ON SCHWARZ TYPE INEQUALITIES

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Abstract. We show Schwarz type inequalities and consider their converses. A continuous function $f : [0, \infty) \to [0, \infty)$ is said to be semi-operator monotone on $(a, b)$ if $\{f(t^2)\}^2$ is operator monotone on $(a^2, b^2)$. Let $T$ be a bounded linear operator on a complex Hilbert space $H$ and $T = U|T|$ be the polar decomposition of $T$. Let $0 \leq A, B \in B(H)$ and $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$ for $x, y \in H$. Then $\langle T(T)^{\alpha + \beta - 1}x, y \rangle \leq \|A^\alpha x\|\|B^\beta y\|$ for $x, y \in H$, where $\alpha, \beta \in [0, 1]$ with $1 \leq \alpha + \beta$.

The Heinz-Kato inequality is the case $\alpha + \beta = 1$. Recently, M. Uchiyama extended this result as follows.

Proposition 1 (Heinz-Kato-Furuta). Let $T = U|T|$ be the polar decomposition of $T \in B(H), 0 \leq A, B \in B(H)$ and $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$ for $x, y \in H$. Then $\langle U|T|g(T)x, y \rangle \leq \|A^\alpha x\|\|g(B)y\|$ for $x, y \in H$.

1. Introduction

Let $H$ be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on $H$. Furuta extended the Heinz-Kato inequality, which is an extension of the Schwarz inequality.

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The Heinz-Kato inequality is the case $\alpha + \beta = 1$. Recently, M. Uchiyama extended this result as follows.

Proposition 2 (M. Uchiyama). Let $f, g : [0, \infty) \to [0, \infty)$ be continuous operator monotone functions. Let $T = U|T|$ be the polar decomposition of $T \in B(H), 0 \leq A, B \in B(H)$ and $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$ for $x, y \in H$. Then $\langle Uf(|T|)g(|T|)x, y \rangle \leq \|A^\alpha x\|\|g(B)y\|$ for $x, y \in H$.

In this paper, we introduce semi-operator monotonicity which assures the conclusion of Proposition 2.

Definition 3. A continuous function $f : [0, \infty) \to [0, \infty)$ is said to be semi-operator monotone on $(a, b)$ if $\{f(t^2)\}^2$ is operator monotone on $(a^2, b^2)$.

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Proposition 4. Let $f : [0, \infty) \to [0, \infty)$ be a continuous function. Then $f$ is semi-operator monotone on $(a, b)$ if and only if $f$ has an analytic continuation for $\Pi_1 = \{z \in \mathbb{C} | 0 < \arg z < \frac{\pi}{2}\}$ with $f(\Pi_1) \subset \Pi_1$.

Proof. Let $f$ be semi-operator monotone on $(a, b)$. Then $g(t) = \{f(t^\frac{1}{2})\}^2$ is operator monotone on $(a^2, b^2)$. Hence $g(t)$ has analytic continuation $g(z) = \{f(z^\frac{1}{2})\}^2$ for $\Pi_+ = \{z \in \mathbb{C} | 0 < \arg z < \pi\}$ with $g(\Pi_+) \subset \Pi_+$. Hence $f(z^\frac{1}{2})$ is analytic for $z \in \Pi_+$ and $f(z^\frac{1}{2}) \in \Pi_1$. This implies that $f$ has an analytic continuation for $\Pi_1$ with $f(\Pi_1) \subset \Pi_1$. The converse is clear. \hfill \qed

Every operator monotone function is semi-operator monotone. But the converse does not hold. For example, let $f(t) = \{\log(1 + t^2)\}^\frac{1}{2} : [0, \infty) \to [0, \infty)$. Then $\{f(t^\frac{1}{2})\}^2 =\log(1+t)$ is operator monotone on $(0, \infty)$. But its analytic continuation $f(z) = \{\log(1 + z^2)\}^\frac{1}{2}$ is singular at $z = i$. Hence $f(t)$ is not operator monotone on $(0, \infty)$. Thus the class of all semi-operator monotone functions strictly includes the class of all operator monotone functions. Also, this example shows that $g(t) = \log(1 + t)$ is operator monotone, but $g(t^\frac{1}{2})$ is not operator monotone. Non-constant semi-operator monotone functions are strictly increasing.


Since every operator monotone function $f : [0, \infty) \to [0, \infty)$ is semi-operator monotone on $(0, \infty)$, we can give a simple proof of Uchiyama’s result as the following Schwarz type inequality.

Theorem 5. Let $f, g : [0, \infty) \to [0, \infty)$ be continuous functions. Let $T = U|T|$ be the polar decomposition of $T \in B(\mathcal{H}), 0 \leq A, B \in B(\mathcal{H})$ and $\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\|$ for $x, y \in \mathcal{H}$. Then, if $f, g$ are semi-operator monotone on $(0, \infty), we have

\begin{equation}
\|\langle Uf(|T|)g(|T|)x, y \rangle\| \leq \|f(A)x\| \|g(B)y\| \text{ for } x, y \in \mathcal{H}.
\end{equation}

Conversely, if $f, g$ satisfy the conclusion (1) and $f(t)g(t) \neq 0$, then $f, g$ are semi-operator monotone on $(0, \infty)$.

Proof. Let $f, g$ be semi-operator monotone on $(0, \infty)$. Since $\|T\|^2 \leq A^2$ and $\|T^*\|^2 \leq B^2$, we have $\{f(|T|)\}^2 \leq \{f(A)\}^2$ and $\{g(|T^*|)\}^2 \leq \{g(B)\}^2$. Hence

$$\|\langle Uf(|T|)g(|T|)x, y \rangle\| = \|\langle g(|T^*|)Uf(|T|)x, y \rangle\|$$

$$= \|\langle Uf(|T|)x, g(|T^*|)y \rangle\|$$

$$\leq \|f(|T|)x\| \|g(|T^*|)y\|$$

$$\leq \|f(A)x\| \|g(B)y\| \text{ for } x, y \in \mathcal{H}.$$

Conversely, take a point $a \in (0, \infty)$ such that $f(a)g(a) > 0$. Let $(c, d)$ be the maximal open interval including $a$ such that $0 < f(t), g(t)$ for $t \in (c, d)$. First we show that $f, g$ are semi-operator monotone on $(c, d)$. Let $\tilde{f}(t) = \{f(t^\frac{1}{2})\}^2, \tilde{g}(t) = \{g(t^\frac{1}{2})\}^2$ for $t \in (c^2, d^2)$. Let $C \leq D, \sigma(C), \sigma(D) \subset (c^2, d^2)$. Then $C, D, g(C^\frac{1}{2})$ are invertible. Let $T = C^\frac{1}{2}, A = D^\frac{1}{2}, B = C^\frac{1}{2}$. Then condition (1) implies

$$\|f(D^\frac{1}{2})x\| \|g(C^\frac{1}{2})y\| \geq \|\langle f(D^\frac{1}{2})g(C^\frac{1}{2})x, y \rangle\|$$

$$= \|\langle f(C^\frac{1}{2})x, g(C^\frac{1}{2})y \rangle\| \text{ for } x, y \in \mathcal{H}.$$
Hence \( \| f(C^{\frac{1}{2}})x \| \leq \| f(D^{\frac{1}{2}})x \| \) for \( x \in \mathcal{H} \) and \( \hat{f}(C) = \{ f(C^{\frac{1}{2}}) \}^2 \leq \{ f(D^{\frac{1}{2}}) \}^2 = \hat{f}(D) \). Thus \( f(t) \) is semi-operator monotone on \((c,d)\). Similarly we can show that \( g(t) \) is semi-operator monotone on \((c,d)\).

We show \( c = 0 \) and \( d = \infty \). If \( d < \infty \), then \( f(d) = 0 \) or \( g(d) = 0 \). But \( f, g \) are positive and semi-operator monotone on \((c,d)\). Hence \( 0 < f(d) \) and \( 0 < g(d) \). This is a contradiction. Hence \( d = \infty \). Assume \( 0 < c \). Then \( f(c) = 0 \) or \( g(c) = 0 \). In this case \( f(0)g(0) = 0 \). Because if \( f(0)g(0) > 0 \), then \( f(t) > 0, g(t) > 0 \) for \( t \in (0, \infty) \) by the preceding argument. Let

\[
A = B = \begin{pmatrix} \sqrt{2c} & 0 \\ 0 & c \end{pmatrix}, T = \frac{2c}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.
\]

Then \( 0 \leq A, B, T \) and \( T^2 \leq A^2 = B^2 \). Let \( x = y = (\frac{1}{2}) \). Then

\[
\langle Uf(|T|)g(|T|)x, y \rangle = \frac{1}{2} f \left( \frac{2c}{\sqrt{3}} \right) g \left( \frac{2c}{\sqrt{3}} \right) \neq 0
\]

and

\[
\| f(A)x \| \| g(B)y \| = \left\| \begin{pmatrix} 0 \\ f(c) \end{pmatrix} \right\| \left\| \begin{pmatrix} 0 \\ g(c) \end{pmatrix} \right\| = 0.
\]

This is a contradiction. Thus \( c = 0 \) and \( f, g \) are semi-operator monotone on \((0, \infty)\).

Next we show a direct extension of the Heinz-Kato inequality.

**Theorem 6.** Let \( T \in B(\mathcal{H}), 0 \leq A, B \in B(\mathcal{H}) \) and \( \| Tx \| \leq \| Ax \|, \| T^*y \| \leq \| By \| \) for \( x, y \in \mathcal{H} \). Then, if a non-zero function \( f \) is semi-operator monotone on \((0, \infty)\), then we have

\[
\| (Tx, y) \| \leq \| f(A)x \| \| g(B)y \|
\]

for \( x, y \in \mathcal{H} \), where \( g(t) = t/f(t) \).

Conversely, if continuous functions \( f, g : [0, \infty) \to [0, \infty) \) satisfy conclusion (2) and \( f(t)g(t) = t \) for \( t \in (0, \infty) \), then \( f, g \) are semi-operator monotone on \((0, \infty)\).

**Proof.** Let \( f \) be semi-operator monotone on \((0, \infty)\). Then \( \{ f(t^{\frac{1}{2}}) \}^2 \) is operator monotone on \((0, \infty)\) and \( 0 < f(t^{\frac{1}{2}}) \) for \( t \in (0, \infty) \). Hence \( \{ g(t^{\frac{1}{2}}) \}^2 = t/\{ f(t^{\frac{1}{2}}) \}^2 \) is operator monotone on \((0, \infty)\) by [4, Corollary 2.6]. Thus \( g \) is semi-operator monotone on \((0, \infty)\). Hence

\[
\| (Tx, y) \| = \| (U|T|x, y) \|
\]

\[
= \| (Uf(|T|)g(|T|)x, y) \|
\]

\[
\leq \| f(A)x \| \| g(B)y \| \quad \text{for} \ x, y \in \mathcal{H}
\]

by Theorem 5. The converse is easy from Theorem 5. \( \square \)

**Remark 7.** For example, \( f(t) = \sqrt{1 + t^2} \) is semi-operator monotone on \((0, \infty)\). Hence, if \( \| Tx \| \leq \| Ax \|, \| T^*y \| \leq \| By \| \) for \( x, y \in \mathcal{H} \), then

\[
\| (Tx, y) \| \leq \left\| \sqrt{1 + A^2}x \right\| \left\| \frac{B}{\sqrt{1 + B^2}}y \right\| \quad \text{for} \ x, y \in \mathcal{H}.
\]
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