ON THE RINGS WHOSE INJECTIVE HULLS ARE FLAT

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Abstract. Let $R$ be a commutative Noetherian ring with nonzero identity and let the injective envelope of $R$ be flat. We characterize these kinds of rings and obtain some results about modules with nonzero injective cover over these rings.

1. Introduction

Let $R$ be a commutative Noetherian ring with nonzero identity and let $E(R)$ be an injective envelope of $R$. It is well known that for the ring $R$ the condition that $E(R)$ be flat generalizes the condition that $R$ be Gorenstein (see for example [10, 5.1.2 (1), (4)]). In [2, Theorem 3], Cheathan and Enochs gave a characterization of these kinds of rings. In fact, they showed that $E(R)$ is flat if and only if for any flat $R$-module $F$, the injective envelope $E(F)$ is flat and these are equivalent to, for any injective module $E$, the flat cover $F(E)$ is injective. In this paper, we focus on the rings $R$ for which $E(R)$ is flat. In section two, we extend the above equivalent conditions (see Theorem 2.2).

On the other hand, in [6], Golan and Teply showed that every module admits an injective (pre)cover. Of course, this may be zero. So the natural problem is when a nonzero module has nonzero injective cover. In [10, 2.4.8], it was shown that whenever every nonzero module over a ring $R$ has a nonzero injective cover, the ring $R$ must be Artinian. In section three we obtain some results about modules with nonzero injective cover over the rings with $E(R)$ flat. For example we show that if an $R$-module $M$ has nonzero injective cover, then $\text{Ass}R \cap \text{Coass}M \neq \emptyset$. The converse is true if $M$ is Artinian.

Throughout this paper, $R$ is commutative Noetherian with nonzero identity and all modules are unitary. For an $R$-module $X$, $\text{inj.dim}_RX$ stands for the injective dimension of $X$, $\text{f.dim}_RX$ stands for flat dimension of $X$, $\text{proj.dim}_RX$ stands for projective dimension of $X$, $E(X)$ stands for injective envelope of $X$ and $F(X)$ stands for its flat cover. Finally we use $\mathbb{N}$ to denote the set of positive integers. All other notations are standard.

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2. Characterizing the rings with $E(R)$ flat

We recall that a module $C$ is called cotorsion if $\text{Ext}^1_R(F, C) = 0$ for any flat module $F$; it is strongly cotorsion if $\text{Ext}^1_R(X, C) = 0$ for any module $X$ of finite flat dimension. Also $C$ is called strongly torsion free if $\text{Tor}^1_R(X, C) = 0$ for all modules $X$ of finite flat dimension. If $F$ is flat and cotorsion, it was proved in \cite{1}, p.183 that $F$ can be uniquely written in the form $F = \prod T_p$, where $T_p$ is a completion of a free $R_p$-module with respect to $p$-adic topology. Note that flat covers are known to exist over any rings (see \cite[7.4.4]{11}).

A module $M$ is Gorenstein injective if there is an exact sequence

$$\cdots \rightarrow E^{-2} \rightarrow E^{-1} \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots$$

of injective modules such that $M = \ker(E^0 \rightarrow E^1)$ and such that $\text{Hom}_R(E, -)$ leaves the sequence exact when $E$ is injective. If $M$ is Gorenstein injective, then $\text{Ext}^1_R(L, M) = 0$ for all $i \in \mathbb{N}$ and $L$ such that $\text{inj.dim}L < \infty$. Also an $R$-module $N$ is called Gorenstein flat if there is an exact complex

$$\cdots \rightarrow F^{-2} \rightarrow F^{-1} \rightarrow F^0 \rightarrow F^1 \rightarrow F^2 \rightarrow \cdots$$

with $F^3$ flat and $N = \ker(F^0 \rightarrow F^1)$ such that $E \otimes_R -$ leaves it exact for every injective module $E$.

**Lemma 2.1.** Let $R$ be commutative Noetherian with finite Krull dimension $d$. If $M$ is Gorenstein injective, then $M$ is strongly cotorsion.

**Proof.** Since $M$ is Gorenstein injective, there exists a sequence

$$\cdots \rightarrow E^{-2} \delta^{-2} \rightarrow E^{-1} \delta^{-1} \rightarrow E^0 \delta^0 \rightarrow E^1 \delta^1 \rightarrow E^2 \rightarrow \cdots$$

of injective modules such that $M = \ker \delta^0$ and such that $\text{Hom}_R(E, -)$ leaves the sequence exact when $E$ is injective. For each $i \in \mathbb{N}$, put $N_i := \ker \delta^{-i}$ and break the above long exact sequence into short exact sequences. We have that

$$\text{Ext}^1_R(F, M) \cong \text{Ext}^2_R(F, N_1) \cong \cdots \cong \text{Ext}^{d+1}_R(F, N_d).$$

Now, by Gruson and Jensen’s theorem in \cite{2}, proj.dim $F \leq d$ and so $\text{Ext}^{d+1}_R(F, N_d) = 0$ and the proof is complete.

There have been attempts to dualize the theory of associated primes (see \cite{12}, \cite{1} and \cite{11}). However, these dualizing concepts are equivalent \cite{11}. A prime ideal $p$ of $R$ is said to be a coassociated prime of $M$ if there exists an Artinian homomorphic image $L$ of $M$ with $p = 0 :_R L$. The set of coassociated prime ideals of $M$ is denoted by $\text{Coass}(M)$.

**Theorem 2.2.** Let $R$ be commutative Noetherian. The following conditions are equivalent:

1. $E(R)$ is flat;
2. $E(R)$ has finite flat dimension;
3. The injective envelope $E(F)$ is flat for any flat module $F$;
4. The flat cover $F(E)$ is injective for any injective module $E$;
5. The flat cover $F(M)$ is injective for any strongly cotorsion module $M$;
6. The injective envelope $E(M)$ is flat for any strongly torsion free module $M$;
7. The injective envelope $E(M)$ is flat for any Gorenstein flat module $M$;
8. If $p \in \text{Coass}(E)$ for an injective $R$-module $E$, then $\hat{R}_p$ is injective.
If moreover the Krull dimension of $R$ is finite, then the above conditions are equivalent to:

(9) The flat cover $F(M)$ is injective for any Gorenstein injective module $M$.

Proof. The implications $(1) \iff (3) \iff (4)$ follow from [2, Theorem 3].

We will show that $(2) \implies (1) \implies (7) \implies (2), (3) \implies (5) \implies (6) \implies (3), (1) \implies (8) \iff (4)$ and $(5) \iff (9)$.

$(2) \implies (1)$ Let $p \in \text{Ass}(R)$. Since $E(R/p)$ is a summand of $E(R)$, it has finite flat dimension as an $R$-module and hence as an $R_p$-module. So, by [9, Proposition 2.1(3), (2)] $R_p$ is Gorenstein ring and $\text{f.dim}_{R_p} E(R/p) = 0$. Therefore $\text{f.dim}_R E(R/p) = 0$ which, in turn, implies that $E(R)$ is a flat $R$-module.

$(1) \implies (7)$ Let $p \in \text{Ass}(R)$. Since $E(R/p)$ is a summand of $E(R)$, it is flat. Now, suppose that $M$ is a Gorenstein flat $R$-module. Then $M$ can be embedded to a flat $R$-module $F$. Hence $\text{Ass}(M) \subseteq \text{Ass}(F) \subseteq \text{Ass}(R)$. Thus $E(M) = \bigoplus_{p \in \text{Ass}(R)} \mu_0([p, M]) E(R/p)$ is flat.

$(7) \implies (2)$ This is clear.

$(3) \implies (5)$ Suppose that $M$ is a strongly cotorsion $R$-module. Consider the full pushout diagram of $F \rightarrow M$ and $F \rightarrow E$ where $F$ is a flat cover of $M$ and $E$ is an injective envelope of $F$:

Since $E$ is flat, $\text{f.dim}X \leq 2$. Hence the exact sequence

$$0 \rightarrow M \rightarrow C \rightarrow X \rightarrow 0$$

is split. Thus, by [10, 1.2.10], the flat cover of $M$ is a summand of the flat cover of $C$. Now, since $E$ is flat and, by Nakamura’s Lemma (see for example [10, 2.1.1]), $N$ is cotorsion, $E$ is a flat precover of $C$. Hence, by [10, 1.2.7], the flat cover of $C$ is injective and so the flat cover of $M$ is injective.
(5) $\implies$ (6) Let $C$ be an injective cogenerator of $R$-modules. Set $D(-) := \text{Hom}_R(-, C)$. Since $M$ is strongly torsion free, it follows from the equality
\[ D(\text{Tor}_1^R(X, M)) = \text{Ext}_1^R(X, D(M)), \]
for all $R$-modules $X$, that $D(M)$ is strongly cotorsion. Let $F \to D(M) \to 0$ be a flat cover of $D(M)$. Then $F$ is injective. Hence we have an exact sequence $0 \to D(D(M)) \to D(F)$ in which $D(F)$ is flat. Thus the injective envelope $E(D(D(M)))$ is flat and so $E(M)$ is too.

(6) $\implies$ (3) This is clear.

(1) $\implies$ (8) Let $E$ be an injective $R$-module. Let $p \in \text{Coass} E$. Then, by [11, 1.7], there is a maximal ideal $m$ of $R$ such that $p \in \text{Ass} \text{Hom}_R(E, E(R/m))$. Also $\text{Hom}_R(E, E(R/m))$ is flat. Hence $E(R/p)$ is flat which implies that
\[ \text{Hom}_R(E(R/p), E(R/p)) \cong \hat{R}_p \]
is injective.

(8) $\implies$ (4) Let $E$ be an injective $R$-module and let $T_p \neq 0$ appear in $F(E)$ for some prime ideal $p$ of $R$. We will show that $T_p$ is injective and so $F(E)$ is injective. Since, by [5, 2.2],
\[ 0 \neq k(p) \otimes_{R_p} \text{Hom}_R(R_p, E) \cong \text{Hom}_R(\text{Hom}_{R_p}(k(p), R_p), E), \]
we have that $p \in \text{Ass} R$. Now, let $m$ be a maximal ideal of $R$ such that $p \subseteq m$. Since $E(R/m)$ is Artinian, by [8, 2.1] and [11],
\[ \{ q \in \text{Ass} R : q \subseteq m \} = \text{Att} E(R/m) = \text{Coass} E(R/m). \]
Hence $p \in \text{Coass} E(R/m)$ and so, by our assumption, $\hat{R}_p$ is injective. Now, it follows from the fact that $T_p$ is a summand of a product of copies of $\hat{R}_p$ (see for example [10, 4.1.10]) that $T_p$ is injective.

(5) $\implies$ (9) This follows from Lemma 2.1.

(9) $\implies$ (5) This is clear.

3. Results about modules over the rings with $E(R)$ flat

Theorem 3.1. Let $R$ be commutative Noetherian. If the injective envelope $E(R)$ is flat, then for any $R$-module $M$, the following conditions are equivalent:

1. $F(M)$ is injective;
2. every injective precover $E \xrightarrow{\varphi} M$ of $M$ is surjective and $\ker \varphi$ is cotorsion;
3. there is an injective precover $E \xrightarrow{\varphi} M$ of $M$ such that $\varphi$ is surjective and $\ker \varphi$ is cotorsion.

Proof. (1) $\implies$ (2) Since $F(M)$ is injective, every injective precover $E \xrightarrow{\varphi} M$ is surjective. Set $K := \ker \varphi$. We must show that $K$ is cotorsion. To do this, consider the full pullback diagram of $F(M) \to M$ and $E \to M$: 

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0 0
\downarrow \downarrow
H \rightarrow H

0 \rightarrow K \rightarrow C \rightarrow F(M) \rightarrow 0

0 \rightarrow K \rightarrow E \rightarrow M \rightarrow 0

0 \rightarrow 0

Since \( \varphi : E \rightarrow M \) is an injective precover, \( \text{Ext}^1_R(X, K) = 0 \) for every injective module \( X \). In particular, \( \text{Ext}^1_R(F(M), K) = 0 \). So the upper exact row is split. Hence \( K \) is a summand of \( C \). On the other hand, since \( H \) is cotorsion and \( E \) is injective, \( C \) is cotorsion. Hence \( K \) is cotorsion.

(2) \( \Rightarrow \) (3) This is clear.

(3) \( \Rightarrow \) (1) Let \( M \) be an \( R \)-module and let

\[
0 \rightarrow K \rightarrow E \rightarrow M \rightarrow 0
\]

be an exact sequence such that \( E \) is an injective precover of \( M \) and \( K \) is cotorsion. Then, it is easy to see that \( F(E) \) is a flat precover of \( M \). Thus \( F(M) \) is a summand of \( F(E) \) and so \( F(M) \) is injective because \( F(E) \) is injective.

The following example, which was suggested by the referee, provides a noninjective module whose flat cover is injective.

**Example 3.2.** Let \( k \) be a field. Let \( R = k[[x^3, x^4, x^5]] \). Then \( R \) is not Gorenstein since 3, 4 and 5 do not generate a symmetric submonoid of \( \mathbb{N} \). Note that \( R \) is a cotorsion (in fact pure injective) \( R \)-module. On the other hand, \( k((x)) \) (the field of fractions of \( R \)) is a flat and injective \( R \)-module. Set \( M := k((x))/R \). So \( k(x) \rightarrow M \) is a flat (pre)cover of \( M \). Therefore \( F(M) \) is injective but \( M \) is not injective, since \( R \) is not Gorenstein.

Recall that whenever \( R \) is Gorenstein, \( E(R) \) is flat (see for example [10, 5.1.2]).

**Theorem 3.3.** Let \( R \) be a Gorenstein ring. Then \( M \) is strongly cotorsion if and only if \( M \) is cotorsion and Gorenstein injective.

**Proof.** \((\Rightarrow)\) It follows from [9] Lemma 4.1.

\((\Leftarrow)\) First of all we show that \( F(M) \) is injective. To do this, we only need to show that whenever \( T_p \neq 0 \) appear in the flat cover of \( M \), it is injective. Using the same notations as we used in the proof of Lemma 2.1, since \( \text{inj.dim} \, R_p < \infty \), it is easy to see that \( \text{Ext}^1_R(R_p, N_i) = 0 \) for all \( i \). Hence, for all \( i \), \( \text{Hom}_R(R_p, N_i) \) and \( \text{Hom}_R(R_p, M) \) are Gorenstein injective \( R_p \)-modules. So, in view of Theorem...
3.1 and Lemma 2.1, $F(\text{Hom}_R(R_p, M))$ is injective. On the other hand, by [3, 2.2], $T_p$ appears in the flat cover of $\text{Hom}_R(R_p, M)$. Therefore $T_p$ is injective and $F(M)$ is too. Since $E^{-1}$ is an injective precovers of $M$, by Lemma 2.1, $N_1$ is cotorsion. Similarly, for each $i$, $N_i$ is cotorsion. Let $L$ be an $R$-module with finite flat dimension $n$. Then, by using the induction on $n$, it is easy to see that $\text{Ext}^i_R(L, N) = 0$ for all $i > n$ and all cotorsion modules $N$ and so $\text{Ext}^{n+1}_R(L, N_n) = 0$. Hence $\text{Ext}^1_R(L, M) \cong \text{Ext}^2_R(L, N_1) \cong \cdots \cong \text{Ext}^{n+1}_R(L, N_n) = 0$. Therefore $M$ is strongly cotorsion.

**Theorem 3.4.** Suppose that $E(R)$ is flat. If the $R$-module $M$ has a nonzero injective cover, then $\text{Ass} R \cap \text{Coass} M \neq \emptyset$. The converse is true if $M$ is Artinian.

**Proof.** Let $M$ be an $R$-module with nonzero injective cover $E \rightarrow M$. Since $\text{Coass} \varphi(E) \neq \emptyset$, it is enough to show that $\text{Coass} \varphi(E) \subseteq \text{Ass} R \cap \text{Coass} M$. Let $p \in \text{Coass} \varphi(E)$. Then, by [11, 1.14], $p \in \text{Att} E \subseteq \text{Ass} R$. Thus, by [2] Theorem 3], $\text{ht} p = 0$. Therefore, by [11, 2.6], $p \in \text{Coass} M$.

For the proof of the last statement let $M$ be an Artinian module and let $p \in \text{Ass} R \cap \text{Att} M \neq \emptyset$. In view of [11, 1.7], there exists a maximal ideal $m$ of $R$ such that $p \subseteq m$ and $\text{Hom}_R(R/p, E(R/m))$ is a homomorphic image of $M$. Now, let $F$ be a flat cover of $\text{Hom}_R(R/p, E(R/m)))$ and let $q$ be a prime ideal of $R$. By using the isomorphisms

$$k(q) \otimes_{R_q} \text{Hom}_R(R_q, \text{Hom}_R(R/p, E(R/m))) \cong k(q) \otimes_{R_q} \text{Hom}_R(R_q \otimes_R R/p, E(R/m))$$

$$\cong \text{Hom}_R(\text{Hom}_R(k(q), R_q \otimes_R R/p), E(R/m))$$

in conjunction with [3, 2.2], it is easy to see that $F = T_p$. Since $p \in \text{Ass} R$, we have that $E(R/p)$ is flat. Thus $\text{Hom}_R(E(R/p), E(R/p)) \cong R_p$ is injective and so $T_p = F$ is injective. Therefore it is enough to show that $\text{Hom}_R(T_p, M) \neq 0$. To do this consider the exact sequence

$$0 \rightarrow K \rightarrow M \rightarrow \text{Hom}_R(R/p, E(R/m)) \rightarrow 0.$$  

Since $K$ is Artinian, it is thus cotorsion, and so the sequence

$$\text{Hom}_R(T_p, M) \rightarrow \text{Hom}_R(T_p, \text{Hom}_R(R/p, E(R/m))) \rightarrow 0$$

is exact. Now, since $T_p$ is a flat cover of $\text{Hom}_R(R/p, E(R/m))$, $\text{Hom}_R(T_p, M) \neq 0$ and the proof is complete.

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**References**


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