COMPACT EINSTEIN WARPED PRODUCT SPACES WITH NONPOSITIVE SCALAR CURVATURE

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Dedicated to Professor Bang-yen Chen on the occasion of his sixtieth birthday

Abstract. We study Einstein warped product spaces. As a result, we prove the following: if $M$ is an Einstein warped product space with nonpositive scalar curvature and compact base, then $M$ is simply a Riemannian product space.

0. Introduction

Let $B = (B^m, g_B)$ and $F = (F^k, g_F)$ be two Riemannian manifolds. We denote by $\pi$ and $\sigma$ the projections of $B \times F$ onto $B$ and $F$, respectively. For a positive smooth function $f$ on $B$ the warped product $M = B \times_f F$ is the product $M = B \times F$ furnished with the metric tensor $g$ defined by $g = \pi^* g_B + f^2 \sigma^* g_F$, where $*$ denotes the pull back. The function $f$ is referred to as the warping function. The notion of warped product $B \times_f F$ generalizes that of a surface of revolution. It was introduced in \cite{3} for studying manifolds of negative curvature.

A Riemannian manifold $M$ is called Einstein if its Ricci tensor Ric is proportional to the metric, that is, $\text{Ric} = \lambda g$, where $\lambda$ is a constant on $M$. Obviously the Riemannian product $M = B \times F$ is Einstein if $B$ and $F$ are Einstein with the same scalar curvature. A warped product $B \times_f F$ with a constant warping function $f$ can be considered as a Riemannian product.

In search of a new compact Einstein space in \cite{2} (p. 265), A. L. Besse asked the following:

"Does there exist a compact Einstein warped product with nonconstant warping function?"

In this article, we give a negative partial answer as follows (cf. \cite{1}):

Theorem 1. Let $M = B \times_f F$ be an Einstein warped product space with base $B$ a compact space. If $M$ has nonpositive scalar curvature, then the warped product is simply a Riemannian product.
1. Proofs

We denote by $\text{Ric}^B, \text{Ric}^F$ the lifts to $M$ of the Ricci curvatures of $B$ and $F$, respectively. Then we have the following (\cite{6}):

**Proposition 2.** The Ricci curvature $\text{Ric}$ of the warped product $M = B \times_f F$ with $k = \dim F$ satisfies

1. $\text{Ric}(X,Y) = \text{Ric}^B(X,Y) - \frac{k}{f}H^f(X,Y)$,
2. $\text{Ric}(X,V) = 0$,
3. $\text{Ric}(V,W) = \text{Ric}^F(V,W) - g(V,W)f^#$, $f^# = \frac{-\Delta f}{f} + \frac{k-1}{2f}g_B(\nabla f, \nabla f)$ for any horizontal vectors $X,Y$ and any vertical vectors $V,W$, where $H^f$ and $\Delta f$ denote the Hessian of $f$ and the Laplacian of $f$ given by $-\text{tr}(H^f)$, respectively.

Hence the Einstein equations become

**Corollary 3.** The warped product $M = B \times_f F$ is Einstein with $\text{Ric} = \lambda g$ if and only if

1. $\text{Ric}^B = \lambda g_B + \frac{k}{f}H^f$,
2. $(F,g_F)$ is Einstein with $\text{Ric}^F = \mu g_F$,
3. $-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu$.

Now we prove a lemma.

**Lemma 4.** Let $f$ be a smooth function on a Riemannian manifold $B$. Then for any vector $X$, the divergence of the Hessian tensor $H^f$ satisfies

$$\text{div}(H^f)(X) = \text{Ric}(\nabla f, X) - \Delta(df)(X),$$

where $\Delta = d\delta + \delta d$ denotes the Laplacian on $B$ acting on differential forms.

**Proof.** The well-known Ricci identity implies (cf. \cite{4}, p. 159)

$$D^2df(X,Y,Z) - D^2df(Y,X,Z) = df(R_{XY}Z)$$

for all vector fields $X,Y,$ and $Z$ where $D^2_{XY} = D_XD_Y - D_{[X,Y]}$ denotes the second order covariant differential operator and $R_{XY} = -D_XD_Y + D_YD_X + D_{[X,Y]}$ is the curvature tensor acting on tensors as a derivation. Since $df$ is closed, it is easily proved that

$$D^2df(X,Y,Z) = D^2df(X,Z,Y)$$

for any vector fields $X,Y,$ and $Z$.

For a fixed $p \in B$ we may choose a local orthonormal frame $E_1, E_2, \cdots, E_m$ of the space $B$ such that $D_{E_i}E_j(p) = 0$ for all $i,j$. Also, we may assume $D_{E_i}Y(p) = 0$ for a vector field $Y$. Taking the trace with respect to $X$ and $Z$ in (1.5) and using (1.6), we have

$$\sum_i (D^2df)(E_i, E_i, Y) = -d\Delta f(Y) + \text{Ric}(Y, \nabla f)$$

at $p$. Since $\text{div}H^f(Y) = \sum_i (D^2df)(E_i, E_i, Y)$ is straightforward, (1.4) is proved. \hfill $\Box$

**Proposition 5.** Let $(B^m, g_B)$ be a compact Riemannian manifold of dimension $m \geq 2$. Suppose that $f$ is a nonconstant smooth function on $B$ satisfying (1.1) for a constant $\lambda \in R$ and a natural number $k \in N$. Then $f$ satisfies (1.3) for a constant $\mu \in R$. Hence for a compact Einstein space $(F,g_F)$ of dimension $k$ with $\text{Ric}^F = \lambda g_F$...
\( \mu g_F \), we can make a compact Einstein warped product space \( M = B \times_f F \) with \( \text{Ric} = \lambda g \).

**Proof.** By taking the trace of both sides of (1.1), we have

\[
S = m\lambda - \frac{k}{f}\Delta f,
\]

where \( S \) denotes scalar curvature of \( B \) given by \( \text{tr}(\text{Ric}) \). Note that the second Bianchi identity implies (1.8)

\[
dS = 2 \text{div}(\text{Ric}).
\]

From (1.7) and (1.8), we obtain

\[
\text{div}(\frac{1}{f} H^f)(X) = \frac{1}{2f^2}((k - 1)d(|\nabla f|^2) - 2f\Delta f + 2\lambda f)\}
\]

for any vector field \( X \) on \( B \). Hence, from (1.1) and (1.4) it follows that

\[
\text{div}(\frac{1}{f} H^f)(X) = \frac{1}{2f^2}((k - 1)d(|\nabla f|^2) - 2f\Delta f + 2\lambda f).
\]

But, (1.1) gives \( \text{div}\text{Ric} = \text{div}(\frac{k}{f} H^f) \). Therefore, (1.9) and (1.10) imply that \( d(-f\Delta f + (k - 1)|\nabla f|^2 + \lambda f^2) = 0 \), that is, \( -f\Delta f + (k - 1)|\nabla f|^2 + \lambda f^2 = \mu \) for some constant \( \mu \). Thus the first part of the proposition is proved. For a compact Einstein manifold \( (F,g_F) \) of dimension \( k \) with \( \text{Ric}_F = \mu g_F \), we can construct a compact Einstein warped product \( M = B \times_f F \) by the sufficiencies of Corollary 3.

Now we give the proof of Theorem 1. Note that (1.3) becomes

\[
\text{div}(f\nabla f) + (k - 2)|\nabla f|^2 + \lambda f^2 = \mu.
\]

By integrating (1.11) over \( B \) we have

\[
\mu = \frac{k - 2}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B f^2,
\]

where \( V(B) \) denotes the volume of \( B \).

1) Suppose \( k \geq 3 \). Let \( p \) be a maximum point of \( f \) on \( B \). Then, we have \( f(p) > 0, \nabla f(p) = 0 \) and \( \Delta f(p) \geq 0 \). Hence from (1.3) and (1.12) we obtain the following:

\[
0 \leq f(p)\Delta f(p)
= \lambda f(p)^2 - \mu
= \frac{2 - k}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B (f(p)^2 - f^2)
\leq 0.
\]
The last inequality follows from the hypothesis on $\lambda$. Thus, $f$ is constant.

2) When $k = 1, 2$, we choose $q$ as a minimum point of $f$ on $B$. Then, we have $f(q) > 0$, $\nabla f(q) = 0$ and $\Delta f(q) \leq 0$. Hence we obtain from (1.3) and (1.12)

$$0 \geq f(q)\Delta f(q)$$
$$= \lambda f(q)^2 - \mu$$
$$= \frac{2-k}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B (f(q)^2 - f^2)$$
$$\geq 0.$$ (1.13)

As in case 1), the last inequality follows from the hypothesis on $\lambda$. If $k = 1$ or $\lambda < 0$, then (1.13) shows that $f$ is constant. If $k = 2$ and $\lambda = 0$, (1.11) and (1.12) imply that $f^2$ is harmonic on $B$, and hence $f$ is constant. This completes the proof of the theorem.

In a similar manner, we may prove the following (cf. [1]):

**Remark 6.** Let $(M, g)$ be a compact Riemannian manifold. If the Ricci tensor satisfies $\text{Ric} = \lambda g + Hf$ for a nonpositive constant $\lambda \in \mathbb{R}$ and a smooth function $f$ on $M$, then $f$ is constant.

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**References**


