EVERY SET OF FINITE HAUSDORFF MEASURE IS A COUNTABLE UNION OF SETS WHOSE HAUSDORFF MEASURE AND CONTENT COINCIDE

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Abstract. A set \( E \subseteq \mathbb{R}^n \) is \( h \)-straight if \( E \) has finite Hausdorff \( h \)-measure equal to its Hausdorff \( h \)-content, where \( h : [0, \infty) \rightarrow [0, \infty) \) is continuous and non-decreasing with \( h(0) = 0 \). Here, if \( h \) satisfies the standard doubling condition, then every set of finite Hausdorff \( h \)-measure in \( \mathbb{R}^n \) is shown to be a countable union of \( h \)-straight sets. This also settles a conjecture of Foran that when \( h(t) = t^s \), every set of finite \( s \)-measure is a countable union of \( s \)-straight sets.

1. Introduction

In [7], Foran introduced the notion of an \( s \)-straight set, that is, a set whose Hausdorff \( s \)-measure and Hausdorff \( s \)-content are equal (see Definitions 1 and 2 below). In [1] and [2], we continued the first analysis of such sets, among other results proving that in \( \mathbb{R}^2 \) a quarter circle is the countable union of \( 1 \)-straight sets, verifying a conjecture of Foran. In [3] and [4], by different detailed arguments we extended that result, proving that in \( \mathbb{R}^2 \) the graphs of any convex, continuously differentiable, absolutely continuous, or increasing continuous functions, as well as regular sets of finite 1-measure, all consist of such countable unions.

Here we prove the more general result that if a function \( h \) satisfies the standard doubling condition (Condition 1), then every set of finite \( h \)-measure in \( \mathbb{R}^n \) is a countable union of \( h \)-straight sets (Definition 2, Theorem 4, and Theorem 5). This also settles the conjecture of Foran [7] that every set of finite \( s \)-measure is a countable union of \( s \)-straight sets.

Let \( d \) be the standard distance function on \( \mathbb{R}^n \) where \( n \geq 1 \). The diameter of an arbitrary nonempty set \( U \subseteq \mathbb{R}^n \) is defined by \( |U| = \text{sup}\{d(x, y) : x, y \in U\} \), with \( |\emptyset| = 0 \). Given \( 0 < \delta \leq \infty \), let \( C^\delta_n \) represent the collection of subsets of \( \mathbb{R}^n \) with diameter less than \( \delta \). Let \( h : [0, \infty) \rightarrow [0, \infty) \) be continuous and non-decreasing, with \( h(0) = 0 \). (In Condition 1 we restrict \( h(t) \) further.)

Definition 1 ([6, 2.10.1] or [8, p. 60]). For \( E \subseteq \mathbb{R}^n \), let

\[ \mathcal{H}^h_\delta(E) = \inf \left\{ \sum h(|E_i|) : E \subseteq \bigcup E_i \text{ where } E_i \in C^\delta_n \text{ for } i = 1, 2, \ldots \right\} \]
Then $\mathcal{H}_h^h(E)$ is called the Hausdorff $h$-content of $E$, and $\mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}_h^h(E)$ is called the Hausdorff $h$-measure, or just the $h$-measure of $E$.

Note that Hausdorff $h$-measure is Borel regular [6, 2.10.2 (1)]. Also [8, p. 54], since for any $0 < \alpha < \beta \leq \infty$ it follows that $\mathcal{H}_\beta^h(E) \leq \mathcal{H}_\alpha^h(E)$, we always have the inequality

$$\mathcal{H}_\infty^h(E) \leq \mathcal{H}^h(E).$$

**Definition 2.** Define a set $E \subseteq \mathbb{R}^n$ to be $h$-straight if

$$\mathcal{H}^h(E) < \infty.$$

A set which is the countable union of $h$-straight sets is called $\sigma h$-straight. (When $h(t) = t^s$, the terms are $s$-straight and $\sigma s$-straight, [1], [2].)

In [7], Foran proves the following theorem for the case of Hausdorff $s$-measure providing a useful equivalent definition of an $s$-straight set. We make the easy extension of his proof to $h$-straight sets.

**Theorem 1.** Let $E \subseteq \mathbb{R}^n$ have finite $h$-measure. Then, $E$ is $h$-straight if and only if for each $A \subseteq E$

$$\mathcal{H}^h(A) \leq h(|A|).$$

In particular, sets of zero $h$-measure are $h$-straight.

**Proof.** (Based on [7] p. 733.) On the one hand, suppose for each $A \subseteq E$ that $\mathcal{H}^h(A) \leq h(|A|)$. Then

$$\mathcal{H}^h(E) \geq \mathcal{H}^h_\infty(E) = \inf \left\{ \sum h(|E_i|) : E = \bigcup E_i \right\}$$

$$\geq \inf \left\{ \sum \mathcal{H}^h(E_i) : E = \bigcup E_i \right\}$$

$$\geq \mathcal{H}^h(E),$$

where the infima are over countable covers $\{E_i\}$ of $E$. Since $E$ has finite $h$-measure, it then follows that $\mathcal{H}^h_\infty(E) = \mathcal{H}^h(E) < \infty$. Hence $E$ is $h$-straight. Conversely, suppose $\mathcal{H}^h_\infty(E) = \mathcal{H}^h(E) < \infty$. If there were a subset $A \subseteq E$ such that $\mathcal{H}^h(A) > h(|A|)$, then, since $h$ is non-decreasing and $\mathcal{H}^h_\infty(E \setminus A) \leq \mathcal{H}^h(E \setminus A) \leq \mathcal{H}^h(E) < \infty$, we have

$$\mathcal{H}^h(E) = \mathcal{H}^h(A) + \mathcal{H}^h(E \setminus A)$$

$$> h(|A|) + \mathcal{H}^h_\infty(E \setminus A)$$

$$\geq \mathcal{H}^h_\infty(A) + \mathcal{H}^h_\infty(E \setminus A)$$

$$\geq \mathcal{H}^h_\infty(E),$$

contradicting the assumption that $\mathcal{H}^h_\infty(E) = \mathcal{H}^h(E)$. \qed

Theorem 2 is proved in [1], [2], for $s$-straight sets using a standard exhaustion argument. The proof of the generalization here to $h$-straight sets is omitted.

**Theorem 2.** Let $E \subseteq \mathbb{R}^n$ have finite $h$-measure. Then, every $\mathcal{H}^h$-measurable subset of positive $h$-measure of $E$ contains an $h$-straight set of positive $h$-measure if and only if $E$ is $\sigma h$-straight.
2. Main result

To prepare for the proof of the main result, Theorem 3, we begin with a well-known definition.

**Definition 3** ([6, p. 21]). Let $E \subseteq \mathbb{R}^n$ have finite $h$-measure, and $x \in E$. The upper convex density of $E$ at $x$ is defined to be

$$\overline{D}_c^h(E, x) = \lim_{\eta \to 0} \sup \{ \frac{\mathcal{H}^h(E \cap S)}{h(|S|)} : x \in S \text{ and } 0 < |S| \leq \eta \}$$

where the supremum is over all convex sets $S \subseteq \mathbb{R}^n$.

Note that since any set is contained in a convex set of the same diameter, Definition 3 may be interpreted as taking the supremum over all sets $S \subseteq \mathbb{R}^n$. We also assume $h$ satisfies the following standard doubling condition.

**Condition 1.** For some $c < \infty$, $h(2t) \leq c \cdot h(t)$ for all $t > 0$.

Theorem 3 is the restriction we need here of a result of Federer, [6, 2.10.18 (3)].

**Theorem 3** (Corollary to [6, 2.10.18 (3)]). Let $E \subseteq \mathbb{R}^n$ have finite $h$-measure, where $h$ satisfies Condition 1. Then for $\mathcal{H}^h$-almost all $x \in E$ it follows that

$$\overline{D}_c^h(E, x) \leq 1.$$

Lemma 1 is then a straightforward consequence of Theorem 3. The proof, though a standard argument, is included for completeness.

**Lemma 1.** Let $E \subseteq \mathbb{R}^n$ have finite $h$-measure, where $h$ satisfies Condition 1. Let $\{\varepsilon_j\}_{j=0}^{\infty}$ be a positive decreasing sequence of real numbers such that $\lim_{j \to \infty} \varepsilon_j = 0$. Then there exists $A \subseteq E$ with positive $h$-measure satisfying the condition that for each $j \geq 0$ there is $\rho_j > 0$ so that for all $S \subseteq \mathbb{R}^n$ with $0 < |S| \leq \rho_j$ we have

$$\mathcal{H}^h(A \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|).$$

**Proof.** Suppose $h$ is such a function, $E \subseteq \mathbb{R}^n$ satisfies $\mathcal{H}^h(E) < \infty$, and $\{\varepsilon_j\}_{j=0}^{\infty}$ is a positive decreasing sequence of real numbers such that $\lim_{j \to \infty} \varepsilon_j = 0$. By Theorem 3 for $\mathcal{H}^h$-almost all $x \in E$ it follows that

$$\overline{D}_c^h(E, x) = \lim_{\eta \to 0} \sup \{ \frac{\mathcal{H}^h(E \cap S)}{h(|S|)} : x \in S \text{ and } 0 < |S| \leq \eta \} \leq 1.$$

So given $\varepsilon_j > 0$ there exists a corresponding $\eta_j(x) > 0$ such that

$$\sup \{ \frac{\mathcal{H}^h(E \cap S)}{h(|S|)} : x \in S \text{ and } 0 < |S| \leq \eta_j(x) \} \leq 1 + \varepsilon_j.$$

Then for each $S \subseteq \mathbb{R}^n$ with $x \in E \cap S$ and $0 < |S| \leq \eta_j(x)$ we have

$$\mathcal{H}^h(E \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|).$$

Now let $\{\rho_j\}_{j=0}^{\infty}$ be a positive decreasing sequence of real numbers such that $\lim_{j \to \infty} \rho_j = 0$. Define

$$A_j = \{x \in \mathbb{R}^n : \text{ If } x \in S \text{ and } 0 < |S| \leq \rho_j, \text{ then } \mathcal{H}^h(E \cap S) \leq (1 + \varepsilon_j) \cdot h(|S|).\}$$

Then $A_j$ is a Borel (in fact, closed) set, hence $\mathcal{H}^h$-measurable. Since $\overline{D}_c^h(E, x) \leq 1$ for $\mathcal{H}^h$-almost all $x \in E$, we may further choose $\rho_j$ small enough so that given $\delta$ satisfying $0 < \delta < \mathcal{H}^h(E)$, we have $\mathcal{H}^h(E \setminus A_j) < \frac{\delta}{\rho_j}$. Hence, $\sum_{j=0}^{\infty} \mathcal{H}^h(E \setminus A_j) < \frac{\delta}{c}$. Hence, $\sum_{j=0}^{\infty} \mathcal{H}^h(E \setminus A_j) < \frac{\delta}{c}$.
\[\sum_{j=0}^{\infty} \frac{\epsilon_j}{2^j} = \delta.\] For such choices of \(\rho_j\), define \(A = \bigcap_{j=0}^{\infty} A_j\). By the definition of \(A\) as an intersection, for each \(j \geq 0\) there is \(\rho_j > 0\), so for all \(S \subseteq \mathbb{R}^n\) with \(0 < |S| \leq \rho_j\) we have that \(A\) satisfies

\[\mathcal{H}^h(A \cap S) \leq \mathcal{H}^h(E \cap S) \leq (1 + \epsilon_j) \cdot h(|S|).\]

Also, \(A\) has positive \(h\)-measure since

\[\mathcal{H}^h(A) = \mathcal{H}^h \left( \bigcap_{j=0}^{\infty} A_j \right) = \mathcal{H}^h (E) - \mathcal{H}^h \left( \bigcup_{j=0}^{\infty} (E \setminus A_j) \right) \geq \mathcal{H}^h (E) - \sum_{j=0}^{\infty} \mathcal{H}^h (E \setminus A_j) > \mathcal{H}^h (E) - \delta > 0.\]

So, \(A \subseteq E\) is the desired set. \(\square\)

The argument used in the proof of Theorem 4 below was developed from a suggestion of Preiss [10] for sets of finite \(s\)-measure made after reading a preprint of [3].

**Theorem 4.** If \(E \subseteq \mathbb{R}^n\) has finite \(h\)-measure where \(h\) satisfies Condition 11 then there exists \(B \subseteq E\) with positive \(h\)-measure such that \(B\) is \(h\)-straight.

**Proof.** Let \(E \subseteq \mathbb{R}^n\) have finite \(h\)-measure, where \(h\) is such a function. Let \(\{\epsilon_j\}_{j=0}^{\infty}\) be a positive, decreasing sequence of real numbers such that \(\sum_{j=0}^{\infty} \epsilon_j < 1\). Hence, \(\lim_{j \to \infty} \epsilon_j = 0\). So, by Lemma 11 there exists \(A \subseteq E\) with positive \(h\)-measure and corresponding \(\rho_j > 0\) such that for all \(S \subseteq \mathbb{R}^n\) with \(0 < |S| \leq \rho_j\) we have

\[\mathcal{H}^h(A \cap S) \leq (1 + \epsilon_j) \cdot h(|S|).\]

For \(j \geq 0\) replace as needed the \(\rho_j\) with values \(r_j\) small enough so that \(\lim_{j \to \infty} r_j = 0\), with both \(2r_0 < \rho_0\) and

\[2r_{j+1} < \min (r_j, \rho_{j+1}).\]

By the continuity of \(h\) we may also assume the \(r_j\) small enough so that whenever \(r_j \leq t < r_{j-1}\), we have

\[h(t + 2r_{j+1}) < (1 + \epsilon_j^2) \cdot h(t).\]

We may further assume that

\[0 < \mathcal{H}^h(A) \leq h(r_0)\]

by the continuity of \(h\)-measure, since \(0 < \mathcal{H}^h(A) \leq \mathcal{H}^h(E) < \infty\).

Now, for each fixed \(j \geq 1\), let \(\{A_{j,i}\}_i\) be a partition of \(A\) into (relatively) Borel subsets such that \(|A_{j,i}| < r_{j+1}\) for every \(i\). Then, for \(j \geq 1\) let \(B_{j,i} \subseteq A_{j,i}\) be slightly smaller in measure, satisfying

\[\mathcal{H}^h(B_{j,i}) = (1 - \epsilon_{j-1}) \cdot \mathcal{H}^h(A_{j,i}).\]

It then follows for each fixed \(j \geq 1\) that since the \(A_{j,i}\) partition \(A\),

\[\mathcal{H}^h \left( \bigcup_i B_{j,i} \right) = \sum_i \mathcal{H}^h (B_{j,i}) = (1 - \epsilon_{j-1}) \cdot \sum_i \mathcal{H}^h (A_{j,i}) = (1 - \epsilon_{j-1}) \cdot \mathcal{H}^h (A).\]
Define $B = \bigcap_{j \geq 1} \bigcup_i B_{j,i}$. Observe that using the subadditivity of $\mathcal{H}^h$, we have

$$\mathcal{H}^h(A \setminus B) = \mathcal{H}^h\left(\bigcup_{j \geq 1} \left[A \setminus \bigcup_i B_{j,i}\right]\right) \leq \sum_{j \geq 1} \left[\mathcal{H}^h(A) - \mathcal{H}^h\left(\bigcup_i B_{j,i}\right)\right] = \left(\sum_{j \geq 1} \varepsilon_{j-1}\right) \cdot \mathcal{H}^h(A) < \mathcal{H}^h(A).$$

Thus $\mathcal{H}^h(B) = \mathcal{H}^h(A) - \mathcal{H}^h(A \setminus B) > 0$, so $B$ has positive $h$-measure. Using Theorem 1 we show that $B$ is the desired $h$-straight subset of $E$ by showing that $\mathcal{H}^h(B \cap S) \leq h(|S|)$ for every $S \subseteq \mathbb{R}^n$.

First, for any $S \subseteq \mathbb{R}^n$ such that $|S| > r_0$, since $B \cap S \subseteq A$ and $h$ is non-decreasing we immediately have that

$$\mathcal{H}^h(B \cap S) \leq \mathcal{H}^h(A) \leq h(r_0) \leq h(|S|).$$

So, consider all other sets $S \subseteq \mathbb{R}^n$ such that $r_j < |S| \leq r_{j-1}$ for some $j \geq 1$. Define $S' = \{x \in \mathbb{R}^n : d(x, S) \leq r_{j+1}\}$.

So $S \subseteq S'$. By the restrictions on $\{r_k\}_{k=0}^\infty$ we then have that

$$0 < |S'| \leq |S| + 2r_{j+1} < r_{j-1} + r_j < 2r_{j-1} < \rho_{j-1}.$$

By definition for each $j \geq 1$ we have $B \cap A_{j,i} \subseteq B_{j,i}$. Denote by $\bigcup'$ the union and by $\sum'$ the summation over those indices $i$ for which $S \cap B_{j,i} \neq \emptyset$. So for $j \geq 1$ we have $B \cap S \subseteq \bigcup' B_{j,i}$. Also, because $|A_{j,i}| < r_{j+1}$ it follows that each $A_{j,i} \subseteq A \cap S'$ for those same indices $i$. Then, for any fixed $j \geq 1$, we have

$$\mathcal{H}^h(B \cap S) \leq \sum_i \mathcal{H}^h(B_{j,i}) = (1 - \varepsilon_{j-1}) \cdot \sum_i \mathcal{H}^h(A_{j,i}) = (1 - \varepsilon_{j-1}) \cdot \mathcal{H}^h\left(\bigcup_i A_{j,i}\right)$$

(**)

$$\leq (1 - \varepsilon_{j-1}) \cdot \mathcal{H}^h(A \cap S').$$

Since $r_j < |S| \leq r_{j-1}$ and $h$ is non-decreasing, again by the restrictions on $\{r_k\}_{k=0}^\infty$ we have

$$h(|S'|) \leq h(|S| + 2r_{j+1}) < (1 + \varepsilon_{j-1}) \cdot h(|S|).$$

Since also $0 < |S'| < \rho_{j-1}$, by inequality (**) we further have

$$\mathcal{H}^h(A \cap S') \leq (1 + \varepsilon_{j-1}) \cdot h(|S'|).$$

So, by inequality (**), we finally conclude

$$\mathcal{H}^h(B \cap S) \leq (1 - \varepsilon_{j-1}) \cdot (1 + \varepsilon_{j-1}) \cdot h(|S'|)$$

$$< (1 - \varepsilon_{j-1}^2) \cdot (1 + \varepsilon_{j-1}^2) \cdot h(|S|)$$

$$= (1 - \varepsilon_{j-1}^2) \cdot h(|S|)$$

$$< h(|S|).$$
Therefore in particular if $S \subseteq B$ it follows that $\mathcal{H}^h(S) = \mathcal{H}^h(B \cap S) \leq h(|S|)$. So by Theorem 1 the set $B$ is $h$-straight with positive $h$-measure, as desired. □

**Theorem 5.** If $E \subseteq \mathbb{R}^n$ has finite $h$-measure where $h$ satisfies Condition 1, then $E$ is $\sigma h$-straight.

**Proof.** Let $E \subseteq \mathbb{R}^n$ have finite $h$-measure, where $h$ is such a function. By Theorem 4 every $\mathcal{H}^h$-measurable subset of $E$ of positive $h$-measure contains an $h$-straight subset of positive $h$-measure. By Theorem 2 it then follows that $E$ is $\sigma h$-straight. □

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**References**


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