

## ON SCHWARZ TYPE INEQUALITIES

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(Communicated by Joseph A. Ball)

ABSTRACT. We show Schwarz type inequalities and consider their converses. A continuous function  $f : [0, \infty) \rightarrow [0, \infty)$  is said to be semi-operator monotone on  $(a, b)$  if  $\{f(t^{\frac{1}{2}})\}^2$  is operator monotone on  $(a^2, b^2)$ . Let  $T$  be a bounded linear operator on a complex Hilbert space  $\mathcal{H}$  and  $T = U|T|$  be the polar decomposition of  $T$ . Let  $0 \leq A, B \in B(\mathcal{H})$  and  $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$  for  $x, y \in \mathcal{H}$ . (1) If a non-zero function  $f$  is semi-operator monotone on  $(0, \infty)$ , then  $|\langle Tx, y \rangle| \leq \|f(A)x\| \|g(B)y\|$  for  $x, y \in \mathcal{H}$ , where  $g(t) = t/f(t)$ . (2) If  $f, g$  are semi-operator monotone on  $(0, \infty)$ , then  $|\langle Uf(|T|)g(|T|x), y \rangle| \leq \|f(A)x\| \|g(B)y\|$  for  $x, y \in \mathcal{H}$ . Also, we show converses of these inequalities, which imply that semi-operator monotonicity is necessary.

### 1. INTRODUCTION

Let  $\mathcal{H}$  be a complex Hilbert space and  $B(\mathcal{H})$  be the algebra of all bounded linear operators on  $\mathcal{H}$ . Furuta [3] extended the Heinz-Kato [5], [6] inequality, which is an extension of the Schwarz inequality.

**Proposition 1** (Heinz-Kato-Furuta). *Let  $T = U|T|$  be the polar decomposition of  $T \in B(\mathcal{H}), 0 \leq A, B \in B(\mathcal{H})$  and  $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$  for  $x, y \in \mathcal{H}$ . Then*

$$|\langle T|T|^{\alpha+\beta-1}x, y \rangle| \leq \|A^\alpha x\| \|B^\beta y\|$$

for  $x, y \in \mathcal{H}, \alpha, \beta \in [0, 1]$  with  $1 \leq \alpha + \beta$ .

The Heinz-Kato inequality is the case  $\alpha + \beta = 1$ . Recently, M. Uchiyama [7] extended this result as follows.

**Proposition 2** (M. Uchiyama). *Let  $f, g : [0, \infty) \rightarrow [0, \infty)$  be continuous operator monotone functions. Let  $T = U|T|$  be the polar decomposition of  $T \in B(\mathcal{H}), 0 \leq A, B \in B(\mathcal{H})$  and  $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$  for  $x, y \in \mathcal{H}$ . Then*

$$|\langle Uf(|T|)g(|T|x), y \rangle| \leq \|f(A)x\| \|g(B)y\| \quad \text{for } x, y \in \mathcal{H}.$$

In this paper, we introduce semi-operator monotonicity which assures the conclusion of Proposition 2.

**Definition 3.** A continuous function  $f : [0, \infty) \rightarrow [0, \infty)$  is said to be semi-operator monotone on  $(a, b)$  if  $\{f(t^{\frac{1}{2}})\}^2$  is operator monotone on  $(a^2, b^2)$ .

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Received by the editors December 17, 2001 and, in revised form, March 29, 2002.  
2000 *Mathematics Subject Classification.* Primary 47A30, 47A63, 47B15.  
*Key words and phrases.* Schwarz inequality, Heinz-Kato-Furuta inequality.  
This research was supported by Grant-in-Aid Research No. 12640187.

**Proposition 4.** *Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a continuous function. Then  $f$  is semi-operator monotone on  $(a, b)$  if and only if  $f$  has an analytic continuation for  $\Pi_1 = \{z \in \mathbb{C} \mid 0 < \arg z < \frac{\pi}{2}\}$  with  $f(\Pi_1) \subset \Pi_1$ .*

*Proof.* Let  $f$  be semi-operator monotone on  $(a, b)$ . Then  $g(t) = \{f(t^{\frac{1}{2}})\}^2$  is operator monotone on  $(a^2, b^2)$ . Hence  $g(t)$  has analytic continuation  $g(z) = \{f(z^{\frac{1}{2}})\}^2$  for  $\Pi_+ = \{z \in \mathbb{C} \mid 0 < \arg z < \pi\}$  with  $g(\Pi_+) \subset \Pi_+$ . Hence  $f(z^{\frac{1}{2}})$  is analytic for  $z \in \Pi_+$  and  $f(z^{\frac{1}{2}}) \in \Pi_1$ . This implies that  $f$  has an analytic continuation for  $\Pi_1$  with  $f(\Pi_1) \subset \Pi_1$ . The converse is clear.  $\square$

Every operator monotone function is semi-operator monotone. But the converse does not hold. For example, let  $f(t) = \{\log(1 + t^2)\}^{\frac{1}{2}} : [0, \infty) \rightarrow [0, \infty)$ . Then  $\{f(t^{\frac{1}{2}})\}^2 = \log(1 + t)$  is operator monotone on  $(0, \infty)$ . But its analytic continuation  $f(z) = \{\log(1 + z^2)\}^{\frac{1}{2}}$  is singular at  $z = i$ . Hence  $f(t)$  is not operator monotone on  $(0, \infty)$ . Thus the class of all semi-operator monotone functions strictly includes the class of all operator monotone functions. Also, this example shows that  $g(t) = \log(1 + t)$  is operator monotone, but  $\{g(t^2)\}^{\frac{1}{2}}$  is not operator monotone. Non-constant semi-operator monotone functions are strictly increasing.

J. S. Aujla [1] and J. I. Fujii, M. Fujii [2] studied semi-operator monotone function on  $(0, \infty)$  and gave several characterizations. Moreover J. I. Fujii and M. Fujii [2] studied  $n$ -operator monotone function on  $(0, \infty)$ , which generalize semi-operator monotonicity.

Since every operator monotone function  $f : [0, \infty) \rightarrow [0, \infty)$  is semi-operator monotone on  $(0, \infty)$ , we can give a simple proof of Uchiyama's result as the following Schwarz type inequality.

**Theorem 5.** *Let  $f, g : [0, \infty) \rightarrow [0, \infty)$  be continuous functions. Let  $T = U|T|$  be the polar decomposition of  $T \in B(\mathcal{H})$ ,  $0 \leq A, B \in B(\mathcal{H})$  and  $\|Tx\| \leq \|Ax\|$ ,  $\|T^*y\| \leq \|By\|$  for  $x, y \in \mathcal{H}$ . Then, if  $f, g$  are semi-operator monotone on  $(0, \infty)$ , we have*

$$(1) \quad |\langle Uf(|T|)g(|T|x), y \rangle| \leq \|f(A)x\| \|g(B)y\| \quad \text{for } x, y \in \mathcal{H}.$$

*Conversely, if  $f, g$  satisfy the conclusion (1) and  $f(t)g(t) \not\equiv 0$ , then  $f, g$  are semi-operator monotone on  $(0, \infty)$ .*

*Proof.* Let  $f, g$  be semi-operator monotone on  $(0, \infty)$ . Since  $|T|^2 \leq A^2$  and  $|T^*|^2 \leq B^2$ , we have  $\{f(|T|)\}^2 \leq \{f(A)\}^2$  and  $\{g(|T^*|)\}^2 \leq \{g(B)\}^2$ . Hence

$$\begin{aligned} |\langle Uf(|T|)g(|T|x), y \rangle| &= |\langle g(|T^*|)Uf(|T|x), y \rangle| \\ &= |\langle Uf(|T|x), g(|T^*|)y \rangle| \\ &\leq \|f(|T|x)\| \|g(|T^*|)y\| \\ &\leq \|f(A)x\| \|g(B)y\| \quad \text{for } x, y \in \mathcal{H}. \end{aligned}$$

Conversely, take a point  $a \in (0, \infty)$  such that  $f(a)g(a) > 0$ . Let  $(c, d)$  be the maximal open interval including  $a$  such that  $0 < f(t), g(t)$  for  $t \in (c, d)$ . First we show that  $f, g$  are semi-operator monotone on  $(c, d)$ . Let  $\tilde{f}(t) = \{f(t^{\frac{1}{2}})\}^2$ ,  $\tilde{g}(t) = \{g(t^{\frac{1}{2}})\}^2$  for  $t \in (c^2, d^2)$ . Let  $C \leq D$ ,  $\sigma(C), \sigma(D) \subset (c^2, d^2)$ . Then  $C, D, g(C^{\frac{1}{2}})$  are invertible. Let  $T = C^{\frac{1}{2}}$ ,  $A = D^{\frac{1}{2}}$ ,  $B = C^{\frac{1}{2}}$ . Then condition (1) implies

$$\begin{aligned} \|f(D^{\frac{1}{2}})x\| \|g(C^{\frac{1}{2}})y\| &\geq |\langle f(C^{\frac{1}{2}})g(C^{\frac{1}{2}})x, y \rangle| \\ &= |\langle f(C^{\frac{1}{2}})x, g(C^{\frac{1}{2}})y \rangle| \quad \text{for } x, y \in \mathcal{H}. \end{aligned}$$

Hence  $\|f(C^{\frac{1}{2}})x\| \leq \|f(D^{\frac{1}{2}})x\|$  for  $x \in \mathcal{H}$  and  $\tilde{f}(C) = \{f(C^{\frac{1}{2}})\}^2 \leq \{f(D^{\frac{1}{2}})\}^2 = \tilde{f}(D)$ . Thus  $f(t)$  is semi-operator monotone on  $(c, d)$ . Similarly we can show that  $g(t)$  is semi-operator monotone on  $(c, d)$ .

We show  $c = 0$  and  $d = \infty$ . If  $d < \infty$ , then  $f(d) = 0$  or  $g(d) = 0$ . But  $f, g$  are positive and semi-operator monotone on  $(c, d)$ . Hence  $0 < f(d)$  and  $0 < g(d)$ . This is a contradiction. Hence  $d = \infty$ . Assume  $0 < c$ . Then  $f(c) = 0$  or  $g(c) = 0$ . In this case  $f(0)g(0) = 0$ . Because if  $f(0)g(0) > 0$ , then  $f(t) > 0, g(t) > 0$  for  $t \in (0, \infty)$  by the preceding argument. Let

$$A = B = \begin{pmatrix} \sqrt{2}c & 0 \\ 0 & c \end{pmatrix}, T = \frac{2c}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then  $0 \leq A, B, T$  and  $T^2 \leq A^2 = B^2$ . Let  $x = y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then

$$\langle Uf(|T|)g(|T|x), y \rangle = \frac{1}{2}f\left(\frac{2c}{\sqrt{3}}\right)g\left(\frac{2c}{\sqrt{3}}\right) \neq 0$$

and

$$\|f(A)x\| \|g(B)y\| = \left\| \begin{pmatrix} 0 \\ f(c) \end{pmatrix} \right\| \left\| \begin{pmatrix} 0 \\ g(c) \end{pmatrix} \right\| = 0.$$

This is a contradiction. Thus  $c = 0$  and  $f, g$  are semi-operator monotone on  $(0, \infty)$ . □

Next we show a direct extension of the Heinz-Kato inequality.

**Theorem 6.** *Let  $T \in B(\mathcal{H}), 0 \leq A, B \in B(\mathcal{H})$  and  $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$  for  $x, y \in \mathcal{H}$ . Then, if a non-zero function  $f$  is semi-operator monotone on  $(0, \infty)$ , then we have*

$$(2) \quad |\langle Tx, y \rangle| \leq \|f(A)x\| \|g(B)y\|$$

for  $x, y \in \mathcal{H}$ , where  $g(t) = t/f(t)$ .

*Conversely, if continuous functions  $f, g : [0, \infty) \rightarrow [0, \infty)$  satisfy conclusion (2) and  $f(t)g(t) = t$  for  $t \in (0, \infty)$ , then  $f, g$  are semi-operator monotone on  $(0, \infty)$ .*

*Proof.* Let  $f$  be semi-operator monotone on  $(0, \infty)$ . Then  $\{f(t^{\frac{1}{2}})\}^2$  is operator monotone on  $(0, \infty)$  and  $0 < f(t^{\frac{1}{2}})$  for  $t \in (0, \infty)$ . Hence  $\{g(t^{\frac{1}{2}})\}^2 = t/\{f(t^{\frac{1}{2}})\}^2$  is operator monotone on  $(0, \infty)$  by [4, Corollary 2.6]. Thus  $g$  is semi-operator monotone on  $(0, \infty)$ . Hence

$$\begin{aligned} |\langle Tx, y \rangle| &= |\langle U|T|x, y \rangle| \\ &= |\langle Uf(|T|)g(|T|x), y \rangle| \\ &\leq \|f(A)x\| \|g(B)y\| \quad \text{for } x, y \in \mathcal{H} \end{aligned}$$

by Theorem 5. The converse is easy from Theorem 5. □

*Remark 7.* For example,  $f(t) = \sqrt{1+t^2}$  is semi-operator monotone on  $(0, \infty)$ . Hence, if  $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$  for  $x, y \in \mathcal{H}$ , then

$$|\langle Tx, y \rangle| \leq \left\| \sqrt{1+A^2}x \right\| \left\| \frac{B}{\sqrt{1+B^2}}y \right\| \quad \text{for } x, y \in \mathcal{H}.$$

## ACKNOWLEDGEMENTS

We express sincere thanks to Professor M. Fujii, who kindly pointed out to us paper [2]. Also, we express sincere thanks to the referee for his helpful comments.

## REFERENCES

- [1] J. S. Aujla, *Perturbation bounds for certain operator functions*, Math. Inequal. Appl., **4** (2001), 609–617. MR **2002h**:47015
- [2] J. I. Fujii and M. Fujii, *An analogue to Hansen's theory of generalized Löwner's functions*, Math. Japonica, **35** (1990), 327–330. MR **91d**:47015
- [3] T. Furuta, *An extension of the Heinz-Kato theorem*, Proc. Amer. Math. Soc., **120** (1994), 785–787. MR **94e**:47034
- [4] E. Hansen and G. K. Pedersen, *Jensen's inequality for operators and Löwner's theorem*, Math. Ann., **258** (1982), 229–241.
- [5] E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann., **123** (1951), 415–438. MR **13**:471f
- [6] T. Kato, *Notes on some inequalities for linear operators*, Math. Ann., **125** (1952), 208–212. MR **14**:766e
- [7] M. Uchiyama, *Further extension of the Heinz-Kato-Furuta inequality*, Proc. Amer. Math. Soc., **127** (1999), 2899–2904. MR **2000a**:47045

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