

ON A CLASS OF SUBLINEAR QUASILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We establish existence and multiplicity of positive solutions to the quasilinear boundary value problem

$$\begin{aligned} \operatorname{div}(|\nabla u|^{p-2}\nabla u) &= -\lambda f(u) \text{ in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $f : [0, \infty) \rightarrow R$ is continuous and p -sublinear at ∞ , and λ is a large parameter.

1. INTRODUCTION

Consider the quasilinear elliptic boundary value problem

$$(I) \quad \begin{aligned} \Delta_p u &= -\lambda f(u) \text{ in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $f : [0, \infty) \rightarrow R$, and λ is a positive parameter.

Problem (I) has been studied extensively during recent years (see [2]–[11] and the references therein). In [5], Guo and Webb proved existence results for (I) for λ large when f is a smooth, nondecreasing positive function on $(0, \infty)$, $\lim_{s \rightarrow \infty} s^{-\mu} f(s) = \beta > 0$ for some $\mu \in (0, p - 1)$, and $\liminf_{x \rightarrow 0^+} \frac{f(s)}{s^{p-1}} > 0$, using Serrin's sweeping principle. They also considered uniqueness of positive solutions when $\frac{f(x)}{x^{p-1}}$ is decreasing on $(0, \infty)$. Multiplicity results were established in [4] when the above condition of f at 0 is replaced by $\lim_{s \rightarrow 0} \frac{f(s)}{s^{p-1}} = 0$. Related results for the radial case in a ball using ordinary differential equations techniques can be found in [6]. In this paper, we shall establish existence results to (I) for λ large when f is merely continuous and p -sublinear at ∞ . In particular, we do not assume any conditions of f near 0, nor monotonicity and smoothness on f as in [4, 5]. We also establish a multiplicity result for (I) under additional assumptions. Our approach depends on $C^{1,\beta}$ regularity results in [7, 9] to create ordered sub- and supersolutions that provide a solution in between via the Schauder fixed point theorem. Note that our sub- and supersolutions are defined in the operator context and are not weak sub- and supersolutions in general.

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2. EXISTENCE RESULTS

We shall make the following assumptions:

(A.1) $f : [0, \infty) \rightarrow R$ is continuous and there exist positive numbers K, L_1 such that $f(x) > 2L_1$ for $x > K$.

(A.2) $\lim_{x \rightarrow \infty} \frac{f(x)}{x^{p-1}} = 0$.

Our main result is

Theorem 1. *Let (A.1)-(A.2) hold. Then there exists $\lambda_0 > 0$ such that (I) has a positive solution for $\lambda > \lambda_0$. If, in addition, $f \geq 0$ on $(0, \infty)$ and*

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x^{p-1}} = 0,$$

then (I) has at least two positive solutions for $\lambda > \lambda_0$.

We first need some preliminary results. We shall denote by $\|\cdot\|_p$ and $|\cdot|_p$ the norms in $L^p(\Omega)$ and $C^p(\bar{\Omega})$ respectively.

The following lemma gives positivity of the p-Laplacian operator.

Lemma 1 ([10]). *Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ be two points in R^n , and let $|\cdot|$ and (\cdot, \cdot) denote the Euclidean norm and the corresponding inner product in R^n respectively.*

Then

(i) If $p \geq 2$,

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq \left(\frac{1}{2}\right)^{p-1}|x - y|^p.$$

(ii) If $1 < p \leq 2$,

$$(|x| + |y|)^{2-p}(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq (p - 1)|x - y|^2.$$

Next, let $\phi \in C^1(\bar{\Omega})$ be the solution of

$$\begin{cases} \Delta_p \phi = -1 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we have

Lemma 2. *Let $g \in L^\infty(\Omega)$ and let $u \in W_0^{1,p}(\Omega)$ satisfy*

$$\begin{cases} \Delta_p u = -g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq M|\phi|_0,$$

where $M = \|g\|_\infty^{\frac{1}{p-1}}$.

Proof. Since $\Delta_p(M\phi) = -M^{p-1} = -\|g\|_\infty$ and $\Delta_p u = -g$, it follows from the weak maximum principle [8] that $-M\phi \leq u \leq M\phi$, from which the lemma follows.

Lemma 3. *Let C, L_0, L_1 , and L be positive numbers with $L_0 \leq L \leq L_1$. Suppose that there exist $h, h_{\lambda,L} \in L^\infty(\Omega)$ such that*

$$(2.1) \quad \|h_{\lambda,L}\|_\infty \leq C$$

for every $\lambda > 0, L_0 \leq L \leq L_1$, and

$$\lim_{\lambda \rightarrow \infty} \|h_{\lambda,L} - h\|_2 = 0$$

uniformly for $L \in [L_0, L_1]$. Let $u_{\lambda,L}$ and u be solutions of

$$\Delta_p u_{\lambda,L} = -h_{\lambda,L} \text{ in } \Omega, \quad u_{\lambda,L} = 0 \text{ on } \partial\Omega$$

and

$$\Delta_p u = -h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

respectively. Then

$$(2.2) \quad \lim_{\lambda \rightarrow \infty} \|u_{\lambda,L} - u\|_1 = 0$$

uniformly for $L \in [L_0, L_1]$.

Proof. By (2.1) and Lemma 2, it follows that $u_{\lambda,L}$ are bounded in $L^\infty(\Omega)$ independent of λ, L . It then follows from regularity results in [7, 9] that there exist numbers $\beta \in (0, 1)$ and $C_1 > 0$ such that $u_{\lambda,L}, u \in C^{1,\beta}(\bar{\Omega})$ and

$$(2.3) \quad \|u_{\lambda,L}\|_1 < C_1$$

for all $\lambda > 0, L \in [L_0, L_1]$.

Multiplying the equation

$$\Delta_p u_{\lambda,L} - \Delta_p u = -(h_{\lambda,L} - h)$$

by $u_{\lambda,L} - u$ and integrating gives

$$(2.4) \quad \begin{aligned} & \int_{\Omega} (|\nabla u_{\lambda,L}|^{p-2} \nabla u_{\lambda,L} - |\nabla u|^{p-2} \nabla u, \nabla u_{\lambda,L} - \nabla u) dx \\ &= \int_{\Omega} (h_{\lambda,L} - h)(u_{\lambda,L} - u) dx \\ &\leq \|h_{\lambda,L} - h\|_2 \|u_{\lambda,L} - u\|_2. \end{aligned}$$

On the other hand, Lemma 1 and (2.3) imply

$$(2.5) \quad \int_{\Omega} (|\nabla u_{\lambda,L}|^{p-2} \nabla u_{\lambda,L} - |\nabla u|^{p-2} \nabla u, \nabla u_{\lambda,L} - \nabla u) dx \geq C_2 \|\nabla(u_{\lambda,L} - u)\|_2^q,$$

where $q = \max(p, 2)$, and C_2 is a positive constant depending only on p, C_1 .

Combining (2.4), (2.5), and Poincaré's inequality, we obtain

$$(2.6) \quad \lim_{\lambda \rightarrow \infty} \|u_{\lambda,L} - u\|_2 = 0$$

uniformly for $L \in [L_0, L_1]$.

Suppose to the contrary that (2.2) does not hold. Then there exist a positive number ε and sequences $(L_n) \subset [L_0, L_1], (\lambda_n) \rightarrow \infty$ such that

$$\|u_{\lambda_n, L_n} - u\|_1 \geq \varepsilon$$

for all n . Since (u_{λ_n, L_n}) is bounded in $C^{1,\beta}(\bar{\Omega})$, there exist $v \in C^1(\bar{\Omega})$ and a subsequence of (u_{λ_n, L_n}) , which we still denote by (u_{λ_n, L_n}) , such that

$$\lim_{n \rightarrow \infty} \|u_{\lambda_n, L_n} - v\|_1 = 0.$$

From this and (2.6), we deduce that $u = v$, a contradiction. This completes the proof of Lemma 3.

For each $\lambda > 0$, define the operator A_λ on $C(\bar{\Omega})$ by $A_\lambda v = u$ if

$$\Delta_p u = -\lambda f(v) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where we define $f(x) = f(0)$ if $x < 0$. Then $A_\lambda : C(\bar{\Omega}) \rightarrow C(\bar{\Omega})$ is completely continuous, and fixed points of A_λ are solutions of (I).

The next lemma produces subsolutions to (I).

Lemma 4. *Let $0 < L_0 \leq L \leq L_1$, where L_1 is given by (A.1), and let $\phi_{\lambda,L} = (\lambda L)^{\frac{1}{p-1}}\phi$. Then there exists a positive number λ_0 independent of L such that for $\lambda \geq \lambda_0$,*

$$v \geq \phi_{\lambda,L} \Rightarrow A_\lambda v \geq \phi_{\lambda,L}.$$

Proof. Let $v \in C(\bar{\Omega})$ with $v \geq \phi_{\lambda,L}$, and let $u_\lambda = A_\lambda v$, $\tilde{u}_{\lambda,L} = \frac{u_\lambda}{(2\lambda L)^{\frac{1}{p-1}}}$. Then

$$v(x) \geq (\lambda L_0)^{\frac{1}{p-1}}\phi(x) > K \text{ if } \phi(x) > \frac{K}{(\lambda L_0)^{\frac{1}{p-1}}},$$

where K is given by (A.1). Define

$$h_{\lambda,L}(x) = \begin{cases} 1 & \text{if } \phi(x) > \frac{K}{(\lambda L_0)^{\frac{1}{p-1}}}, \\ \frac{M_0}{2L} & \text{if } \phi(x) \leq \frac{K}{(\lambda L_0)^{\frac{1}{p-1}}}, \end{cases}$$

where M_0 is a number such that $f(x) \geq M_0$ for all x .

By (A.1),

$$\Delta_p(\tilde{u}_{\lambda,L}) = -\frac{f(v)}{2L} < -h_{\lambda,L} \text{ in } \Omega,$$

and the maximum principle implies that $\tilde{u}_{\lambda,L} \geq \bar{u}_{\lambda,L}$, where $\bar{u}_{\lambda,L}$ is the solution of

$$\Delta_p(\bar{u}_{\lambda,L}) = -h_{\lambda,L} \text{ in } \Omega, \quad \bar{u}_{\lambda,L} = 0 \text{ on } \partial\Omega.$$

Note that

$$\|h_{\lambda,L} - 1\|_2 \leq \left(\frac{|M_0|}{2L} + 1\right) \left(\int_{\left\{x \in \Omega: \phi(x) \leq \frac{K}{(\lambda L_0)^{\frac{1}{p-1}}}\right\}} dx\right)^{\frac{1}{2}} \rightarrow 0$$

as $\lambda \rightarrow \infty$ uniformly for $L \in [L_0, L_1]$. Since $\Delta_p\phi = -1$ in Ω , it follows from Lemma 3 that

$$\lim_{\lambda \rightarrow \infty} |\bar{u}_{\lambda,L} - \phi|_1 = 0$$

uniformly for $L \in [L_0, L_1]$. Let $\gamma > 0$. Then by the mean value theorem

$$|\bar{u}_{\lambda,L}(x) - \phi(x)| \leq |\bar{u}_{\lambda,L} - \phi|_1 d(x, \partial\Omega) < \gamma c\phi(x), \quad x \in \Omega,$$

for λ large, where $c > 0$ is such that $d(x, \partial\Omega) \leq c\phi(x)$ for all $x \in \Omega$. Hence for such λ ,

$$\bar{u}_{\lambda,L}(x) \geq (1 - \gamma c)\phi(x), \quad x \in \Omega,$$

from which it follows that

$$u_\lambda(x) \geq (2\lambda L)^{\frac{1}{p-1}}\bar{u}_{\lambda,L}(x) \geq (\lambda L)^{\frac{1}{p-1}}\phi(x) = \phi_{\lambda,L}(x), \quad x \in \Omega,$$

if γ is sufficiently small. This completes the proof of Lemma 4.

We are now in a position to give the

Proof of Theorem 1. Let $\lambda > \lambda_0$, where λ_0 is given by Lemma 4.

Let $\bar{\phi}_\lambda = M_\lambda\phi$ where $M_\lambda \gg 1$. Let $v \in C(\bar{\Omega})$ satisfy $v \leq \bar{\phi}_\lambda$. We claim that $\tilde{u}_\lambda \equiv A_\lambda v \leq \bar{\phi}_\lambda$. Indeed, let $\tilde{f}(x) = \sup_{z \leq x} f(z)$. Then \tilde{f} is nondecreasing and

$$\Delta_p \tilde{u}_\lambda = -\lambda f(v) \geq -\lambda \tilde{f}(M_\lambda|\phi|_0) \geq -M_\lambda^{p-1} \text{ in } \Omega,$$

where we use the fact that $\lim_{x \rightarrow \infty} \frac{\tilde{f}(x)}{x^{p-1}} = 0$. Since $\Delta_p(\bar{\phi}_\lambda) = -M_\lambda^{p-1}$, the maximum principle implies $\tilde{u}_\lambda \leq \bar{\phi}_\lambda$, as claimed.

Thus $A_\lambda : [\phi_{\lambda,L}, \bar{\phi}_\lambda] \rightarrow [\phi_{\lambda,L}, \bar{\phi}_\lambda]$, where $\phi_{\lambda,L}$ is defined in Lemma 4. The Schauder fixed point theorem then gives a fixed point u of A_λ in $[\phi_{\lambda,L}, \bar{\phi}_\lambda]$, which is a positive solution of (I). This completes the proof of the first part of Theorem 1. Suppose next that $f \geq 0$ and $\lim_{x \rightarrow 0^+} \frac{f(x)}{x^{p-1}} = 0$. Let $\phi_\varepsilon = \varepsilon\phi$. We claim that for $\varepsilon > 0$ small,

$$v \leq \phi_\varepsilon \Rightarrow u_\lambda \equiv A_\lambda v \leq \phi_\varepsilon.$$

Indeed, we have

$$\Delta_p u_\lambda = -\lambda f(v) \text{ in } \Omega,$$

and for $\varepsilon > 0$ small,

$$\lambda f(v) \leq \lambda \tilde{f}(v) \leq \lambda \tilde{f}(\varepsilon|\phi|_0) \leq \varepsilon^{p-1} = -\Delta_p \phi_\varepsilon \text{ in } \Omega,$$

where $\tilde{f}(x) = \sup_{z \leq x} f(z)$. Here we have used the fact that $\lim_{x \rightarrow \infty} \frac{\tilde{f}(x)}{x^{p-1}} = 0$. By the maximum principle, $u_\lambda \leq \phi_\varepsilon$, as claimed. By decreasing ε if necessary, we assume $\phi_\varepsilon < \phi_{\lambda,L_0}$ in Ω . Define

$$X = C_0^1(\bar{\Omega}), X_1 = [0, \phi_\varepsilon], X_2 = [\phi_{\lambda,L_0}, \bar{\phi}_\lambda], \text{ and } X_3 = [0, \bar{\phi}_\lambda],$$

where $[z, w] = \{u \in X : z \leq u \leq w\}$. Then $A_\lambda : X_k \rightarrow X_k$ and A_λ has a fixed point \tilde{u}_2 in X_2 . By the maximum principle [11], every nontrivial fixed point of A_λ in X_3 is a positive solution of (I). Since 0 is an interior point of X_1 with respect to X_3 , if A_λ has a fixed point u_1 on the boundary of X_1 in X_3 , then $0 < u_1 < \tilde{u}_2$ in Ω and thus (I) has two distinct positive solutions. Since $A_\lambda : [\phi_{\lambda,L_1}, \bar{\phi}_\lambda] \rightarrow [\phi_{\lambda,L_1}, \bar{\phi}_\lambda]$, A_λ has a fixed point \tilde{u} in $[\phi_{\lambda,L_1}, \bar{\phi}_\lambda]$. Note that every point in $[\phi_{\lambda,L_1}, \bar{\phi}_\lambda]$ is an interior point of X_2 in X_3 . If A_λ has a fixed point u_2 on the boundary of X_2 in X_3 , then $u_2 \neq \tilde{u}$ and we have two solutions. Finally, if A_λ does not have any fixed points on the boundaries of X_1 and X_2 in X_3 , then by a result of Amann [1], there exists a fixed point \bar{u} of A_λ such that $\bar{u} \in X_3 \setminus (X_1 \cup X_2)$. In particular, \bar{u} is a second positive solution of (I). This completes the proof of Theorem 1.

Remark 1. (i) Conditions $\lim_{x \rightarrow \infty} \frac{f(x)}{x^{p-1}} = 0$ and $\lim_{x \rightarrow 0^+} \frac{f(x)}{x^{p-1}} = 0$ in Theorem 1 can be replaced by $\liminf_{x \rightarrow \infty} \frac{f(x)}{x^{p-1}} = 0$ and $\liminf_{x \rightarrow 0^+} \frac{f(x)}{x^{p-1}} = 0$ respectively, provided that f is nondecreasing.

(ii) The multiplicity result in Theorem 1 was established in [4] under additional assumptions that $f \in C^1(R^+)$ is strictly increasing on R^+ , and there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \leq \alpha_1 + \alpha_2 s^\mu$ for some $\mu \in (0, p - 1)$.

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