

## COMPACT EINSTEIN WARPED PRODUCT SPACES WITH NONPOSITIVE SCALAR CURVATURE

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*Dedicated to Professor Bang-yen Chen on the occasion of his sixtieth birthday*

ABSTRACT. We study Einstein warped product spaces. As a result, we prove the following: if  $M$  is an Einstein warped product space with nonpositive scalar curvature and compact base, then  $M$  is simply a Riemannian product space.

### 0. INTRODUCTION

Let  $B = (B^m, g_B)$  and  $F = (F^k, g_F)$  be two Riemannian manifolds. We denote by  $\pi$  and  $\sigma$  the projections of  $B \times F$  onto  $B$  and  $F$ , respectively. For a positive smooth function  $f$  on  $B$  the warped product  $M = B \times_f F$  is the product  $M = B \times F$  furnished with the metric tensor  $g$  defined by  $g = \pi^*g_B + f^2\sigma^*g_F$ , where  $*$  denotes the pull back. The function  $f$  is referred to as the warping function. The notion of warped product  $B \times_f F$  generalizes that of a surface of revolution. It was introduced in [3] for studying manifolds of negative curvature.

A Riemannian manifold  $M$  is called Einstein if its Ricci tensor Ric is proportional to the metric  $g$ , that is,  $\text{Ric} = \lambda g$ , where  $\lambda$  is a constant on  $M$ . Obviously the Riemannian product  $M = B \times F$  is Einstein if  $B$  and  $F$  are Einstein with the same scalar curvature. A warped product  $B \times_f F$  with a constant warping function  $f$  can be considered as a Riemannian product.

In search of a new compact Einstein space in [2] (p. 265), A. L. Besse asked the following:

“Does there exist a compact Einstein warped product with nonconstant warping function?”

In this article, we give a negative partial answer as follows (cf. [1]):

**Theorem 1.** *Let  $M = B \times_f F$  be an Einstein warped product space with base  $B$  a compact space. If  $M$  has nonpositive scalar curvature, then the warped product is simply a Riemannian product.*

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1. PROOFS

We denote by  $\text{Ric}^B, \text{Ric}^F$  the lifts to  $M$  of the Ricci curvatures of  $B$  and  $F$ , respectively. Then we have the following ([6]):

**Proposition 2.** *The Ricci curvature Ric of the warped product  $M = B \times_f F$  with  $k = \dim F$  satisfies*

- (1)  $\text{Ric}(X, Y) = \text{Ric}^B(X, Y) - \frac{k}{f}H^f(X, Y)$ ,
- (2)  $\text{Ric}(X, V) = 0$ ,
- (3)  $\text{Ric}(V, W) = \text{Ric}^F(V, W) - g(V, W)f^\#, f^\# = \frac{-\Delta f}{f} + \frac{k-1}{f^2}g_B(\nabla f, \nabla f)$  for any horizontal vectors  $X, Y$  and any vertical vectors  $V, W$ , where  $H^f$  and  $\Delta f$  denote the Hessian of  $f$  and the Laplacian of  $f$  given by  $-\text{tr}(H^f)$ , respectively.

Hence the Einstein equations become

**Corollary 3.** *The warped product  $M = B \times_f F$  is Einstein with  $\text{Ric} = \lambda g$  if and only if*

- (1.1)  $\text{Ric}_B = \lambda g_B + \frac{k}{f}H^f$ ,
- (1.2)  $(F, g_F)$  is Einstein with  $\text{Ric}_F = \mu g_F$ ,
- (1.3)  $-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu$ .

Now we prove a lemma.

**Lemma 4.** *Let  $f$  be a smooth function on a Riemannian manifold  $B$ . Then for any vector  $X$ , the divergence of the Hessian tensor  $H^f$  satisfies*

$$(1.4) \quad \text{div}(H^f)(X) = \text{Ric}(\nabla f, X) - \Delta(df)(X),$$

where  $\Delta = d\delta + \delta d$  denotes the Laplacian on  $B$  acting on differential forms.

*Proof.* The well-known Ricci identity implies (cf. [5], p. 159)

$$(1.5) \quad D^2df(X, Y, Z) - D^2df(Y, X, Z) = df(R_{XY}Z)$$

for all vector fields  $X, Y$ , and  $Z$  where  $D^2_{XY} = D_X D_Y - D_{D_X Y}$  denotes the second order covariant differential operator and  $R_{XY} = -D_X D_Y + D_Y D_X + D_{[X, Y]}$  is the curvature tensor acting on tensors as a derivation. Since  $df$  is closed, it is easily proved that

$$(1.6) \quad D^2df(X, Y, Z) = D^2df(X, Z, Y)$$

for any vector fields  $X, Y$  and  $Z$ .

For a fixed  $p \in B$  we may choose a local orthonormal frame  $E_1, E_2, \dots, E_m$  of the space  $B$  such that  $D_{E_i} E_j(p) = 0$  for all  $i, j$ . Also, we may assume  $D_{E_i} Y(p) = 0$  for a vector field  $Y$ . Taking the trace with respect to  $X$  and  $Z$  in (1.5) and using (1.6), we have

$$\sum_i (D^2df)(E_i, E_i, Y) = -d\Delta f(Y) + \text{Ric}(Y, \nabla f)$$

at  $p$ . Since  $\text{div}H^f(Y) = \sum_i (D^2df)(E_i, E_i, Y)$  is straightforward, (1.4) is proved. □

**Proposition 5.** *Let  $(B^m, g_B)$  be a compact Riemannian manifold of dimension  $m \geq 2$ . Suppose that  $f$  is a nonconstant smooth function on  $B$  satisfying (1.1) for a constant  $\lambda \in \mathbb{R}$  and a natural number  $k \in \mathbb{N}$ . Then  $f$  satisfies (1.3) for a constant  $\mu \in \mathbb{R}$ . Hence for a compact Einstein space  $(F, g_F)$  of dimension  $k$  with  $\text{Ric}_F =$*

$\mu g_F$ , we can make a compact Einstein warped product space  $M = B \times_f F$  with  $\text{Ric} = \lambda g$ .

*Proof.* By taking the trace of both sides of (1.1), we have

$$(1.7) \quad S = m\lambda - \frac{k}{f}\Delta f,$$

where  $S$  denotes scalar curvature of  $B$  given by  $\text{tr}(\text{Ric})$ . Note that the second Bianchi identity implies ([6], p. 88)

$$(1.8) \quad dS = 2\text{div}(\text{Ric}).$$

From (1.7) and (1.8), we obtain

$$(1.9) \quad \text{div Ric}(X) = \frac{k}{2f^2}\{\Delta f df - fd(\Delta f)\}(X).$$

On the other hand, by definition we have

$$\text{div}\left(\frac{1}{f}H^f\right)(X) = \sum_i (D_{E_i}\left(\frac{1}{f}H^f\right))(E_i, X) = -\frac{1}{f^2}H^f(\nabla f, X) + \frac{1}{f}\text{div}H^f(X)$$

for any vector field  $X$  and an orthonormal frame  $E_1, E_2, \dots, E_m$  of  $B$ . Since  $H^f(X, \nabla f) = (D_X df)(\nabla f) = \frac{1}{2}d(|\nabla f|^2)(X)$ , the last equation becomes

$$\text{div}\left(\frac{1}{f}H^f\right)(X) = -\frac{1}{2f^2}d(|\nabla f|^2)(X) + \frac{1}{f}\text{div}H^f(X)$$

for a vector field  $X$  on  $B$ . Hence, from (1.1) and (1.4) it follows that

$$(1.10) \quad \text{div}\left(\frac{1}{f}H^f\right) = \frac{1}{2f^2}\{(k-1)d(|\nabla f|^2) - 2fd(\Delta f) + 2\lambda f df\}.$$

But, (1.1) gives  $\text{div Ric} = \text{div}\left(\frac{k}{f}H^f\right)$ . Therefore, (1.9) and (1.10) imply that  $d(-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2) = 0$ , that is,  $-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu$  for some constant  $\mu$ . Thus the first part of the proposition is proved. For a compact Einstein manifold  $(F, g_F)$  of dimension  $k$  with  $\text{Ric}_F = \mu g_F$ , we can construct a compact Einstein warped product  $M = B \times_f F$  by the sufficiencies of Corollary 3.  $\square$

Now we give the proof of Theorem 1. Note that (1.3) becomes

$$(1.11) \quad \text{div}(f\nabla f) + (k-2)|\nabla f|^2 + \lambda f^2 = \mu.$$

By integrating (1.11) over  $B$  we have

$$(1.12) \quad \mu = \frac{k-2}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B f^2,$$

where  $V(B)$  denotes the volume of  $B$ .

1) Suppose  $k \geq 3$ . Let  $p$  be a maximum point of  $f$  on  $B$ . Then, we have  $f(p) > 0, \nabla f(p) = 0$  and  $\Delta f(p) \geq 0$ . Hence from (1.3) and (1.12) we obtain the following:

$$\begin{aligned} 0 &\leq f(p)\Delta f(p) \\ &= \lambda f(p)^2 - \mu \\ &= \frac{2-k}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B (f(p)^2 - f^2) \\ &\leq 0. \end{aligned}$$

The last inequality follows from the hypothesis on  $\lambda$ . Thus,  $f$  is constant.

2) When  $k = 1, 2$ , we choose  $q$  as a minimum point of  $f$  on  $B$ . Then, we have  $f(q) > 0, \nabla f(q) = 0$  and  $\Delta f(q) \leq 0$ . Hence we obtain from (1.3) and (1.12)

$$\begin{aligned}
 (1.13) \quad 0 &\geq f(q)\Delta f(q) \\
 &= \lambda f(q)^2 - \mu \\
 &= \frac{2-k}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B (f(q)^2 - f^2) \\
 &\geq 0.
 \end{aligned}$$

As in case 1), the last inequality follows from the hypothesis on  $\lambda$ . If  $k = 1$  or  $\lambda < 0$ , then (1.13) shows that  $f$  is constant. If  $k = 2$  and  $\lambda = 0$ , (1.11) and (1.12) imply that  $f^2$  is harmonic on  $B$ , and hence  $f$  is constant. This completes the proof of the theorem.

In a similar manner, we may prove the following (cf. [4]):

*Remark 6.* Let  $(M, g)$  be a compact Riemannian manifold. If the Ricci tensor satisfies  $\text{Ric} = \lambda g + H^f$  for a nonpositive constant  $\lambda \in \mathbb{R}$  and a smooth function  $f$  on  $M$ , then  $f$  is constant.

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