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ON A PROPOSED CHARACTERIZATION OF SCHATTEN-CLASS COMPOSITION OPERATORS

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ABSTRACT. For an analytic function φ which maps the open unit disc D to itself, let C_φ be the operator of composition with φ on the Bergman space $L^2_a(D, dA)$. It has been a longstanding problem to determine whether or not the membership of C_φ in the Schatten class $\mathcal{C}_p$, $1 < p < \infty$, is equivalent to the condition that the function $z \mapsto \{(1-|z|^2)/(1-|\varphi(z)|^2)\}^p$ has a finite integral with respect to the Möbius-invariant measure $d\lambda(z) = (1-|z|^2)^{-2}dA(z)$ on D. We show that the answer is negative when $2 < p < \infty$.

1. INTRODUCTION

Let D be the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$ in the complex plane and let dA be the area measure on D normalized in such a way that $A(D) = 1$. Throughout the paper, $d\lambda$ denotes the Möbius-invariant measure on D, i.e.,

$$d\lambda(z) = (1-|z|^2)^{-2}dA(z). \tag{1.1}$$

Recall that the Bergman space $L^2_a(D, dA)$ is defined to be $\{f \in L^2(D, dA) : f$ is analytic on $D\}$. For an analytic function $\varphi : D \to D$, let C_φ denote the operator of composition with φ on $L^2_a(D, dA)$. That is,

$$(C_\varphi f)(z) = f(\varphi(z)), \quad f \in L^2_a(D, dA).$$

It is a consequence of Littlewood's subordination theorem that the operator C_φ is bounded on $L^2_a(D, dA)$ for any analytic function $\varphi : D \to D$ [7, Theorem 10.3.2].

Recall that, for any $1 \le p < \infty$, the Schatten p-class $\mathcal{C}_p$ consists of operators T satisfying the condition $\|T\|_p < \infty$, where the p-norm is defined by the formula

$$\|T\|_p = \{\mathrm{tr}(|T|^p)\}^{1/p} = \{\mathrm{tr}((T^*T)^{p/2})\}^{1/p}.$$

The main interest of this paper concerns a proposed characterization of the membership of C_φ in $\mathcal{C}_p$. Let us briefly review the relevant background for the benefit of the reader, even though the problem itself is well known. In [5], D. Luecking and K. Zhu proved that $C_\varphi \in \mathcal{C}_p$ if and only if the function $z \mapsto \{\log(1/|z|)\}^{-2}N_{\varphi,2}(z)$ belongs to $L^{p/2}(D, d\lambda)$, where $d\lambda$ is given by (1.1) and $N_{\varphi,2}(z) = \sum_{w\in\varphi^{-1}\{z\}}\{\log(1/|w|)\}^2$, which is a generalized Nevanlinna counting function for φ. But such a characterization of the membership $C_\varphi \in \mathcal{C}_p$ was not considered to be completely satisfactory

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(see [7, page 226]) because φ enters in an indirect way, namely through the counting function $N_{\varphi,2}$. An alternate criterion for $C_\varphi \in \mathcal{C}_p$, one which not only involves φ directly but also appears to be aesthetically more pleasing, was proposed and debated in [4, 5, 7]. More precisely, the following problem arose (in chronological order) on page 363 of [4], on pages 226-228 of [7], and in Section 7 of [5].

Problem 1. *Let $1 < p < \infty$ and let $\varphi : D \to D$ be an analytic function. Is it true that the composition operator $C_\varphi : L_a^2(D, dA) \to L_a^2(D, dA)$ belongs to the Schatten class $\mathcal{C}_p$ if and only if*

$$\int_D \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^p d\lambda(z) < \infty? \tag{1.2}$$

There are several reasons for proposing this problem. First of all,

$$\{(1 - |z|^2)/(1 - |\varphi(z)|^2)\}^p = \{\langle C_\varphi C_\varphi^* k_z, k_z \rangle\}^{p/2},$$

where $k_z(w) = (1 - |z|^2)/(1 - \bar{z}w)^2$ (see [7, Proposition 10.3.4]) and the function $z \mapsto \langle C_\varphi C_\varphi^* k_z, k_z \rangle = \|C_\varphi^* k_z\|^2$ is the *Berezin symbol* of $C_\varphi C_\varphi^*$. Secondly, it is known that C_φ is compact if and only if

$$\lim_{|z| \uparrow 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0$$

(see [6] or [1, Theorem 3.22]). Thirdly, (1.2) is sufficient for $C_\varphi \in \mathcal{C}_p$ when $1 < p \le 2$ and necessary for $C_\varphi \in \mathcal{C}_p$ when $2 \le p < \infty$ (see [5, Section 7]). In particular, $C_\varphi \in \mathcal{C}_2$ if and only if (1.2) holds with $p = 2$. Thus, when $2 < p < \infty$, (1.2) appears to be the natural "interpolation" between the compactness criterion and the Hilbert-Schmidt-class criterion for C_φ. Furthermore, Zhu recently proved the following:

Theorem 2 ([8, Theorem 1.1])**.** *Let $2 \le p < \infty$. Let $\varphi : D \to D$ be an analytic function which has bounded valence, i.e., there is a positive integer N such that card$\{w \in D : \varphi(w) = z\} \le N$ for every $z \in D$. Then $C_\varphi \in \mathcal{C}_p$ if and only if* (1.2) *holds.*

The purpose of this paper is to report that, however pleasing or natural (1.2) may appear, this condition in fact is *not* sufficient for the membership $C_\varphi \in \mathcal{C}_p$ when $2 < p < \infty$. Thus Luecking's comment on page 363 of [4] turns out to be prophetic after all. The following is our main result:

Theorem 3. *For any $2 < p < \infty$, there exists an analytic function $\varphi : D \to D$ such that*

$$\int_D \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^p d\lambda(z) < \infty \tag{1.3}$$

and such that the composition operator $C_\varphi : L_a^2(D, dA) \to L_a^2(D, dA)$ does not belong to the Schatten class $\mathcal{C}_p$.

This result leads to the following contrast between the Berezin symbol of $C_\varphi^* C_\varphi$ and the Berezin symbol of $C_\varphi C_\varphi^*$ which seems to be interesting. Consider the conditions

$$\text{(a)} \quad \int_D \|C_\varphi k_z\|^p d\lambda(z) < \infty, \qquad \text{(b)} \quad \int_D \|C_\varphi^* k_z\|^p d\lambda(z) < \infty.$$

Because $\|C_\varphi k_z\|^2$ is the Berezin symbol of $C_\varphi^* C_\varphi$ and because $C_\varphi^* C_\varphi$ is a Toeplitz operator as defined on page 106 of [7], condition (a) implies $C_\varphi \in \mathcal{C}_p$ [8, Lemmas 2.1 and 2.2]. On the other hand, as we have already mentioned, the Berezin symbol of $C_\varphi C_\varphi^*$ equals $\|C_\varphi^* k_z\|^2 = (1-|z|^2)^2/(1-|\varphi(z)|^2)^2$. Theorem 3 tells us that condition (b) does not imply $C_\varphi \in \mathcal{C}_p$ when $2 < p < \infty$. Thus conditions (a) and (b) are not equivalent.

The proof of Theorem 3 is technical but self-contained. In fact, the only thing the reader needs to known about $L_a^2(D, dA)$ is that $\{(\ell+1)^{1/2} z^\ell : \ell = 0, 1, 2, ...\}$ is an orthonormal basis. To ensure that (1.3) holds, we only need to control the *modulus* $|\varphi|$ of φ. Thanks to a new lower bound for $\operatorname{tr}((C_\varphi^* C_\varphi)^{p/2})$ (see (3.4) below), our proof that $C_\varphi \notin \mathcal{C}_p$ also involves the modulus $|\varphi|$ only. But the main obstacle in the proof of Theorem 3 lies in the fact that it is difficult to prescribe the modulus of an analytic function. In other words, the analyticity requirement is a severe handicap for the construction of φ. Fortunately, by carefully exploiting the distribution of the Poisson kernel, we are able to find just enough control on D to make such a construction possible.

The rest of the paper is organized as follows. In Section 2 we construct the required φ and establish the estimates which are necessary for the proof of Theorem 3. The proof itself is completed in Section 3.

2. Construction and estimates

The construction of the desired φ begins with the intervals

$$T_n = (2^{-(n+1)}, 2^{-n}], \qquad S_n = ((4/3)2^{-(n+1)}, (5/3)2^{-(n+1)}]$$

in $\mathbf{R}$, $n = 1, 2, \dots$. That is, S_n is the middle third of T_n. Let $t_n = (4/3)2^{-(n+1)}$, which is the left end-point of S_n, $n \in \mathbf{N}$.

Let $2 < p < \infty$ be given. We choose a *rational number* ϵ such that

$$0 < \epsilon < p^{-1}. \tag{2.1}$$

The requirement $\epsilon < p^{-1}$ ensures $\lim_{k\to\infty} 2^{-(p^{-1}-\epsilon)k} = 0$. This allows us to choose a strictly increasing sequence $k(1) < \dots < k(n) < \dots$ of positive integers such that

$$2^{-(p^{-1}+\epsilon)k(n)} \cdot 2 \cdot 2^{2\epsilon k(n)} \le (1/3)2^{-(n+1)} \tag{2.2}$$

for every $n \in \mathbf{N}$ and such that every $\epsilon k(n)$ is an *integer*. While the requirement that both inequalities in (2.1) be strict is indispensable to the proof of Theorem 3, the stipulation that ϵ be rational, which makes it possible for $\epsilon k(n)$ to be an integer, is not. But the fact that $\epsilon k(n)$ is chosen to be an integer does allow us to avoid certain unnecessary trivialities.

Next we subdivide S_n. For integers $n \ge 1$ and $1 \le j \le 2^{2\epsilon k(n)}$, define the intervals

$$J_{n,j} = (t_n + 2^{-(p^{-1}+\epsilon)k(n)} \cdot 2 \cdot (j-1), t_n + 2^{-(p^{-1}+\epsilon)k(n)} \cdot 2 \cdot j),$$
$$I_{n,j} = (t_n + 2^{-(p^{-1}+\epsilon)k(n)} \cdot 2 \cdot (j-1), t_n + 2^{-(p^{-1}+\epsilon)k(n)} \cdot (2j-1)),$$

where, as we recall, t_n is the left end-point of S_n. Thus, $I_{n,j}$ is the left half of $J_{n,j}$ and the $J_{n,j}$'s are pairwise disjoint. (2.2) ensures that $\bigcup_{j=1}^{2^{2\epsilon k(n)}} J_{n,j} \subset S_n$. Keep in mind that the length of $I_{n,j}$ equals $2^{-(p^{-1}+\epsilon)k(n)}$. We now define a measurable function u on the unit circle $T = \{\tau \in \mathbf{C} : |\tau| = 1\}$ as follows:

$$u(e^{it}) = 2^{-k(n)} \quad \text{if } t \in \bigcup_{j=1}^{2^{2\epsilon k(n)}} I_{n,j},\ n \geq 1,$$
$$u(e^{it}) = 1 \quad \text{if } t \in (-\pi, \pi] \backslash \left\{ \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{2\epsilon k(n)}} I_{n,j} \right\}.$$

The harmonic extension of u to D will be denoted by the same symbol. Finally, define

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it}) dt,$$
$$\varphi(z) = \exp(-h(z)), \quad z \in D. \tag{2.3}$$

Obviously, $\mathrm{Re}\{h(z)\} = u(z) > 0$ for every $z \in D$. Therefore φ maps D to itself. Our remaining task is to verify that φ satisfies (1.3) and has the property $C_\varphi \notin \mathcal{C}_p$. This verification is based on a number of estimates of the modulus $|\varphi(z)|$, which will take up the rest of the section.

We begin with the Poisson kernel $P(z; \tau) = (1 - |z|^2)/|1 - z\bar{\tau}|^2$, $\tau \in T$ and $z \in D$. There exist constants $0 < \alpha < \beta < \infty$ such that if $1/2 \leq r < 1$ and $|\theta - t| \leq 5$, then

$$\frac{\alpha(1-r)}{(1-r)^2 + (\theta - t)^2} \leq \frac{1}{2\pi} P(re^{i\theta}; e^{it}) \leq \frac{\beta(1-r)}{(1-r)^2 + (\theta - t)^2}. \tag{2.4}$$

This follows from the identity $|1 - re^{i(\theta - t)}|^2 = (1-r)^2 + 2r(1 - \cos(\theta - t))$ and the fact that there are $0 < a < b < \infty$ such that $a \leq x^{-2}(1 - \cos x) \leq b$ when $0 < x \leq 5$.

Lemma 4. *For any $n \in \mathbf{N}$ and $1 \leq j \leq 2^{2\epsilon k(n)}$, define*

$$G_{n,j} = \{re^{i\theta} : \theta \in I_{n,j},\ 0 < 1 - r \leq 2^{-(p^{-1}+\epsilon)k(n)}\}. \tag{2.5}$$

Then there is a constant C_4 which is independent of n, j such that

$$\int_{G_{n,j}} \frac{(1 - |z|)^{p-2}}{(1 - |\varphi(z)|)^p} dA(z) \leq C_4 2^{-\epsilon p k(n)}.$$

Proof. Given such a pair of n, j, we have $G_{n,j} = \bigcup_{\nu=0}^{k(n)} G_{n,j}^{\nu}$, where

$$G_{n,j}^0 = \{re^{i\theta} : \theta \in I_{n,j},\ 0 < 1 - r \leq 2^{-(1+p^{-1}+\epsilon)k(n)}\},$$
$$G_{n,j}^{\nu} = \{re^{i\theta} : \theta \in I_{n,j},\ 2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^{(\nu-1)} < 1 - r \leq 2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^{\nu}\}$$

for $1 \leq \nu \leq k(n)$. Since $|\varphi(z)| = e^{-\mathrm{Re}\{h(z)\}} = e^{-u(z)}$, the proof hinges on an estimate of $u(z)$ on each $G_{n,j}^{\nu}$. For this purpose let us write $I_{n,j} = (a_{n,j}, b_{n,j})$ and $J_{n,j} = (a_{n,j}, c_{n,j})$. Then $c_{n,j} - b_{n,j} = b_{n,j} - a_{n,j} = 2^{-(p^{-1}+\epsilon)k(n)}$, which will be denoted by ρ_n. Let us first consider the case $z \in G_{n,j}^{\nu}$ where $1 \leq \nu \leq k(n)$. For such a z, write $z = re^{i\theta}$ with $\theta \in I_{n,j}$. Since $u(e^{it}) = 1$ for $t \in J_{n,j} \backslash I_{n,j} = [b_{n,j}, c_{n,j})$, recalling (2.4), we have

$$\begin{aligned} u(re^{i\theta}) &\geq \frac{1}{2\pi}\int_{J_{n,j}\setminus I_{n,j}} P(re^{i\theta};e^{it})dt \geq \alpha\int_{b_{n,j}}^{c_{n,j}} \frac{1-r}{(1-r)^2+(t-\theta)^2}dt \\ &\geq \alpha\int_{b_{n,j}}^{c_{n,j}} \frac{1-r}{(1-r)^2+(t-a_{n,j})^2}dt \\ &= \alpha\int_{\rho_n}^{2\rho_n} \frac{1-r}{(1-r)^2+s^2}ds = \alpha\int_{\rho_n/(1-r)}^{2\rho_n/(1-r)} \frac{dx}{1+x^2}. \end{aligned}$$

The condition $re^{i\theta} \in G^{\nu}_{n,j}$ implies $1-r \leq 2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^{\nu} = \rho_n \cdot 2^{-k(n)+\nu}$. In particular, $\rho_n/(1-r) \geq 1$. Hence $(1+x^2)^{-1} \geq (2x^2)^{-1}$ if $x \geq \rho_n/(1-r)$. The condition $re^{i\theta} \in G^{\nu}_{n,j}$ also requires $2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^{(\nu-1)} < 1-r$, i.e., $2^{-k(n)+\nu-1} < (1-r)/\rho_n$. Thus

$$u(re^{i\theta}) \geq \frac{\alpha}{2}\int_{\rho_n/(1-r)}^{2\rho_n/(1-r)} \frac{1}{x^2}dx = \frac{\alpha}{4}\cdot\frac{1-r}{\rho_n} \geq \frac{\alpha}{4}2^{-k(n)+\nu-1} = \frac{\alpha}{8}2^{-k(n)+\nu} \tag{2.6}$$

if $re^{i\theta} \in G^{\nu}_{n,j}$ and $1 \leq \nu \leq k(n)$.

Now suppose that $re^{i\theta} \in G^{0}_{n,j}$, where $\theta \in I_{n,j}$. In this case we use the fact that $u(e^{it}) \geq 2^{-k(n)}$ for $t \in T_n$. Obviously, $[\theta-\rho_n, \theta+\rho_n] \subset T_n$. Therefore

$$\begin{aligned} u(re^{i\theta}) &\geq 2^{-k(n)}\frac{1}{2\pi}\int_{\theta-\rho_n}^{\theta+\rho_n} P(re^{i\theta};e^{it})dt \geq 2^{-k(n)}\alpha\int_{\theta-\rho_n}^{\theta+\rho_n} \frac{1-r}{(1-r)^2+(t-\theta)^2}dt \\ &= 2^{-k(n)}\alpha\int_{-\rho_n}^{\rho_n} \frac{1-r}{(1-r)^2+s^2}ds \geq 2^{-k(n)}\alpha\int_{-1}^{1}\frac{dx}{1+x^2}, \end{aligned} \tag{2.7}$$

where the last $\geq$ is due to the fact that $1-r \leq 2^{-(1+p^{-1}+\epsilon)k(n)} = 2^{-k(n)}\rho_n$.

Combining (2.7) with (2.6), we see that there is a constant $0 < c < 1$ which is independent of n, j such that $u(z) \geq c2^{-k(n)+\nu}$ if $z \in G^{\nu}_{n,j}$ and $0 \leq \nu \leq k(n)$. This implies that $1-|\varphi(z)| = 1-e^{-u(z)} \geq 1-\exp(-c2^{-k(n)+\nu})$ if $z \in G^{\nu}_{n,j}$ and $0 \leq \nu \leq k(n)$. Thus, if we let $\delta = \inf_{0<x\leq 1} x^{-1}(1-e^{-x})$, then

$$\inf_{z\in G^{\nu}_{n,j}} (1-|\varphi(z)|)^p \geq (\delta c)^p \cdot 2^{-pk(n)} \cdot 2^{p\nu}, \quad 0 \leq \nu \leq k(n). \tag{2.8}$$

Because $p-2 > 0$, it follows from the definition of $G^{\nu}_{n,j}$ that

$$\sup_{z\in G^{\nu}_{n,j}} (1-|z|)^{p-2} \leq 2^{-(p-2)(1+p^{-1}+\epsilon)k(n)} \cdot 2^{(p-2)\nu}, \quad 0 \leq \nu \leq k(n). \tag{2.9}$$

From the definitions of $G^{\nu}_{n,j}$ and $I_{n,j}$ it obviously follows that

$$A(G^{\nu}_{n,j}) \leq 2^{-(p^{-1}+\epsilon)k(n)} \cdot 2^{-(1+p^{-1}+\epsilon)k(n)} \cdot 2^{\nu}, \quad 0 \leq \nu \leq k(n). \tag{2.10}$$

Combining (2.8)-(2.10) and simplifying the exponents involved, we find that

$$\int_{G^{\nu}_{n,j}} \frac{(1-|z|)^{p-2}}{(1-|\varphi(z)|)^p}dA(z) \leq \frac{\sup_{z\in G^{\nu}_{n,j}}(1-|z|)^{p-2}}{\inf_{z\in G^{\nu}_{n,j}}(1-|\varphi(z)|)^p}A(G^{\nu}_{n,j}) \leq (c\delta)^{-p}\cdot 2^{-\nu}\cdot 2^{-\epsilon p k(n)} \tag{2.11}$$

for all $0 \leq \nu \leq k(n)$. Recall that $G_{n,j} = \bigcup_{\nu=0}^{k(n)} G^{\nu}_{n,j}$. Because of the factor $2^{-\nu}$ that appears on the right-hand side of (2.11), if we sum both sides of this inequality as ν ranges from 0 to $k(n)$, we see that the conclusion of the lemma holds for $C_4 = 2(c\delta)^{-p}$. □

Lemma 5. *For any $n \geq 1$ and $1 \leq j \leq 2^{2\epsilon k(n)}$, let $B_{n,j}$ be the middle third of $I_{n,j}$. That is, $B_{n,j} = (3^{-1}(b_{n,j} + 2a_{n,j}), 3^{-1}(2b_{n,j} + a_{n,j}))$, where $a_{n,j} < b_{n,j}$ are the end-points of $I_{n,j}$. Furthermore, for such n and j, define*

$$E_{n,j} = \{re^{it} : t \in B_{n,j},\ 0 < 1 - r \leq 2^{-(1+p^{-1}+\epsilon)k(n)}\}. \tag{2.12}$$

Then $\sup_{z \in E_{n,j}} u(z) \leq (1+6\beta)2^{-k(n)}$, where β is the constant that appears in (2.4).

Proof. Let $n \geq 1$ and $1 \leq j \leq 2^{2\epsilon k(n)}$ be given. Consider an arbitrary $z = re^{i\theta} \in E_{n,j}$, where $\theta \in B_{n,j}$. Since $B_{n,j}$ is the middle third of $I_{n,j}$, we have $|\theta - t| \geq \rho_n/3$ if $t \in \mathbf{R}\backslash I_{n,j}$, where $\rho_n = 2^{-(p^{-1}+\epsilon)k(n)}$ is the length of $I_{n,j}$. The condition $\theta \in B_{n,j}$ also implies $0 < \theta < 1/2$. Therefore $(-\pi, \pi]\backslash I_{n,j} \subset \{t \in \mathbf{R} : \rho_n/3 \leq |\theta - t| \leq \pi + (1/2)\}$. Now, because $u(e^{it}) = 2^{-k(n)}$ for $t \in I_{n,j}$ and $u(e^{it}) \leq 1$ for $t \in (-\pi, \pi]\backslash I_{n,j}$, we have

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\theta}; e^{it})u(e^{it})dt \leq 2^{-k(n)} + \frac{1}{2\pi}\int_{(-\pi,\pi]\backslash I_{n,j}} P(re^{i\theta}; e^{it})dt$$

$$\leq 2^{-k(n)} + \beta \int_{\rho_n/3 \leq |\theta - t| \leq \pi + (1/2)} \frac{1-r}{(1-r)^2 + (\theta - t)^2} dt, \tag{2.13}$$

where β is the constant that appears in (2.4). But

$$\begin{aligned} \int_{|\theta - t| \geq \rho_n/3} \frac{1-r}{(1-r)^2 + (\theta-t)^2} dt &= 2\int_{\rho_n/3}^{\infty} \frac{1-r}{(1-r)^2 + s^2} ds \\ &= 2\int_{\rho_n/3(1-r)}^{\infty} \frac{1}{1+x^2} dx \\ &\leq 6(1-r)/\rho_n. \end{aligned} \tag{2.14}$$

The condition $re^{i\theta} \in E_{n,j}$ requires $1 - r \leq 2^{-(1+p^{-1}+\epsilon)k(n)} = 2^{-k(n)}\rho_n$. Thus it follows from (2.13) and (2.14) that $u(z) \leq (1 + 6\beta)2^{-k(n)}$ for every $z \in E_{n,j}$. □

Lemma 6. *Let $U = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{2\epsilon k(n)}} I_{n,j}$. Then for any $x \in \mathbf{R}\backslash U$ and $0 < a < \infty$, we have*

$$m((x-a, x+a)\backslash U) \geq (2/3)a, \tag{2.15}$$

where m is the standard Lebesgue measure on $\mathbf{R}$.

Proof. The case where either $(x-a, x) \cap U = \emptyset$ or $(x, x+a) \cap U = \emptyset$ is trivial. Thus let us assume that $x \in \mathbf{R}\backslash U$ and $a > 0$ are such that $(x-a, x) \cap U \neq \emptyset$ and $(x, x+a) \cap U \neq \emptyset$. Because $x \notin U$, this means that there are at least two intervals in the family

$$\mathcal{F} = \{I_{n,j} : (x-a, x+a) \cap I_{n,j} \neq \emptyset,\ n \geq 1,\ 1 \leq j \leq 2^{2\epsilon k(n)}\}.$$

Let Y be the smallest open interval such that

$$Y \cap U = (x-a, x+a) \cap U.$$

Since $(x-a, x+a)\backslash U = \{Y\backslash U\} \cup \{(x-a, x+a)\backslash Y\}$, (2.15) will follow once we show that $m(Y\backslash U) \geq (1/3)m(Y)$.

From the arrangement $I_{n,j} \subset S_n \subset T_n = (2^{-(n+1)}, 2^{-n}]$ it is easy to see that we can re-enumerate the family $\mathcal{F}$ as $\{(a_\nu, b_\nu) : 1 \leq \nu < N\}$, where $3 \leq N \leq \infty$, such that $b_{\nu+1} < a_\nu$ for all $1 \leq \nu < N-1$. Thus $Y \cap U = \bigcup_{1 \leq \nu < N} I_\nu$, where $I_1 = (a_1, b_1')$ with $a_1 < b_1' \leq b_1$, $I_\nu = (a_\nu, b_\nu)$ if $2 \leq \nu < N-1$, and, in the event $N < \infty$, $I_{N-1} = (a'_{N-1}, b_{N-1})$ with $a_{N-1} \le a'_{N-1} < b_{N-1}$. The definition of the $I_{n,j}$'s ensures that the length of the gap between $(a_{\nu+1}, b_{\nu+1})$ and (a_ν, b_ν) is greater than or equal to the length of (a_ν, b_ν). That is, $a_\nu - b_{\nu+1} \ge b_\nu - a_\nu$, $1 \le \nu < N-1$. Also, $b_\nu - a_\nu \ge b_{\nu+1} - a_{\nu+1}$ by the definition of the $I_{n,j}$'s. Since $Y\backslash U = \bigcup_{1\le\nu<N-1}[b_{\nu+1}, a_\nu]$, we have

$$\begin{aligned} m(Y\backslash U) &= \sum_{1\le\nu<N-1}(a_\nu - b_{\nu+1}) \ge \sum_{1\le\nu<N-1}(b_\nu - a_\nu) \\ &\ge \frac{1}{2}\sum_{1\le\nu<N}(b_\nu - a_\nu) \ge \frac{1}{2}m(Y\cap U). \end{aligned}$$

Adding $(1/2)m(Y\backslash U)$ to both sides, we find that $(3/2)m(Y\backslash U) \ge (1/2)m(Y)$. Thus $m(Y\backslash U) \ge (1/3)m(Y)$ as promised. □

Lemma 7. *There is a $c_7 > 0$ such that*

$$u(z) \ge c_7 \quad \textit{for every } z \in D\backslash\left\{\bigcup_{n=1}^{\infty}\bigcup_{j=1}^{2^{2\epsilon k(n)}} G_{n,j}\right\},$$

where $G_{n,j}$ is defined by (2.5).

Proof. Let $W = \{re^{it} : 3/4 < r < 1,\ t \in (-1/4, 3/4)\}$. Since $u(e^{it}) = 1$ when $t \in (-\pi,\pi]\backslash(0,1/2]$, we have $\lim_{r\uparrow 1} u(re^{it}) = 1$ *uniformly* for $t \in (-\pi,\pi]\backslash(-1/8, 5/8)$. Hence it suffices to find a $c_7 > 0$ such that

$$u(z) \ge c_7 \quad \text{for all } z \in W\backslash\left\{\bigcup_{n=1}^{\infty}\bigcup_{j=1}^{2^{2\epsilon k(n)}} G_{n,j}\right\}.$$

For any $0 \le r < 1$ and $\theta \in \mathbf{R}$, define $I(\theta, r) = (\theta - 3(1-r), \theta + 3(1-r))$. Then

$$\frac{1-r}{(1-r)^2 + (\theta - t)^2} = \frac{1}{1-r}\cdot\left\{1 + \left(\frac{\theta - t}{1-r}\right)^2\right\}^{-1} \ge \frac{10^{-1}}{1-r}\chi_{I(\theta,r)}(t). \tag{2.16}$$

Let $\theta \in (-1/4, 3/4)$ and $3/4 < r < 1$. Then $I(\theta, r) \subset (-\pi,\pi]$ and $u(e^{it}) = 1$ for $t \in I(\theta,r)\backslash U$, where $U = \bigcup_{n=1}^{\infty}\bigcup_{j=1}^{2^{2\epsilon k(n)}} I_{n,j}$ as in Lemma 6. By (2.4) and (2.16),

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\theta}; e^{it})u(e^{it})dt \ge \frac{\alpha}{10}\cdot\frac{1}{1-r}\int_{I(\theta,r)} u(e^{it})dt \\ &\ge \frac{\alpha}{10}\cdot\frac{m(I(\theta,r)\backslash U)}{1-r}. \end{aligned} \tag{2.17}$$

Let us further assume $re^{i\theta} \in W\backslash\{\bigcup_{n=1}^{\infty}\bigcup_{j=1}^{2^{2\epsilon k(n)}} G_{n,j}\}$. We consider the following two cases:

(i) If $\theta \in (-1/4, 3/4)\backslash U$, then we apply Lemma 6 to the case where $x = \theta$ and $a = 3(1-r)$ to obtain

$$\begin{aligned} m(I(\theta,r)\backslash U) &= m((\theta - 3(1-r), \theta + 3(1-r))\backslash U) \\ &\ge (2/3)\cdot 3(1-r) = 2(1-r). \end{aligned}$$

By (2.17), we have $u(re^{i\theta}) \ge \alpha/5$ in this case.

(ii) If $\theta \in U$, then there exist an $n \ge 1$ and a $1 \le j \le k(n)$ such that $\theta \in I_{n,j}$. Now, because $re^{i\theta} \notin G_{n,j}$, (2.5) tells us $1 - r > 2^{-(p^{-1}+\epsilon)k(n)} = \rho_n$, the length of $I_{n,j}$. Since the distance between θ and $J_{n,j}\backslash I_{n,j}$ is less than ρ_n, we can pick a $\theta' \in J_{n,j}\backslash I_{n,j}$ such that $|\theta - \theta'| < \rho_n < 1-r$. Thus $I(\theta,r) = (\theta - 3(1-r), \theta + 3(1-r)) \supset (\theta' - 2(1-r), \theta' + 2(1-r))$. Since $J_{n,j}\backslash I_{n,j} \subset \mathbf{R}\backslash U$, we now apply Lemma 6 to the case where $x = \theta'$ and $a = 2(1-r)$ to obtain $m(I(\theta,r)\backslash U) \geq m((\theta' - 2(1-r), \theta' + 2(1-r))\backslash U) \geq (2/3)\cdot 2(1-r)$. By (2.17), we have $u(re^{i\theta}) \geq 2\alpha/15$ in this case. This completes the proof. $\square$

3. Proof of Theorem 3

We must show that the analytic function $\varphi : D \to D$ defined by (2.3) satisfies (1.3) and has the property that $C_\varphi \notin \mathcal{C}_p$. Let us first verify (1.3).

Let $G = \bigcup_{n=1}^\infty \bigcup_{j=1}^{2^{2\epsilon k(n)}} G_{n,j}$, where $G_{n,j}$ is given by (2.5). By Lemma 7, $|\varphi(z)| = e^{-\mathrm{Re}\{h(z)\}} = e^{-u(z)} \leq e^{-c_7}$ when $z \in D\backslash G$. Therefore

$$\int_{D\backslash G} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^p d\lambda(z) = \int_{D\backslash G} \frac{(1-|z|^2)^{p-2}}{(1-|\varphi(z)|^2)^p} dA(z) < \infty. \tag{3.1}$$

Invoking Lemma 4, we have

$$\begin{aligned}\int_G \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^p d\lambda(z) &\leq \sum_{n=1}^\infty \sum_{j=1}^{2^{2\epsilon k(n)}} \int_{G_{n,j}} \frac{2^{p-2}(1-|z|)^{p-2}}{(1-|\varphi(z)|)^p} dA(z) \\ &\leq 2^{p-2} C_4 \sum_{n=1}^\infty 2^{-\epsilon p k(n)} \cdot 2^{2\epsilon k(n)} = 2^{p-2} C_4 \sum_{n=1}^\infty 2^{-\epsilon(p-2)k(n)}. \end{aligned}\tag{3.2}$$

Because $p > 2$ and $\epsilon > 0$, the above is finite and (1.3) follows from (3.1) and (3.2).

To show that $C_\varphi \notin \mathcal{C}_p$ or, what amounts to the same, $\mathrm{tr}((C_\varphi^* C_\varphi)^{p/2}) = \infty$, we need the following inequality: For any $1 < \rho < \infty$ and $0 < x < 1$, we have

$$\sum_{\ell=0}^\infty (\ell+1)^\rho x^\ell \geq \frac{1}{(1-x)^{\rho+1}}. \tag{3.3}$$

The proof of (3.3) is elementary. We begin with the identity

$$(1-x)^{-2} = \sum_{\ell=0}^\infty (\ell+1) x^\ell = \sum_{\ell=0}^\infty \{(\ell+1)^\rho x^\ell\}^{1/\rho} \cdot \{x^\ell\}^{(\rho-1)/\rho}$$

and apply Hölder's inequality with conjugate exponents ρ and $\rho/(\rho-1)$. This gives us $(1-x)^{-2} \leq (\sum_{\ell=0}^\infty (\ell+1)^\rho x^\ell)^{1/\rho} (1-x)^{-(\rho-1)/\rho}$. Multiplying both sides by $(1-x)^{(\rho-1)/\rho}$, we find that $(1-x)^{-(\rho+1)/\rho} \leq (\sum_{\ell=0}^\infty (\ell+1)^\rho x^\ell)^{1/\rho}$. Clearly, (3.3) follows from this.

Let $e_\ell(z) = (\ell+1)^{1/2} z^\ell$, $\ell = 0, 1, 2, \ldots$. Recall that $\{e_\ell : \ell \geq 0\}$ is the standard orthonormal basis for the Bergman space $L_a^2(D, dA)$. Because $p/2 > 1$ and $\|e_\ell\| = 1$, it follows from the spectral decomposition of $C_\varphi^* C_\varphi$ and Hölder's inequality that $\langle (C_\varphi^* C_\varphi)^{p/2} e_\ell, e_\ell\rangle \geq \{\langle C_\varphi^* C_\varphi e_\ell, e_\ell\rangle\}^{p/2} = \|C_\varphi e_\ell\|^p$ [7, Proposition 6.3.3]. But $C_\varphi e_\ell = (\ell+1)^{1/2}\varphi^\ell$. Therefore

$$\langle (C_\varphi^* C_\varphi)^{p/2} e_\ell, e_\ell\rangle \geq (\ell+1)^{p/2} \|\varphi^\ell\|^p \geq (\ell+1)^{p/2} \{\sum_{n=1}^\infty \int_{E_n} |\varphi|^{2\ell} dA\}^{p/2},$$

where $E_n = \bigcup_{j=1}^{2^{2\epsilon k(n)}} E_{n,j}$, $n \in \mathbf{N}$, and $E_{n,j}$ is defined by (2.12). Define

$$f_n = \inf_{z\in E_n} |\varphi(z)|, \qquad n \in \mathbf{N}.$$

It is elementary that $\{\sum_n a_n\}^\rho \geq \sum_n a_n^\rho$ if $\rho > 1$ and $a_n \geq 0$. Hence

$$\begin{aligned}\langle (C_\varphi^* C_\varphi)^{p/2} e_\ell, e_\ell\rangle &\geq (\ell+1)^{p/2}\{\sum_{n=1}^\infty f_n^{2\ell} A(E_n)\}^{p/2}\\ &\geq (\ell+1)^{p/2}\sum_{n=1}^\infty (f_n^{2\ell}A(E_n))^{p/2}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{tr}((C_\varphi^* C_\varphi)^{p/2}) &= \sum_{\ell=0}^\infty \langle (C_\varphi^* C_\varphi)^{p/2} e_\ell, e_\ell\rangle \geq \sum_{\ell=0}^\infty (\ell+1)^{p/2} \sum_{n=1}^\infty (f_n^{2\ell}A(E_n))^{p/2}\\ &= \sum_{n=1}^\infty (A(E_n))^{p/2}\sum_{\ell=0}^\infty (\ell+1)^{p/2}(f_n^p)^\ell \geq \sum_{n=1}^\infty \frac{(A(E_n))^{p/2}}{(1-f_n^p)^{1+(p/2)}}, \qquad (3.4)\end{aligned}$$

where the second $\geq$ follows from an application of (3.3) with $\rho = p/2$. Thus the conclusion $C_\varphi \notin \mathcal{C}_p$ will follow once we show that there is a $d > 0$ such that

$$\frac{(A(E_n))^{p/2}}{(1-f_n^p)^{1+(p/2)}} \geq d \quad \text{for every} \quad n \geq 1. \tag{3.5}$$

To prove (3.5), we estimate $A(E_n)$ and $1 - f_n^p$. It follows from (2.12) that $A(E_{n,j}) \geq (2\pi)^{-1} 2^{-(1+p^{-1}+\epsilon)k(n)} m(B_{n,j}) = (6\pi)^{-1} 2^{-(1+(2/p)+2\epsilon)k(n)}$. Hence

$$A(E_n) = \sum_{j=1}^{2^{2\epsilon k(n)}} A(E_{n,j}) \geq \frac{1}{6\pi} 2^{-(1+(2/p)+2\epsilon)k(n)} \cdot 2^{2\epsilon k(n)} = \frac{1}{6\pi} 2^{-(1+(2/p))k(n)}. \tag{3.6}$$

Since $|\varphi(z)| = e^{-u(z)}$, it follows from Lemma 5 that $|\varphi(z)| \geq \exp(-M2^{-k(n)})$ for every $z \in E_n$, where $M = 1 + 6\beta$. That is, $f_n \geq \exp(-M2^{-k(n)})$. Therefore

$$1 - f_n^p \leq 1 - \exp(-pM2^{-k(n)}) \leq pM2^{-k(n)}, \tag{3.7}$$

where the second $\leq$ follows from the inequality $x^{-1}(1 - e^{-x}) \leq 1$ for $0 < x < \infty$, which holds because the function $x \mapsto x^{-1}(1 - e^{-x})$ is decreasing on $(0, \infty)$. Combining (3.6) and (3.7), we see that (3.5) holds for $d = (pM)^{-1-(p/2)}(6\pi)^{-p/2}$. This completes the proof of Theorem 3.

References

1. C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, 1995. MR **97i:**47056
2. J. Garnett, *Bounded analytic functions*, Academic Press, New York-London, 1981. MR **83g:**30037
3. S. Li, *Trace ideal criteria for composition operators on Bergman spaces*, Amer. J. Math. **117** (1995), 1299-1323. MR **96g:**47023
4. D. Luecking, *Trace ideal criteria for Toeplitz operators*, J. Funct. Anal. **73** (1987), 345-368. MR **88m:**47046
5. D. Luecking and K. Zhu, *Composition operators belonging to the Schatten ideals*, Amer. J. Math. **114** (1992), 1127-1145. MR **93i:**47032
6. B. MacCluer and J. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces*, Canad. J. Math. **38** (1986), 878-906. MR **87h:**47048

7. K. Zhu, *Operator theory in function spaces*, Marcel Dekker, New York, 1990. MR **92c:**47031
8. K. Zhu, *Schatten class composition operators on weighted Bergman spaces of the disk*, J. Operator Theory **46** (2001), 173-181. MR **2002h:**47039

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