

ON THE RINGS WHOSE INJECTIVE HULLS ARE FLAT

K. KHASHYARMANESH AND SH. SALARIAN

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ABSTRACT. Let R be a commutative Noetherian ring with nonzero identity and let the injective envelope of R be flat. We characterize these kinds of rings and obtain some results about modules with nonzero injective cover over these rings.

1. INTRODUCTION

Let R be a commutative Noetherian ring with nonzero identity and let $E(R)$ be an injective envelope of R . It is well known that for the ring R the condition that $E(R)$ be flat generalizes the condition that R be Gorenstein (see for example [10, 5.1.2 (1), (4)]). In [2, Theorem 3], Cheathan and Enochs gave a characterization of these kinds of rings. In fact, they showed that $E(R)$ is flat if and only if for any flat R -module F , the injective envelope $E(F)$ is flat and these are equivalent to, for any injective module E , the flat cover $F(E)$ is injective. In this paper, we focus on the rings R for which $E(R)$ is flat. In section two, we extend the above equivalent conditions (see Theorem 2.2).

On the other hand, in [6], Golan and Teply showed that every module admits an injective (pre)cover. Of course, this may be zero. So the natural problem is when a nonzero module has nonzero injective cover. In [10, 2.4.8], it was shown that whenever every nonzero module over a ring R has a nonzero injective cover, the ring R must be Artinian. In section three we obtain some results about modules with nonzero injective cover over the rings with $E(R)$ flat. For example we show that if an R -module M has nonzero injective cover, then $\text{Ass}R \cap \text{Coass}M \neq \emptyset$. The converse is true if M is Artinian.

Throughout this paper, R is commutative Noetherian with nonzero identity and all modules are unitary. For an R -module X , $\text{inj.dim}_R X$ stands for the injective dimension of X , $\text{f.dim}_R X$ stands for flat dimension of X , $\text{proj.dim}_R X$ stands for projective dimension of X , $E(X)$ stands for injective envelope of X and $F(X)$ stands for its flat cover. Finally we use \mathbb{N} to denote the set of positive integers. All other notations are standard.

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2. CHARACTERIZING THE RINGS WITH $E(R)$ FLAT

We recall that a module C is called cotorsion if $\text{Ext}_R^1(F, C) = 0$ for any flat module F ; it is strongly cotorsion if $\text{Ext}_R^1(X, C) = 0$ for any module X of finite flat dimension. Also C is called strongly torsion free if $\text{Tor}_1^R(X, C) = 0$ for all modules X of finite flat dimension. If F is flat and cotorsion, it was proved in [3, p.183] that F can be uniquely written in the form $F = \prod T_{\mathfrak{p}}$, where $T_{\mathfrak{p}}$ is a completion of a free $R_{\mathfrak{p}}$ -module with respect to \mathfrak{p} -adic topology. Note that flat covers are known to exist over any rings (see [4, 7.4.4]).

A module M is Gorenstein injective if there is an exact sequence

$$\dots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \dots$$

of injective modules such that $M = \ker(E^0 \longrightarrow E^1)$ and such that $\text{Hom}_R(E, -)$ leaves the sequence exact when E is injective. If M is Gorenstein injective, then $\text{Ext}_R^i(L, M) = 0$ for all $i \in \mathbb{N}$ and L such that $\text{inj.dim} L < \infty$. Also an R -module N is called Gorenstein flat if there is an exact complex

$$\dots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \dots$$

with F^i flat and $N = \ker(F^0 \longrightarrow F^1)$ such that $E \otimes_R -$ leaves it exact for every injective module E .

Lemma 2.1. *Let R be commutative Noetherian with finite Krull dimension d . If M is Gorenstein injective, then M is strongly cotorsion.*

Proof. Since M is Gorenstein injective, there exists a sequence

$$\dots \longrightarrow E^{-2} \xrightarrow{\delta^{-2}} E^{-1} \xrightarrow{\delta^{-1}} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} E^2 \longrightarrow \dots$$

of injective modules such that $M = \ker \delta^0$ and such that $\text{Hom}_R(E, -)$ leaves the sequence exact when E is injective. For each $i \in \mathbb{N}$, put $N_i := \ker \delta^{-i}$ and break the above long exact sequence into short exact sequences. We have that

$$\text{Ext}_R^1(F, M) \cong \text{Ext}_R^2(F, N_1) \cong \dots \cong \text{Ext}_R^{d+1}(F, N_d).$$

Now, by Gruson and Jensen's theorem in [7], $\text{proj.dim} F \leq d$ and so $\text{Ext}_R^{d+1}(F, N_d) = 0$ and the proof is complete.

There have been attempts to dualize the theory of associated primes (see [12], [1] and [11]). However, these dualizing concepts are equivalent [11]. A prime ideal \mathfrak{p} of R is said to be a coassociated prime of M if there exists an Artinian homomorphic image L of M with $\mathfrak{p} = 0 :_R L$. The set of coassociated prime ideals of M is denoted by $\text{Coass}(M)$.

Theorem 2.2. *Let R be commutative Noetherian. The following conditions are equivalent:*

- (1) $E(R)$ is flat;
- (2) $E(R)$ has finite flat dimension;
- (3) The injective envelope $E(F)$ is flat for any flat module F ;
- (4) The flat cover $F(E)$ is injective for any injective module E ;
- (5) The flat cover $F(M)$ is injective for any strongly cotorsion module M ;
- (6) The injective envelope $E(M)$ is flat for any strongly torsion free module M ;
- (7) The injective envelope $E(M)$ is flat for any Gorenstein flat module M ;
- (8) If $\mathfrak{p} \in \text{Coass}(E)$ for an injective R -module E , then $\hat{R}_{\mathfrak{p}}$ is injective.

If moreover the Krull dimension of R is finite, then the above conditions are equivalent to:

(9) The flat cover $F(M)$ is injective for any Gorenstein injective module M .

Proof. The implications (1) \iff (3) \iff (4) follow from [2, Theorem 3].

We will show that (2) \implies (1) \implies (7) \implies (2), (3) \implies (5) \implies (6) \implies (3), (1) \implies (8) \implies (4) and (5) \iff (9).

(2) \implies (1) Let $\mathfrak{p} \in \text{Ass}(R)$. Since $E(R/\mathfrak{p})$ is a summand of $E(R)$, it has finite flat dimension as an R -module and hence as an $R_{\mathfrak{p}}$ -module. So, by [9, Proposition 2.1(3), (2)] $R_{\mathfrak{p}}$ is Gorenstein ring and $\text{f.dim}_{R_{\mathfrak{p}}} E(R/\mathfrak{p}) = 0$. Therefore $\text{f.dim}_R E(R/\mathfrak{p}) = 0$ which, in turn, implies that $E(R)$ is a flat R -module.

(1) \implies (7) Let $\mathfrak{p} \in \text{Ass}(R)$. Since $E(R/\mathfrak{p})$ is a summand of $E(R)$, it is flat. Now, suppose that M is a Gorenstein flat R -module. Then M can be embedded to a flat R -module F . Hence $\text{Ass}(M) \subseteq \text{Ass}(F) \subseteq \text{Ass}(R)$. Thus $E(M) = \bigoplus_{\mathfrak{p} \in \text{Ass}(R)} \mu_0(\mathfrak{p}, M)E(R/\mathfrak{p})$ is flat.

(7) \implies (2) This is clear.

(3) \implies (5) Suppose that M is a strongly cotorsion R -module. Consider the full pushout diagram of $F \rightarrow M$ and $F \rightarrow E$ where F is a flat cover of M and E is an injective envelope of F :

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & C \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & X & \xlongequal{\quad} & X \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Since E is flat, $\text{f.dim} X \leq 2$. Hence the exact sequence

$$0 \longrightarrow M \longrightarrow C \longrightarrow X \longrightarrow 0$$

is split. Thus, by [10, 1.2.10], the flat cover of M is a summand of the flat cover of C . Now, since E is flat and, by Wakamutsu's Lemma (see for example [10, 2.1.1]), N is cotorsion, E is a flat precover of C . Hence, by [10, 1.2.7], the flat cover of C is injective and so the flat cover of M is injective.

(5) \implies (6) Let C be an injective cogenerator of R -modules. Set $D(-) := \text{Hom}_R(-, C)$. Since M is strongly torsion free, it follows from the equality

$$D(\text{Tor}_1^R(X, M)) = \text{Ext}_R^1(X, D(M)),$$

for all R -modules X , that $D(M)$ is strongly cotorsion. Let $F \rightarrow D(M) \rightarrow 0$ be a flat cover of $D(M)$. Then F is injective. Hence we have an exact sequence $0 \rightarrow D(D(M)) \rightarrow D(F)$ in which $D(F)$ is flat. Thus the injective envelope $E(D(D(M)))$ is flat and so $E(M)$ is too.

(6) \implies (3) This is clear.

(1) \implies (8) Let E be an injective R -module. Let $\mathfrak{p} \in \text{Coass}E$. Then, by [11, 1.7], there is a maximal ideal \mathfrak{m} of R such that $\mathfrak{p} \in \text{AssHom}_R(E, E(R/\mathfrak{m}))$. Also $\text{Hom}_R(E, E(R/\mathfrak{m}))$ is flat. Hence $E(R/\mathfrak{p})$ is flat which implies that

$$\text{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})) \cong \hat{R}_{\mathfrak{p}}$$

is injective.

(8) \implies (4) Let E be an injective R -module and let $T_{\mathfrak{p}} \neq 0$ appear in $F(E)$ for some prime ideal \mathfrak{p} of R . We will show that $T_{\mathfrak{p}}$ is injective and so $F(E)$ is injective. Since, by [5, 2.2],

$$0 \neq k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \text{Hom}_R(R_{\mathfrak{p}}, E) \cong \text{Hom}_R(\text{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), R_{\mathfrak{p}}), E),$$

we have that $\mathfrak{p} \in \text{Ass}R$. Now, let \mathfrak{m} be a maximal ideal of R such that $\mathfrak{p} \subseteq \mathfrak{m}$. Since $E(R/\mathfrak{m})$ is Artinian, by [8, 2.1] and [11],

$$\{\mathfrak{q} \in \text{Ass}R : \mathfrak{q} \subseteq \mathfrak{m}\} = \text{Att}E(R/\mathfrak{m}) = \text{Coass}E(R/\mathfrak{m}).$$

Hence $\mathfrak{p} \in \text{Coass}E(R/\mathfrak{m})$ and so, by our assumption, $\hat{R}_{\mathfrak{p}}$ is injective. Now, it follows from the fact that $T_{\mathfrak{p}}$ is a summand of a product of copies of $\hat{R}_{\mathfrak{p}}$ (see for example [10, 4.1.10]) that $T_{\mathfrak{p}}$ is injective.

(5) \implies (9) This follows from Lemma 2.1.

(9) \implies (5) This is clear.

3. RESULTS ABOUT MODULES OVER THE RINGS WITH $E(R)$ FLAT

Theorem 3.1. *Let R be commutative Noetherian. If the injective envelope $E(R)$ is flat, then for any R -module M , the following conditions are equivalent:*

- (1) $F(M)$ is injective;
- (2) every injective precover $E \xrightarrow{\varphi} M$ of M is surjective and $\ker\varphi$ is cotorsion;
- (3) there is an injective precover $E \xrightarrow{\varphi} M$ of M such that φ is surjective and $\ker\varphi$ is cotorsion.

Proof. (1) \implies (2) Since $F(M)$ is injective, every injective precover $E \xrightarrow{\varphi} M$ is surjective. Set $K := \ker\varphi$. We must show that K is cotorsion. To do this, consider the full pullback diagram of $F(M) \rightarrow M$ and $E \rightarrow M$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & H & \xlongequal{\quad} & H & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & F(M) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & M \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Since $\varphi : E \rightarrow M$ is an injective precover, $\text{Ext}_R^1(X, K) = 0$ for every injective module X . In particular, $\text{Ext}_R^1(F(M), K) = 0$. So the upper exact row is split. Hence K is a summand of C . On the other hand, since H is cotorsion and E is injective, C is cotorsion. Hence K is cotorsion.

- (2) \implies (3) This is clear.
- (3) \implies (1) Let M be an R -module and let

$$0 \rightarrow K \rightarrow E \rightarrow M \rightarrow 0$$

be an exact sequence such that E is an injective precover of M and K is cotorsion. Then, it is easy to see that $F(E)$ is a flat precover of M . Thus $F(M)$ is a summand of $F(E)$ and so $F(M)$ is injective because $F(E)$ is injective.

The following example, which was suggested by the referee, provides a noninjective module whose flat cover is injective.

Example 3.2. Let k be a field. Let $R = k[[x^3, x^4, x^5]]$. Then R is not Gorenstein since 3, 4 and 5 do not generate a symmetric submonoid of \mathbb{N} . Note that R is a cotorsion (in fact pure injective) R -module. On the other hand, $k((x))$ (the field of fractions of R) is a flat and injective R -module. Set $M := k((x))/R$. So $k(x) \rightarrow M$ is a flat (pre)cover of M . Therefore $F(M)$ is injective but M is not injective, since R is not Gorenstein.

Recall that whenever R is Gorenstein, $E(R)$ is flat (see for example [10, 5.1.2]).

Theorem 3.3. *Let R be a Gorenstein ring. Then M is strongly cotorsion if and only if M is cotorsion and Gorenstein injective.*

Proof. (\implies) It follows from [9, Lemma 4.1].

(\impliedby) First of all we show that $F(M)$ is injective. To do this, we only need to show that whenever $T_{\mathfrak{p}} \neq 0$ appear in the flat cover of M , it is injective. Using the same notations as we used in the proof of Lemma 2.1, since $\text{inj.dim} R_{\mathfrak{p}} < \infty$, it is easy to see that $\text{Ext}_R^1(R_{\mathfrak{p}}, N_i) = 0$ for all i . Hence, for all i , $\text{Hom}_R(R_{\mathfrak{p}}, N_i)$ and $\text{Hom}_R(R_{\mathfrak{p}}, M)$ are Gorenstein injective $R_{\mathfrak{p}}$ -modules. So, in view of Theorem

3.1 and Lemma 2.1, $F(\text{Hom}_R(R_{\mathfrak{p}}, M))$ is injective. On the other hand, by [5, 2.2], $T_{\mathfrak{p}}$ appears in the flat cover of $\text{Hom}_R(R_{\mathfrak{p}}, M)$. Therefore $T_{\mathfrak{p}}$ is injective and $F(M)$ is too. Since E^{-1} is an injective precover of M , by Lemma 2.1, N_1 is cotorsion. Similarly, for each i , N_i is cotorsion. Let L be an R -module with finite flat dimension n . Then, by using the induction on n , it is easy to see that $\text{Ext}_R^i(L, N) = 0$ for all $i > n$ and all cotorsion modules N and so $\text{Ext}_R^{n+1}(L, N_n) = 0$. Hence $\text{Ext}_R^1(L, M) \cong \text{Ext}_R^2(L, N_1) \cong \cdots \cong \text{Ext}_R^{n+1}(L, N_n) = 0$. Therefore M is strongly cotorsion.

Theorem 3.4. *Suppose that $E(R)$ is flat. If the R -module M has a nonzero injective cover, then $\text{Ass}R \cap \text{Coass}M \neq \emptyset$. The converse is true if M is Artinian.*

Proof. Let M be an R -module with nonzero injective cover $E \xrightarrow{\varphi} M$. Since $\text{Coass}\varphi(E) \neq \emptyset$, it is enough to show that $\text{Coass}\varphi(E) \subseteq \text{Ass}R \cap \text{Coass}M$. Let $\mathfrak{p} \in \text{Coass}\varphi(E)$. Then, by [11, 1.14], $\mathfrak{p} \in \text{Att}E \subseteq \text{Ass}R$. Thus, by [2, Theorem 3], $\text{ht}\mathfrak{p} = 0$. Therefore, by [11, 2.6], $\mathfrak{p} \in \text{Coass}M$.

For the proof of the last statement let M be an Artinian module and let $\mathfrak{p} \in \text{Ass}R \cap \text{Att}M \neq \emptyset$. In view of [11, 1.7], there exists a maximal ideal \mathfrak{m} of R such that $\mathfrak{p} \subseteq \mathfrak{m}$ and $\text{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))$ is a homomorphic image of M . Now, let F be a flat cover of $\text{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))$ and let \mathfrak{q} be a prime ideal of R . By using the isomorphisms

$$\begin{aligned} k(\mathfrak{q}) \otimes_{R_{\mathfrak{p}}} \text{Hom}_R(R_{\mathfrak{q}}, \text{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))) &\cong k(\mathfrak{q}) \otimes_{R_{\mathfrak{p}}} \text{Hom}_R(R_{\mathfrak{q}} \otimes_R R/\mathfrak{p}, E(R/\mathfrak{m})) \\ &\cong \text{Hom}_R(\text{Hom}_R(k(\mathfrak{q}), R_{\mathfrak{q}} \otimes_R R/\mathfrak{p}), E(R/\mathfrak{m})) \end{aligned}$$

in conjunction with [5, 2.2], it is easy to see that $F = T_{\mathfrak{p}}$. Since $\mathfrak{p} \in \text{Ass}R$, we have that $E(R/\mathfrak{p})$ is flat. Thus $\text{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})) \cong \hat{R}_{\mathfrak{p}}$ is injective and so $T_{\mathfrak{p}} = F$ is injective. Therefore it is enough to show that $\text{Hom}_R(T_{\mathfrak{p}}, M) \neq 0$. To do this consider the exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow \text{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m})) \longrightarrow 0.$$

Since K is Artinian, it is thus cotorsion, and so the sequence

$$\text{Hom}_R(T_{\mathfrak{p}}, M) \longrightarrow \text{Hom}_R(T_{\mathfrak{p}}, \text{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))) \longrightarrow 0$$

is exact. Now, since $T_{\mathfrak{p}}$ is a flat cover of $\text{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{m}))$, $\text{Hom}_R(T_{\mathfrak{p}}, M) \neq 0$ and the proof is complete.

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INSTITUTE FOR STUDIES IN THEORETICAL PHYSICS AND MATHEMATICS, P.O. Box 19395-5746,
TEHRAN, IRAN – AND – DEPARTMENT OF MATHEMATICS, DAMGHAN UNIVERSITY, P.O. Box 36715-
364, DAMGHAN, IRAN

E-mail address: `khashyar@mail.ipm.ir`

INSTITUTE FOR STUDIES IN THEORETICAL PHYSICS AND MATHEMATICS, P.O. Box 19395-5746,
TEHRAN, IRAN – AND – DEPARTMENT OF MATHEMATICS, DAMGHAN UNIVERSITY, P.O. Box 36715-
364, DAMGHAN, IRAN