subseteq \mathbb{R}^n$ with $0 < |S| \leq \rho_j$ we have

$$(*) \quad \mathcal{H}^h(A \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|).$$

For $j \geq 0$ replace as needed the ρ_j with values r_j small enough so that $\lim_{j \rightarrow \infty} r_j = 0$, with both $2r_0 < \rho_0$ and

$$2r_{j+1} < \min(r_j, \rho_{j+1}).$$

By the continuity of h we may also assume the r_j small enough so that whenever $r_j \leq t \leq r_{j-1}$, we have

$$h(t + 2r_{j+1}) < (1 + \varepsilon_{j-1}^2) \cdot h(t).$$

We may further assume that

$$0 < \mathcal{H}^h(A) \leq h(r_0)$$

by the continuity of h -measure, since $0 < \mathcal{H}^h(A) \leq \mathcal{H}^h(E) < \infty$.

Now, for each fixed $j \geq 1$, let $\{A_{j,i}\}_i$ be a partition of A into (relatively) Borel subsets such that $|A_{j,i}| < r_{j+1}$ for every i . Then, for $j \geq 1$ let $B_{j,i} \subseteq A_{j,i}$ be slightly smaller in measure, satisfying

$$\mathcal{H}^h(B_{j,i}) = (1 - \varepsilon_{j-1}) \cdot \mathcal{H}^h(A_{j,i}).$$

It then follows for each fixed $j \geq 1$ that since the $A_{j,i}$ partition A ,

$$\mathcal{H}^h\left(\bigcup_i B_{j,i}\right) = \sum_i \mathcal{H}^h(B_{j,i}) = (1 - \varepsilon_{j-1}) \cdot \sum_i \mathcal{H}^h(A_{j,i}) = (1 - \varepsilon_{j-1}) \cdot \mathcal{H}^h(A).$$

Define $B = \bigcap_{j \geq 1} \bigcup_i B_{j,i}$. Observe that using the subadditivity of \mathcal{H}^h , we have

$$\begin{aligned} \mathcal{H}^h(A \setminus B) &= \mathcal{H}^h\left(\bigcup_{j \geq 1} \left[A \setminus \bigcup_i B_{j,i}\right]\right) \\ &\leq \sum_{j \geq 1} \left[\mathcal{H}^h(A) - \mathcal{H}^h\left(\bigcup_i B_{j,i}\right)\right] \\ &= \left(\sum_{j \geq 1} \varepsilon_{j-1}\right) \cdot \mathcal{H}^h(A) < \mathcal{H}^h(A). \end{aligned}$$

Thus $\mathcal{H}^h(B) = \mathcal{H}^h(A) - \mathcal{H}^h(A \setminus B) > 0$, so B has positive h -measure. Using Theorem 1 we show that B is the desired h -straight subset of E by showing that $\mathcal{H}^h(B \cap S) \leq h(|S|)$ for every $S \subseteq \mathbb{R}^n$.

First, for any $S \subseteq \mathbb{R}^n$ such that $|S| > r_0$, since $B \cap S \subseteq A$ and h is non-decreasing we immediately have that

$$\mathcal{H}^h(B \cap S) \leq \mathcal{H}^h(A) \leq h(r_0) \leq h(|S|).$$

So, consider all other sets $S \subseteq \mathbb{R}^n$ such that $r_j < |S| \leq r_{j-1}$ for some $j \geq 1$. Define

$$S' = \{x \in \mathbb{R}^n : d(x, S) \leq r_{j+1}\}.$$

So $S \subseteq S'$. By the restrictions on $\{r_k\}_{k=0}^\infty$ we then have that

$$0 < |S'| \leq |S| + 2r_{j+1} < r_{j-1} + r_j < 2r_{j-1} < \rho_{j-1}.$$

By definition for each $j \geq 1$ we have $B \cap A_{j,i} \subseteq B_{j,i}$. Denote by \bigcup' the union and by \sum' the summation over those indices i for which $S \cap B_{j,i} \neq \emptyset$. So for $j \geq 1$ we have $B \cap S \subseteq \bigcup'_i B_{j,i}$. Also, because $|A_{j,i}| < r_{j+1}$ it follows that each $A_{j,i} \subseteq A \cap S'$ for those same indices i . Then, for any fixed $j \geq 1$, we have

$$\begin{aligned} \mathcal{H}^h(B \cap S) &\leq \sum'_i \mathcal{H}^h(B_{j,i}) \\ &= (1 - \varepsilon_{j-1}) \cdot \sum'_i \mathcal{H}^h(A_{j,i}) = (1 - \varepsilon_{j-1}) \cdot \mathcal{H}^h\left(\bigcup'_i A_{j,i}\right) \\ (**) \quad &\leq (1 - \varepsilon_{j-1}) \cdot \mathcal{H}^h(A \cap S'). \end{aligned}$$

Since $r_j < |S| \leq r_{j-1}$ and h is non-decreasing, again by the restrictions on $\{r_k\}_{k=0}^\infty$ we have

$$h(|S'|) \leq h(|S| + 2r_{j+1}) < (1 + \varepsilon_{j-1}^2) \cdot h(|S|).$$

Since also $0 < |S'| < \rho_{j-1}$, by inequality (*) we further have

$$\mathcal{H}^h(A \cap S') \leq (1 + \varepsilon_{j-1}) \cdot h(|S'|).$$

So, by inequality (**), we finally conclude

$$\begin{aligned} \mathcal{H}^h(B \cap S) &\leq (1 - \varepsilon_{j-1}) \cdot (1 + \varepsilon_{j-1}) \cdot h(|S'|) \\ &< (1 - \varepsilon_{j-1}^2) \cdot (1 + \varepsilon_{j-1}^2) \cdot h(|S|) \\ &= (1 - \varepsilon_{j-1}^4) \cdot h(|S|) \\ &< h(|S|). \end{aligned}$$

Therefore in particular if $S \subseteq B$ it follows that $\mathcal{H}^h(S) = \mathcal{H}^h(B \cap S) \leq h(|S|)$. So by Theorem 1, the set B is h -straight with positive h -measure, as desired. \square

Theorem 5. *If $E \subseteq \mathbb{R}^n$ has finite h -measure where h satisfies Condition 1, then E is σh -straight.*

Proof. Let $E \subseteq \mathbb{R}^n$ have finite h -measure, where h is such a function. By Theorem 4, every \mathcal{H}^h -measurable subset of E of positive h -measure contains an h -straight subset of positive h -measure. By Theorem 2 it then follows that E is σh -straight. \square

ACKNOWLEDGEMENTS

The author thanks James Foran for helpful conversations, and David Preiss [10], Pertti Mattila [9], and the referee for valuable suggestions and improvements.

REFERENCES

- [1] R. Delaware, *Sets Whose Hausdorff Measure Equals Method I Outer Measure*, Ph.D. Dissertation, University of Missouri-Kansas City, 2000.
- [2] R. Delaware, *Sets Whose Hausdorff Measure Equals Method I Outer Measure*, *Real Anal. Exchange*, **27**(2), 2001/2, 535-562.
- [3] R. Delaware, *Graphs of Convex Functions are $\sigma 1$ -straight*, *Rocky Mountain Journal of Mathematics*, to appear.
- [4] R. Delaware and L. Eifler, *Graphs of Functions, Regular Sets, and s -straight Sets*, *Real Anal. Exchange*, **26**(2), 2000/1, 885-892. MR **2002e**:28010
- [5] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, 1985. MR **88d**:28001
- [6] H. Federer, *Geometric Measure Theory*, Springer-Verlag, 1969. MR **41**:1976
- [7] J. Foran, *Measure-Preserving Continuous Straightening of Fractional Dimension Sets*, *Real Anal. Exchange* (2), **21** (1995-96), 732-738. MR **97k**:28013
- [8] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces, Fractals and rectifiability*, Cambridge University Press, 1995. MR **96h**:28006
- [9] P. Mattila, private email communication, 2 April 2001.
- [10] D. Preiss, private email communication, 20 October 2000.

DEPARTMENT OF MATHEMATICS AND STATISTICS, HAAG HALL ROOM 206, UNIVERSITY OF MISSOURI - KANSAS CITY, 5100 ROCKHILL RD., KANSAS CITY, MISSOURI 64110
E-mail address: RDelaware3141@cs.com