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SOME REMARKS ON SPREADING MODELS AND MIXED TSIRELSON SPACES

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Abstract. We prove that if a Banach space with a bimonotone shrinking basis does not contain ℓ_1^ω spreading models but every block sequence of the basis contains a further block sequence which is a $c-\ell_1^n$ spreading model for every $n \in \mathbb{N}$, then every subspace has a further subspace which is arbitrarily distortable. We also prove that a mixed Tsirelson space $T[(\mathcal{S}_n, \theta_n)_n]$, such that $\theta_n \searrow 0$, does not contain $\ell_1^{\omega 2}$ spreading models.

Introduction

A Banach space X with a basis (e_i) is an asymptotic ℓ_1 space if there exists $c > 0$ such that for all n and all $e_n < x_1 < \ldots < x_n$,

$$\|\sum_{i=1}^{n} x_i\| \geq c \sum_{i=1}^{n} \|x_i\| \ .$$

The first non-trivial example of an asymptotic ℓ_1 space was discovered by Tsirelson [17]. Recent results [6], [7], [15], have shown the necessity of studying the higher ordinal structure of an asymptotic ℓ_1 Banach space in order to obtain results on the global structure of its infinite dimensional subspaces. A normalized sequence $(x_n)_n$ in a Banach space X is said to be a $c-\ell_1^\xi$ spreading model if

$$\|\sum_{n\in F} \alpha_n x_n\| \geq c \sum_{n\in F} |\alpha_n| \ \ \forall F \in \mathcal{S}_\xi, \ (a_n)_{n\in F} \subset \mathbb{R},$$

where $\mathcal{S}_\xi$, $\xi < \omega_1$, are the generalized Schreier families defined in [1].

It is well known that if a separable Banach space X does not contain ℓ_1, then there exists $\xi < \omega_1$, such that X does not contain an ℓ_1^ξ spreading model. A complete classification of normalized weakly null sequences, in connection with spreading models, has been provided in [7].

Spreading models is a basic tool for the study of the asymptotic structure of a Banach space. The structure of the spreading models may even determine the geometry of the space [14]. Spreading models have been employed in [10] to prove the existence of strictly singular non-compact operators in certain Hereditarily Indecomposable mixed Tsirelson spaces.

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The idea of investigating the geometry of a Banach space by studying its asymptotic finite-dimensional subspaces emerged naturally in recent studies related to problems of distortion, i.e. the stabilization of equivalent norms on infinite-dimensional subspaces [2], [3], [5], [9], [15].

A Banach space $(X, \|\cdot\|)$ is said to be λ-distortable if there exists an equivalent norm $|\cdot|$ on X so that

$$\inf_Y \sup\{\frac{|x|}{|y|} : x, y \in S(Y, \|\cdot\|)\} \geq \lambda ,$$

where the infimum is taken over all infinite-dimensional subspaces Y of X. X is arbitrarily distortable if it is λ-distortable for all $\lambda > 1$. In section 2 we prove the following:

Theorem. *Let X be a Banach space with a bimonotone shrinking basis (e_i) such that*

(1) *X does not contain an ℓ_1^ω spreading model.*
(2) *For every $n \in \mathbb{N}$, every block sequence of (e_i) contains a further block sequence which is a $c - \ell_1^n$ spreading model.*

Then every subspace of X contains a further subspace which is arbitrarily distortable.

The relation of the distortion problem with spreading models has been studied in [15], [9]. In [9] a criterion has been provided related to ℓ_1^n spreading models, which implies the distortion of certain asymptotic ℓ_1 Banach spaces. The motivation for this theorem was the space constructed in [5], where an example of a mixed Tsirelson space X was given which has $c - \ell_1^n$ spreading models in every block subspace but has no ℓ_1^ω spreading model. The norm of X satisfies, for an appropriate sequence (k_j, θ_j), the implicit equation

$$\|x\| = \max\left\{\|x\|_\infty, \sup\left\{\sum_{k=1}^n \|x|_{[n,+\infty)}\|_{j_k} : n \in \mathbb{N},\ j_1 < j_2 < \ldots < j_n\right\}\right\},$$

where $\|x\|_j = \sup\{\theta_j \sum_{i\in F} \|E_i x\| : (E_i x)_{i\in F}\ \mathcal{S}_{k_j}\text{-admissible}\}$.

To prove the theorem we use some results proved by E.Odell, N.Tomczak-Jaegermann and R.Wagner [15]. In this paper, for a Banach space X with basis (e_i), certain indices $(\delta_\alpha(x_i))_{\alpha<\omega_1}$, for a block sequence (x_i), have been introduced and studied. Roughly speaking, the indices $(\delta_\alpha(x_i))_{\alpha<\omega_1}$ measure the strong presence of ℓ_1 in the subspace $\langle(x_i)\rangle$ in connection with the families $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$. The notion of Δ-spectrum, $\Delta(X)$, is also introduced. Roughly, $\Delta(X)$ is the set of all $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$, where γ_α is the stabilization of $\delta_\alpha(y_i)$ for some block basis (y_i) of (e_i). Using the stabilization result from [15], we prove that every block subspace has a further subspace Y such that: For every $n \in \mathbb{N}$ there exist two asymptotic sets A_n and B_n in Y and a subset A_n^* of X^* such that the equivalent norm

$$|||x||| = \gamma_n\|x\| + \sup\{x^*(x) : x^* \in A_n^*\}$$

is a $\approx \frac{1}{\gamma_n}$ distortion for Y. Since $\gamma_\omega = 0$, from the continuity of the indices $(\gamma_\alpha)_{\alpha<\omega_1}$ [15], we have that Y is arbitrarily distortable.

In section 3 we prove that in the mixed Tsirelson spaces $T[(\mathcal{S}_n, \theta_n)_n]$ the following holds.

Theorem. *Let* $X = T[(\mathcal{S}_n, \theta_n)_n]$ *such that* $\theta_{n+m} \geq \theta_n \theta_m$, $\lim_n \theta_n = 0$. *Then the space* X *does not contain* $\ell_1^{\omega^2}$ *spreading models.*

In [5] it has been proved that, assuming $\lim_n \theta_n^{1/n} = 1$, the space $T[(\mathcal{S}_n, \theta_n)_n]$ contains an ℓ_1^ω spreading model. The existence of ℓ_1^ω spreading models in these spaces is based on the disjoint representability of c_0 in these spaces [4]. Another approach to the existence of ℓ_1^ω in certain mixed Tsirelson spaces has been provided in [10]. The key point for the proof of this theorem is to produce for every normalized block sequence (x_n) of the basis, a vector in the linear span of (x_n), whose norm is arbitrarily small yet its support with respect to (x_n) belongs to $\mathcal{S}_{\omega 2}$. The dual of the original Tsirelson's space [17] contains no ℓ_1^ω spreading model. This is due to the fact that every block sequence is equivalent to a subsequence of the basis.

1. Preliminaries

Notation. Let $(e_i)_{i=1}^\infty$ be a basic sequence. For $x = \sum_{i=1}^\infty a_i e_i$ the *support* of x w.r.t. (e_i) is the set $\operatorname{supp} x = \{i \in \mathbb{N} : a_i \neq 0\}$. The *range* of x, written range(x), is the smallest interval of $\mathbb{N}$ containing the support of x. For finite subsets E, F of $\mathbb{N}$, $E < F$ means $\max E < \min F$ or either E or F is empty. For $n \in \mathbb{N}$, $E \subset \mathbb{N}$, $n < E$ (resp. $E < n$) means $n < \min E$ (resp. $\max E < n$). For x, y in c_{00}, $x < y$ means $\operatorname{supp} x < \operatorname{supp} y$. For $n \in \mathbb{N}$, $x \in c_{00}$, we write $n < x$ (resp. $x < n$) if $n < \operatorname{supp} x$ (resp. $\operatorname{supp} x < n$). We say that the sets $E_i \subset \mathbb{N}$, $i = 1, \ldots, n$, are *successive* if $E_1 < E_2 < \ldots < E_n$. Similarly, the vectors x_i, $i = 1, \ldots n$, are *successive* if $x_1 < x_2 < \ldots < x_n$. If (x_i) is a block sequence of (e_i) we write $(x_i) \prec (e_i)$. For $x = \sum_{i=1}^\infty a_i e_i$ and E a subset of $\mathbb{N}$, we denote by Ex the vector $Ex = \sum_{i \in E} a_i e_i$. For an infinite subset M of $\mathbb{N}$ we denote by $[M]$ the class of infinite subsets of M and by $[M]^{<\omega}$ the class of finite subsets of M.

The *generalized Schreier families* $\{\mathcal{S}_\xi\}_{\xi < \omega_1}$, introduced in [1], are defined by transfinite induction as follows:

$$\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}.$$

Suppose that the families $\mathcal{S}_\alpha$ have been defined for all $\alpha < \xi$.

If $\xi = \zeta + 1$, we set

$$\mathcal{S}_\xi = \{F \in [\mathbb{N}]^{<\omega} : F = \bigcup_{i=1}^n F_i,\ n \in \mathbb{N},\ \forall i \leq n\, F_i \in \mathcal{S}_\zeta \text{ and } n \leq F_1 < \ldots < F_n\} \cup \{\emptyset\}.$$

If ξ is a limit ordinal, let $(\xi_n + 1)_n$ be a sequence of successor ordinals which strictly increases to ξ. We set

$$\mathcal{S}_\xi = \{F \in \mathbb{N}^{<\omega} : \text{for some } n \in \mathbb{N},\ n \leq \min F \text{ and } F \in \mathcal{S}_{\xi_n + 1}\}.$$

If $N = (n_i)_i$ is an infinite subset of $\mathbb{N}$, then we define

$$\mathcal{S}_\xi[N] = \{F : F \subset N, F \in \mathcal{S}_\xi\} \text{ and } \mathcal{S}_\xi(N) = \{(n_i)_{i \in F} : F \in \mathcal{S}_\xi\}\ .$$

Proposition 1.1. (*a*) [2] *Let* $N \in [\mathbb{N}]$. *Then there exists* $L = (\ell_i) \in [N]$ *so that for all* $\alpha < \omega_1$,

$$(\ell_i)_{i \in F} \in \mathcal{S}_\alpha \Rightarrow (\ell_i)_{i \in F \setminus (\min F)} \in \mathcal{S}_\alpha(N)\ .$$

(*b*) [15] *Let* $\beta < \alpha < \omega_1$. *There exists* $n_0 \in \mathbb{N}$ *such that*

$$n_0 < F \in \mathcal{S}_\beta \Rightarrow F \in \mathcal{S}_\alpha\ .$$

(*c*) [15] *Let* $\beta < \alpha < \omega_1$. *There exists* $M \in [\mathbb{N}]$ *such that* $\mathcal{S}_\alpha[\mathcal{S}_\beta](M) \subset \mathcal{S}_{\beta + \alpha}$.

We next pass to the definition of the *repeated averages hierarchy* introduced in [7]. We let (e_n) denote the standard basis of c_{00}. For every countable ordinal ξ and every $M \in [\mathbb{N}]$, we define a convex block sequence $(\xi_n^M)_{n=1}^\infty$ of (e_n) by transfinite induction on ξ in the following manner:

If $\xi = 0$ and $M = (m_n)_{n=1}^\infty$, then $\xi_n^M = e_{m_n}$, for all $n \in \mathbb{N}$.

Assume that $(\zeta_n^M)_{n=1}^\infty$ has been defined for all $\zeta < \xi$ and $M \in [\mathbb{N}]$. Let $\xi = \zeta + 1$. We set

$$\xi_1^M = \frac{1}{m_1} \sum_{i=1}^{m_1} \zeta_i^M$$

where $m_1 = \min M$. Suppose that $\xi_1^M < \ldots < \xi_n^M$ have been defined. Let

$$M_n = \{m \in M : m > \max\operatorname{supp} \xi_n^M\} \quad \text{and} \quad k_n = \min M_n.$$

Set

$$\xi_{n+1}^M = \frac{1}{k_n} \sum_{i=1}^{k_n} \zeta_i^{M_n} = \xi_1^{M_n} .$$

If ξ is a limit ordinal, let $(\xi_n + 1)_n$ be the sequence of ordinals associated to ξ, and also let $M \in [\mathbb{N}]$. Define

$$\xi_1^M = [\xi_{m_1} + 1]_1^M$$

where $m_1 = \min M$. Suppose that $\xi_1^M < \ldots < \xi_n^M$ have been defined. Let

$$M_n = \{m \in M : m > \max\operatorname{supp} \xi_n^M\} \text{ and } k_n = \min M_n.$$

Set

$$\xi_{n+1}^M = [\xi_{k_n} + 1]_1^{M_n} .$$

The inductive definition of $(\xi_n^M)_{n=1}^\infty, M \in [\mathbb{N}]$, is now complete. We note that $\operatorname{supp} \xi_n^M \in \mathcal{S}_\xi$, for all $M \in [\mathbb{N}]$, $\xi < \omega_1$ and $n \in \mathbb{N}$.

Definition 1.2. (a) Let $k \in \mathbb{N}$. A finite sequence $(E_i)_{i=1}^m$ of successive subsets of $\mathbb{N}$ is said to be $\mathcal{S}_k$-admissible if $\{\min E_i\}_{i=1}^m \in \mathcal{S}_k$. A finite block sequence $(x_i)_{i=1}^m$ in c_{00} is said to be $\mathcal{S}_k$-admissible if $(\operatorname{supp} x_i)_{i=1}^m$ is $\mathcal{S}_k$-admissible. If (y_i) is a block sequence of the basis of c_{00}, $(x_i)_i$ is a block sequence of (y_i), and $\widetilde{E}_i$ is the range of x_i with respect to (w.r.t.) the basic sequence (y_i), then the block sequence (x_i) is $\mathcal{S}_n$-admissible w.r.t. (y_i) if $\{\min \widetilde{E}_i\}_i \in \mathcal{S}_n$.

(b) Let $\{k_n\}_n$ be an increasing sequence of integers and $\{\theta_n\} \subset (0,1)$ such that $\theta_n \searrow 0$. The mixed Tsirelson space $X = T[(\mathcal{S}_{k_n}, \theta_n)_{n=1}^\infty]$ is the completion of c_{00} under the norm which satisfies the implicit equation

$$\|x\| = \max\{\|x\|_\infty, \sup_n \theta_n \{\sup \sum_{i=1}^m \|E_i(x)\|\}\} ,$$

where the inside supremum is taken over all $\mathcal{S}_{k_n}$-admissible families $\{E_i\}_{i=1}^m$, $m \in \mathbb{N}$. The space X is a reflexive Banach space, and the sequence (e_i) is a basis on X.

An essential role in our proofs is played by the following special vectors.

Definition 1.3. Let (x_n) be a normalized block sequence of (e_n), $\varepsilon > 0$ and $\zeta < \xi < \omega_1$. Set $m_n = \min \operatorname{supp} x_n$ and $M = (m_n)_n$.

An (ε,ξ,ζ) basic special convex combination (basic s.c.c.) for M is any vector of the form $\xi_1^L = \sum_n \xi_1^L(m_n)e_{m_n}$, $L \in [M]$ and $\|\xi_1^L\|_\zeta < \varepsilon$, where $\|\sum \alpha_i e_i\|_\zeta = \sup\{\sum_{i\in F}|\alpha_i| : F \in \mathcal{S}_\zeta\}$.

An (ε,ξ,ζ) special convex combination, (s.c.c.), of (x_n) is any vector of the form $\sum_n \xi_1^L(m_n)x_n$, such that $\sum_n \xi_1^L(m_n)e_{m_n}$ is an (ε,ξ,ζ)-basic s.c.c.

Proposition 1.4 ([6])**.** *For every $M \in [\mathbb{N}]$, $\varepsilon > 0$ and all ordinals $\zeta < \xi < \omega_1$, there exists $N \in [M]$ such that $\|\xi_1^L\|_\zeta < \varepsilon$ for all $L \in [N]$.*

It is not hard to see that the average n_1^L is a $(\frac{3}{\min L}, n, n-1)$-basic s.c.c. for every $L \in [M]$.

Definition 1.5. Let $\xi < \omega_1$ and $\delta > 0$. A normalized sequence (x_n) in a Banach space is an $\delta - \ell_1^\xi$ spreading model if

$$\|\sum_{i\in F}\alpha_i x_i\| \geq \delta\sum_{i\in F}|\alpha_i|$$

for every $F \in \mathcal{S}_\xi$ and all choices of scalars $(\alpha_i)_{i\in F}$.

(x_n) is called an ℓ_1^ξ spreading model if it is an $\delta - \ell_1^\xi$ spreading model for some $\delta > 0$.

2.

Theorem 2.1. *Let X be a Banach space with a bimonotone shrinking basis (e_i) such that*

1. *X does not contain an ℓ_1^ω spreading model.*
2. *There exists $c > 0$ such that for every $n \in \mathbb{N}$, every block sequence of (e_i) contains a further block sequence which is a $c - \ell_1^n$ spreading model.*

Then every subspace of X contains a further subspace which is arbitrarily distortable.

Our proof is based on the following results of E.Odell, N. Tomczak-Jaegermann and R.Wagner [15].

Definition 2.2 ([15])**.** Let $\mathcal{F}$ be a regular set of finite subsets of $\mathbb{N}$. For a basic sequence $(x_i)_i$ in X we define

$$\delta_{\mathcal{F}} = \sup\{\delta \geq 0 : \|\sum_{i=1}^k y_i\| \geq \delta\sum_{i=1}^k \|y_i\| \text{ whenever } (y_i)_1^k \prec (x_i) \text{ is } \mathcal{F}\text{-admissible w.r.t. } (x_i)\}.$$

For $\alpha < \omega_1$ we set $\delta_\alpha(x_i) = \delta_{\mathcal{S}_\alpha}(x_i)$ and $\delta_\alpha(X) = \delta_{\mathcal{S}_\alpha}(X)$.

Definition 2.3 ([15])**.** Let X be a Banach space with basis (e_i), and let $\gamma = (\gamma_\alpha)_{\alpha<\omega_1} \subset \mathbb{R}$. We say that a basic sequence (x_i) in X Δ-stabilizes γ, if there exists $\varepsilon_n \searrow 0$ so that for every $\alpha < \omega_1$ there exists $m \in \mathbb{N}$ such that for every $n \geq m$ if $(y_i) \prec (x_i)_n^\infty$, then $|\delta_\alpha(y_i) - \gamma_\alpha| < \varepsilon_n$.

The Δ-spectrum of X, $\Delta(X)$, is defined to be the set of all γ's so that there exists $(x_i) \prec (e_i)$ such that (x_i) Δ-stabilizes γ.

Proposition 2.4 ([15, Proposition 4.11])**.** (1) *Let X be a Banach space with a basis (e_i). Then there exists $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ and (x_i) block sequence of (e_i) so that (x_i) Δ-stabilizes γ.*

(2) *$(\gamma_\alpha)_{\alpha<\omega_1}$ is a continuous function of α.*

In the above definitions, the admissibility refers with respect to the block basis (x_i) itself. It has been proved in [2] that, if we consider reference level for admissibility fixed the basis, then these two concepts of spectrum actually coincide.

Proof of Theorem 2.1. Let W be a block subspace of X. Then there exists a block sequence $(y_i)_i$ in W and $(\gamma_\alpha)_{\alpha<\omega_1}$ such that (y_i) Δ-stabilizes $(\gamma_\alpha)_{\alpha<\omega_1}$. Let $\varepsilon_n \searrow 0$ be the sequence in the stabilization of $(\gamma_\alpha)_{\alpha<\omega_1}$ by (y_i), i.e. for every $\alpha<\omega_1$ there exists $m\in\mathbb{N}$ such that for every $n\geq m$ if $(x_i)\prec(y_i)_n^\infty$, then $|\delta_\alpha(x_i)-\gamma_\alpha|<\varepsilon_n$.

Inductively choose a strictly increasing sequence $(m(n))_n$ of integers such that $\varepsilon_{m(n)}<\frac{(\delta_1(y_i))^n}{2}$ and

$$|\delta_n(x_i)-\gamma_n|<\varepsilon_{m(n)} \text{ for all } (x_i)\prec(y_{m(i)})_n^\infty \ .$$

Let $N_0=(\min\operatorname{supp} y_{m(i)})_i=(n_i)_i$. Passing to a subset $N=(n_{k_i})_i$, we may assume that for every $\alpha<\omega_1$, if $(n_{k_i})_{i\in F}\in\mathcal{S}_\alpha$, then $(n_{k_i})_{i\in F\setminus\min(F)}\in\mathcal{S}_\alpha(N)$. In particular $(k_i)_{i\in F\setminus\min(F)}\in\mathcal{S}_\alpha$. We shall prove that the subspace $Y=\overline{\operatorname{span}}\{y_{m(k_i)}\}$ is arbitrarily distortable. Let $y^*_{m(k_i)}\in B^*_X$ such that $y^*_{m(k_i)}(y_{m(k_i)})=\|y_{m(k_i)}\|$ and $\operatorname{range}(y^*_{m(k_i)})=\operatorname{range}(y_{m(k_i)})$ for every $i\in\mathbb{N}$. From the hypothesis we have that X does not contain an ℓ_1^ω spreading model. It follows that $\gamma_\omega=0$, for otherwise there exists $(x_i)\prec(y_{m(k_i)})$ such that $\delta_\omega(x_i)>0$, and therefore (x_i) would be an $\delta_\omega(x_i)-\ell_1^\omega$ spreading model. Since $(\gamma_\alpha)_\alpha$ is a continuous function of α it follows that $\gamma_n\searrow 0$. We set

$$\begin{aligned} A_n^* = \{x^* : x^* = \frac{\gamma_n}{2}\sum_{i\in F} x_i^*,\ x_i^*\in B_{Y^*},& \\ y^*_{m(k_n)} \leq (x_i^*)_{i\in F} \text{ is } \mathcal{S}_n\text{-admissible w.r.t. } (y^*_{m(k_i)})_i\} \ ,& \\ A_n = \{y\in S(Y) : y \text{ is } \frac{1}{6}\text{-normed by } A_n^*\} \ .& \end{aligned}$$

We observe that $A_n^*\subset B_{Y^*}$. Indeed, first we observe that if $(x_i)\prec(y_{m(k_i)})_n^\infty$, then

$$\frac{1}{2}\delta_n(x_i)\leq\delta_n(x_i)-\varepsilon_{m(k_n)}<\gamma_n<\delta_n(x_i)+\varepsilon_{m(k_n)}\leq 2\delta_n(x_i) \ ,$$

since $\delta_n(x_i)\geq(\delta_1(y_i))^n>2\varepsilon_{m(k_n)}$.

Let $x^*=\frac{\gamma_n}{2}\sum_{i\in F}x_i^*\in A_n^*$ and for $i\in F$ we set $E_i=\operatorname{range}(x_i^*)$. For every $y\in S_Y$ we have that

$$\begin{aligned} |x^*(y)| \leq \frac{\gamma_n}{2}\sum_{i\in F}|x_i^*(y)| &\leq \frac{\gamma_n}{2}\sum_{i\in F}\|y_{|\operatorname{range}(x_i^*)}\| \\ &\leq \delta_n((y_{m(k_i)})_n^\infty)\sum_{i\in F}\|E_iy\| \leq \|y\| \ , \end{aligned}$$

since $(E_i(y))_{i\in F}$ is $\mathcal{S}_n$-admissible w.r.t. $(y_{m(k_i)})_n^\infty$, and the basis is bimonotone. Also A_n is an asymptotic set. Indeed, for any block sequence $(x_i)\prec(y_{m(k_i)})_n^\infty$ from the stabilization of γ_n we have that $|\delta_n(x_i)-\gamma_n|<\varepsilon_{m(k_n)}$, from which it follows that $\delta_n(x_i)<\gamma_n+\varepsilon_{m(k_n)}$. It follows from the definition of $\delta_n(x_i)$ that there exists a block sequence $(w_i)_{i\in F}$ of (x_i) which is $\mathcal{S}_n$-admissible w.r.t. (x_i) and therefore w.r.t. $(y_{m(i)})$ as well, such that

$$\delta_n(x_i)\sum_{i\in F}\|w_i\|\leq\|\sum_{i\in F}w_i\|<(\gamma_n+\varepsilon_{m(k_n)})\sum_{i\in F}\|w_i\| \ .$$

Let $x_i^* \in B_{Y^*}$ be such that $x_i^*(w_i) = \|w_i\|$ and $\text{range}(x_i^*) \subset \text{range}(w_i)$. We set $x^* = \frac{\gamma_n}{2}\sum_{i\in F} x_i^* \in A_n^*$, and $y = \frac{\sum_{i\in F} w_i}{\|\sum_{i\in F} w_i\|}$. Then we have that

$$x^*(y) = \frac{\gamma_n}{2}\frac{\sum_{i\in F}\|w_i\|}{\|\sum_{i\in F} w_i\|} \geq \frac{\gamma_n}{2(\gamma_n + \varepsilon_{m(k_n)})} \geq \frac{1}{6} \ , \tag{2.1}$$

since $\gamma_n \geq \delta_n(x_i) - \varepsilon_{m(k_n)} > \frac{(\delta_1(x_i))^n}{2} > \varepsilon_{m(k_n)}$.

From the hypothesis we have that every normalized block sequence has a further block subsequence which is an $c{-}\ell_1^k$ spreading model, $k \in \mathbb{N}$. It follows that for every $\varepsilon > 0$ every block sequence $(z_i) \prec (y_{m(k_i)})$ contains $\frac{c}{2}$-normalized $(\varepsilon, n+1, n)$-s.c.c. Indeed let $0 < \varepsilon < \frac{1}{2}$ and (z_i) be an $c{-}\ell_1^{n+1}$ spreading model, and $\min \text{supp} z_i = n_{l_i}$. Let $[n+1]_1^L = \sum_{i\in F}\alpha_{l_i}e_{n_{l_i}}$ be an $(n+1)$-average of a subset L of (n_{l_i}) with $l_{\min F} > 3/\varepsilon$. Then by the remark following Proposition 1.4, $\sum_{i\in F}\alpha_{l_i}e_{n_{l_i}}$ is an $(\varepsilon, n+1, n)$-basic s.c.c., $(n_{l_i})_{i\in F} \in \mathcal{S}_{n+1}$, and by the properties of the set N, we have that $(l_i)_{i\in F\setminus\min(F)} \in \mathcal{S}_{n+1}$. Let $G = \{l_i : i \in F\}$ and set $x = \sum_{j\in G}\alpha_j z_j$. Then x is an $(\varepsilon, n+1, n)$ s.c.c., and

$$\|x\| \geq c \sum_{j\in G\setminus\min(G)} \alpha_j \geq c(1-\varepsilon) \ ,$$

since (z_i) is a $c - \ell_1^{n+1}$ spreading model. We set

$$B_n = \{b : b \text{ is an } \frac{c}{2}\text{-normalized } (\varepsilon_{m(k_n)}, n+1, n)\text{-s.c.c.}$$
$$\text{of a normalized block sequence of the basis of } Y\}.$$

From the above it follows that B_n is an asymptotic set in Y. For every $b \in B_n$ we have

$$|x^*(b)| \leq 3\gamma_n \text{ for every } \ x^* \in A_n^* \ . \tag{2.2}$$

Indeed, let (z_i) be a normalized block sequence of the basis of Y, $b = \sum_{i\in G} b_i z_i \in B_n$, and $x^* = \gamma_n/2\sum_{k\in F} x_k^* \in A_n^*$. Set $I = \{i \in G : \text{supp} z_i \cap \text{supp} x_k^* \neq \emptyset$ for at most one $k\}$, and $J = G\setminus I$. Also for every $i \in J$, let $K_i = \{k : \text{supp} z_i \cap \text{supp} x_k^* \neq \emptyset\}$. Then $(z_i)_{i\in J}$ is the union of at most two $\mathcal{S}_n$ admissible sets, hence

$$\begin{aligned}\frac{\gamma_n}{2}\sum_k |x_k^*(\sum_i b_i z_i)| &\leq \frac{\gamma_n}{2}\left(\sum_{i\in I} b_i\|z_i\| + \sum_{i\in J} b_i \sum_{k\in K_i} |x_k^*(z_i)|\right)\\ &\leq \frac{\gamma_n}{2}\left(1 + \frac{4\varepsilon_{m(k_n)}}{\gamma_n}\max_i \|z_i\|\right) \leq 3\gamma_n \ .\end{aligned}$$

Combining (2.1) and (2.2), it follows that the equivalent norm $|||y||| = \gamma_n\|y\| + \sup\{x^*(y) : x^* \in A_n^*\}$ gives a $\frac{1}{6\gamma_n(1+6/c)}$ distortion on Y. Since $\inf\{\gamma_n : n \in \mathbb{N}\} = 0$ we have that Y is arbitrarily distortable. □

Remark. The arguments of Theorem 2.1 may give us another approach to the distortion of the space X, constructed by E.Odell and Th.Schlumprecht [13], which does not have any ℓ_p, $1 \leq p \leq \infty$, as a spreading model. Following [15], we have to consider the indices $\delta_{A_n}((x_i)_i)$ for the families $A_n = \{F \subset \mathbb{N} : \#F \leq n\}$ and to prove a stabilization result for the sequence $(\delta_{A_n}((x_i)_i)_n$ for every block subspace. Since ℓ_1 is finitely block representable in X, and X does not contain ℓ_1 spreading models, we may deduce that every subspace has an arbitrarily distortable subspace.

Theorem 2.1 should be compared with the following result from [15]: Let $Y = \langle (y_i) \rangle$ be a subspace of X, and $(\gamma_\alpha)_\alpha$ be stabilized by (y_i). For $\alpha < \omega_1$ let $\hat{\gamma}_\alpha(Y) = \lim_n (\gamma_{\alpha \cdot n}(Y))^{\frac{1}{n}}$. If $\lim_n \gamma_{\alpha \cdot n} \widehat{\gamma}_\alpha(Y)^{-n} = 0$, then Y contains an arbitrarily distortable subspace.

3.

Theorem 3.1. *Let $X = T[(\mathcal{S}_n, \theta_n)_n]$ be such that $\theta_{n+m} \geq \theta_n \theta_m$, $\lim_n \theta_n = 0$. Then the space X does not contain an $\ell_1^{\omega 2}$ spreading model.*

Proof. On the contrary assume that there exists a normalized block sequence $(x_i)_i$ which is an $2c - \ell_1^{\omega 2}$ spreading model for some constant $2c > 0$. We shall prove that for every $n_0 \in \mathbb{N}$ we have that $c \leq 10\theta_{n_0}$, which yields that $c = 0$.

By Proposition 1.1(b) choose $k(0) \in \mathbb{N}$ such that if $k(0) \leq F \in \mathcal{S}_\omega$, then $F \in \mathcal{S}_{\omega 2}$ and $k(n)$ such that if $k(n) \leq F \in \mathcal{S}_{\omega + n}$, then $F \in \mathcal{S}_{\omega 2}$. Without loss of generality we may assume that if $n \leq F \in \mathcal{S}_n$, then $F \in \mathcal{S}_\omega$.

Let $n_0 \in \mathbb{N}$ and set $N_0 = \{\min \operatorname{supp} x_i\}_{i \in \mathbb{N}} = (n_i)_i$. Define a sequence (m_i) by the rule $m_1 = n_1$ and $m_{i+1} = n_{m_i}$, and consider the subset $N = (n_{m_i})_{i \in \mathbb{N}}$ of N_0. Passing to a further subset of N and relabelling we may assume that $m_{\min N} > \max\{k(0), k(n_0)\}$, $\mathcal{S}_{n_0}[\mathcal{S}_\omega](N) \subset \mathcal{S}_{\omega + n_0}$ (by Proposition 1.1(c)), and moreover that the following holds:

$$\forall \ \alpha < \omega_1, \text{ if } (n_{m_i})_{i \in F} \in \mathcal{S}_\alpha, \text{ then } (n_{m_i})_{i \in F \setminus \min(F)} \in \mathcal{S}_\alpha(N_0),$$

hence $(m_i)_{i \in F \setminus \min(F)} \in \mathcal{S}_\alpha$ (by Proposition 1.1(a)).

Let $L_0 = (n_{m_i})_{M_0} \subset N$ be such that $m_{\min L_0} > \min\{n_0, \frac{3}{\theta_{n_0}^2}\}$. Let $n_1 > n_0 + 10$. Passing to further subset $L_1 = (n_{m_i})_{M_1}$ of L_0 we may assume that $m_{\min L_1} > \min\{n_1, \frac{3}{\theta_{n_1}^2}\}$.

Let $[n_1]_1^{L_1} = \sum_{j \in F_1} \alpha_{m_j} e_{n_{m_j}}$ be the first n_1-average of L_1, where (e_i) denotes the unit vector basis of the space X. We have that $[n_1]_1^{L_1}$ is an $(\theta_{n_1}^2, n_1, n_1 - 1)$-basic s.c.c. The set $(n_{m_j})_{j \in F_1} \in \mathcal{S}_{n_1}$ and therefore by the property of the set N we have that $G_1 = \{m_i : i \in F_1 \setminus \min(F_1)\} \in \mathcal{S}_{n_1}$. We set $J_1 = \{m_i : i \in F_1\}$, i.e. $J_1 = G_1 \cup \{m_{\min F_1}\}$. Then $[n_1]_1^{L_1} = \sum_{j \in J_1} \alpha_j e_{n_j}$.

Set $y_1 = \sum_{j \in J_1} \alpha_j x_j$. Then y_1 is a $(\theta_{n_1}^2, n_1, n_1 - 1)$-s.c.c. of (x_i) with $c \leq \|y_1\|$.

Indeed, since $\{\min \operatorname{supp} x_j : j \in J_1\} = \{n_{m_j} : j \in F_1\}$ and $[n_1]_1^{L_1}$ is an $(\theta_{n_1}^2, n_1, n_1 - 1)$ basic s.c.c., we have that y_1 is an $(\theta_{n_1}^2, n_1, n_1 - 1)$ s.c.c. Also since $G_1 \geq n_1$ and $G_1 \in \mathcal{S}_{n_1}$, we have that $G_1 \in \mathcal{S}_\omega$. Also from the choice of L_0, $G_1 > k(0)$. Therefore $G_1 \in \mathcal{S}_{\omega 2}$, and

$$\|y_1\| \geq \| \sum_{j \in G_1} \alpha_j x_j \| \geq 2c \sum_{j \in G_1} \alpha_j \geq 2c \sum_{j \in F_1 \setminus \min(F_1)} \alpha_{m_j} \geq c \ ,$$

since $(x_i)_i$ is a $2c - \ell_1^{\omega 2}$ spreading model.

Assume that we have chosen $y_1 < y_2 < \ldots < y_\ell$ and $n_1 < n_2 < \ldots < n_\ell$ such that

(1) Each $y_r = \sum_{j \in J_r} \alpha_j x_j$ is a c-normalized $(\theta_{n_r}^2, n_r, n_r - 1)$ s.c.c. of (x_i), for $r = 1, \ldots, \ell$.
(2) $n_r < J_r \setminus \min(J_r) = G_r \in \mathcal{S}_{n_r}$ for $r = 1, \ldots, \ell$.
(3) $\|y_r\|_{\ell_1} \leq \frac{\theta_{n_r}}{\theta_{n_{r+1}}}$ for $r = 1, \ldots, \ell - 1$ where $\| \cdot \|_{\ell_1}$ denotes the norm of the space ℓ_1.

Then we choose $n_{\ell+1}$ such that $\|y_\ell\|_{\ell_1} \leq \frac{\theta_{n_\ell}}{\theta_{n_{\ell+1}}}$ and $2 < \theta_{n_\ell}/\theta_{n_{\ell+1}}$. We also choose a subset $L_{\ell+1} = (n_{m_i})_i$ of $L \setminus \bigcup_{r=1}^{\ell} \operatorname{supp}[n_r]_1^{L_r}$ such that $m_{\min L_{\ell+1}} \geq \min\{n_{\ell+1}, \frac{3}{\theta^2_{n_{\ell+1}}}\}$.

Let $[n_{\ell+1}]_1^{L_{\ell+1}} = \sum_{j \in F_{\ell+1}} \alpha_{m_j} e_{n_{m_j}}$ be the first $n_{\ell+1}$-average of the set $L_{\ell+1}$. Then $[n_{\ell+1}]_1^{L_{\ell+1}}$ is $(\theta^2_{n_{\ell+1}}, n_{\ell+1}, n_{\ell+1}-1)$ basic s.c.c. Also $(n_{m_j})_{j \in F_{\ell+1}} \in \mathcal{S}_{n_{\ell+1}}$ and therefore by the property of the set N we have that the set $G_{\ell+1} = \{m_j : j \in F_{\ell+1} \setminus \min(F_{\ell+1})\} \in \mathcal{S}_{n_{\ell+1}}$. We set $J_{\ell+1} = \{m_j : j \in F_{\ell+1}\}$. By the choice of $L_{\ell+1}$, we have that $n_{\ell+1} \leq G_{\ell+1}$, and therefore $G_{\ell+1} \in \mathcal{S}_\omega$. Also we have that $k(0) \leq G_{\ell+1}$, so we deduce that $G_{\ell+1} \in \mathcal{S}_{\omega 2}$.

We set $y_{\ell+1} = \sum_{j \in J_{\ell+1}} \alpha_j x_j$. With the same arguments as for y_1 we have that $y_{\ell+1}$ is a $(\theta^2_{n_{\ell+1}}, n_{\ell+1}, n_{\ell+1}-1)$-s.c.c. of (x_i) with $\|y_{\ell+1}\| \geq c$.

We continue in the same manner to produce a sequence $(y_r)_{r\in\mathbb{N}}$ satisfying properties (1)–(3) for all r.

Now let $M \subset \{\min \operatorname{supp} y_i : i \in \mathbb{N}\} \subset L$ and let $[n_0]_1^M = \sum_{j \in G_0} \gamma_{m_j} e_{n_{m_j}}$ be the first n_0 average of the set M, such that $[n_0]_1^M$ is $(\theta^2_{n_0}, n_0, n_0-1)$-basic s.c.c. Then we have that $[n_0]_1^M = \sum_{j\in G_0} \gamma_j e_{n_j}$, where $G = \{m_j : j \in G_0\}$, and $G \setminus \min(G) \in \mathcal{S}_{n_0}$. Also we have that $\bigcup_{G \setminus \min(G)} G_j \in \mathcal{S}_{n_0}[\mathcal{S}_\omega]$ and therefore

$$\{n_i : i \in \bigcup_{G\setminus\min(G)} G_j\} \in \mathcal{S}_{n_0}[\mathcal{S}_\omega](N) \subset \mathcal{S}_{\omega+n_0} \ ,$$

by the choice of the set N. From the definition of $m_{i+1} = n_{m_i}$ it follows that

$$H = \bigcup_{G\setminus\min(G)} (G_j \setminus \min G_j) \in \mathcal{S}_{\omega+n_0}, \tag{3.1}$$

since the families $(\mathcal{S}_\alpha)_\alpha$ are spreading. Indeed, let $G_j = \{m_{r_1^{(j)}} < m_{r_2^{(j)}} < \ldots < m_{r_j^{(j)}}\}$. Then for every $k \geq 2$, $m_{r_k^{(j)}} = n_{m_{r_k^{(j)}-1}} \geq n_{m_{r_{k-1}^{(j)}}}$ and since $(n_i)_{i \in \bigcup_{G\setminus\min(G)} G_j} \in \mathcal{S}_{\omega+n_0}$ we have (3.1). Also $k(n_0) < H$ and therefore H belongs to $\mathcal{S}_{\omega 2}$.

Set $z = \sum_{j\in G} \gamma_j y_j = \sum_{j \in G} \gamma_j \sum_{i \in J_j} \alpha_i x_i$. Since we have assumed that (x_i) is a $2c - \ell_1^{\omega 2}$ spreading model, setting $z_i = \gamma_j \alpha_i x_i$, for every $i \in G_j$ and every $j \in G$, we have that

$$\begin{aligned} \|z\| \geq \| \sum_{i \in \bigcup_{j\in G\setminus\min(G)} G_j} z_i \| &\geq 2c \sum_{j \in G\setminus\min(G)} \gamma_j \sum_{i \in G_j \setminus \min(G_j)} \beta_i \\ &\geq 2c \sum_{j\in G\setminus\min(G)} \gamma_j (1 - 2\theta^2_{n_j}) \geq 2c(1-\theta^2_{n_0})(1-2\theta^2_{n_1}) \geq c \ . \end{aligned}$$

The vector z is also an $(\theta^2_{n_0}, n_0)$-R.I.s.c.c. That is, z is of the form $z = \sum_{j=1}^k \gamma_j y_j$ such that

(1) y_j is a c-normalized $(\theta^2_{n_j}, n_j, n_j - 1)$-s.c.c.
(2) $n_0 + 2 < n_1 < \ldots < n_k$, and $\{\min \operatorname{supp} y_j\}_j \in \mathcal{S}_{n_0}$.
(3) $\|y_j\|_{\ell_1} \leq \frac{\theta_{n_j}}{\theta_{n_{j+1}}}$ and $2 < \theta_{n_j}/\theta_{n_{j+1}}$.

For such vectors we have that there exists a constant $M \leq 10$ such that

$$\|z\| \leq M\theta_{n_0}\ . \tag{3.2}$$

We refer to [3] (Corollary 2.15), [4] (Proposition 1.15), and [9] for a proof of this estimation. Therefore we have that $c \leq 10\theta_{n_0}$. Since n_0 was arbitrarily chosen we have the result. □

Remark. Let $X = T[(\mathcal{S}_{\xi_n}, \theta_n)_n]$ be such that $\lim \xi_n = \xi$ is a limit ordinal. If the sequence $(\xi_n, \theta_n)_n$ is "appropriate" chosen, quoting the above arguments, for appropriate $(\theta_n^2, \xi_n, \zeta)$-s.c.c., we have that the space X does not contain $\ell_1^{\xi 2}$ spreading models. We do not have a proof of inequality (3.2) for every sequence $(\xi_n) \nearrow \xi$. We refer to [8] for a proof of inequality (3.2) for the appropriate chosen sequence $(\xi_n, \theta_n)_n$.

The same arguments apply in the case of p-spaces, and yield that those spaces contain no ℓ_1^2 spreading model. A space X is said to be p-space if $X = T[(\mathcal{A}_n, \frac{1}{n^{1/q_n}})]$, where $\frac{1}{p_n} + \frac{1}{q_n} = 1$ and $p_n \searrow p \in [1, \infty)$ and $\mathcal{A}_n = \{F \subset \mathbb{N} : \#F \leq n\}$ where the norm is defined similarly to the one in Definition 1.2(b). We can also apply the above argument for $A_k[\mathcal{S}_1]$ sequences to produce R.I.s.c.c. of length k and in this case we know that the norm is less than or equal to Mn_k^{1/p_k} [12]. Let us recall that the existence of ℓ_1 spreading models in Schlumprecht's space [16] has been established in [11].

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