

ON THE FUNDAMENTAL GROUP OF MANIFOLDS WITH ALMOST NONNEGATIVE RICCI CURVATURE

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ABSTRACT. Gromov conjectured that the fundamental group of a manifold with almost nonnegative Ricci curvature is almost nilpotent. This conjecture is proved under the additional assumption on the conjugate radius. We show that there exists a nilpotent subgroup of finite index depending on a lower bound of the conjugate radius.

1. INTRODUCTION

Gromov conjectured that if a manifold has almost nonnegative Ricci curvature, then the fundamental group is almost nilpotent. Fukaya and Yamaguchi proved this conjecture when the manifold has almost nonnegative sectional curvature [FY]. Recall that a group is *almost nilpotent* if and only if it has a nilpotent subgroup of finite index. In [FY], they conjectured that if M is an almost nonnegatively curved manifold, then there exists a nilpotent subgroup Γ of $\pi_1(M)$ such that $[\pi_1(M) : \Gamma] \leq \omega(n)$, where $\omega(n)$ is a constant depending only on n .

For a manifold with almost nonnegative Ricci curvature, if its conjugate radius is bounded below, we have weak coordinates which have a bounded weak $C^{0,\alpha}$ -norm [PWY]. Using the smoothing procedure of Petersen, Wei and Ye [PWY], we can apply the fibration theorem of Fukaya [F]. Then the splitting theorem of Cheeger and Colding [CC] implies Gromov's conjecture immediately under the assumption on the conjugate radius.

We have the following result:

Theorem 1.1. *Let M be an n -dimensional compact Riemannian manifold with $\text{diam}(M) \leq 1$ and $\text{conj}_M \geq c_0 > 0$. Then there exist an $\epsilon > 0$ and an $\omega > 0$ both depending only on n and c_0 such that if $\text{Ric}_M \geq -(n-1)\epsilon^2$, there exists a nilpotent subgroup Λ of $\pi_1(M)$ satisfying $[\pi_1(M) : \Lambda] \leq \omega$.*

We briefly review the definition of weak coordinate and weak norm. The (weak) norm of an n -dimensional Riemannian manifold (M, g) on scale $r > 0$ is defined in [PWY] as follows, where g is the metric of M .

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Definition 1.2. The $C^{k,\alpha}$ -norm of an n -dimensional Riemannian manifold (M, g) on scale $r > 0$, denoted by $\|(M, g)\|_{C^{k,\alpha},r}$, is defined to be the infimum of positive numbers Q such that there exist imbeddings

$$\phi_\tau : B(0, r) \subset \mathbb{R}^n \rightarrow U_\tau \subset M$$

with images U_τ , $\tau \in I$ (an index set), with \rightarrow having the following properties:

- 1) $e^{-2Q}\delta \leq \phi_\tau^*(g) \leq e^{2Q}\delta$, where δ is the Euclidean metric,
- 2) every metric ball $B(p, \frac{r}{10}e^{-Q})$ for $p \in M$ lies in some U_τ ,
- 3) $r^{|l|+\alpha} \|\partial^l g_{\tau,ij}\|_{C^{0,\alpha}} \leq Q$ for all multi-indices l with $0 \leq |l| \leq k$, where $g_\tau = \phi_\tau^*g$.

The weak norm $\|(M, g)\|_{C^{k,\alpha},r}^W$ is identically defined except that ϕ_τ is assumed to be a local diffeomorphism instead of a diffeomorphism. Then we may regard ϕ_τ as a *weak* coordinate chart. If we consider only the case that weak coordinate charts are harmonic, we call such a norm a *weak harmonic norm*. In [PWY], they generalized almost flat manifolds using the bounded weak $C^{0,\alpha}$ -harmonic norm. We also have a generalized almost flat manifold using the bounded weak $C^{0,\alpha}$ -norm (without harmonicity assumption) [Pa]. For a positive function $Q(r)$ satisfying $Q(r) \rightarrow 0$ as $r \rightarrow 0$ and $\alpha > 0$, we define the following class of n -dimensional complete Riemannian manifolds without any assumption on the harmonicity:

$$\mathcal{M}(n, \alpha, Q) = \{(M, g) \mid \|(M, g)\|_{C^{0,\alpha},r}^W \leq Q(r) \text{ for } 0 < r \leq 1\}.$$

Theorem 1.3 ([Pa]). *There exists an $\epsilon(n, \alpha, Q) > 0$ depending on n, α and Q such that if $M \in \mathcal{M}(n, \alpha, Q)$ and $\text{diam}(M) \leq \epsilon(n, \alpha, Q)$, then a finite covering space of M is a nilmanifold.*

As an immediate corollary, we obtain that

Corollary 1.4. *Let M be an n -dimensional compact Riemannian manifold with $\text{Ric}_M \geq -1$ and $\text{conj}_M \geq c_0$. Then there exists an $\epsilon(n, c_0) > 0$ such that if $\text{diam}(M) \leq \epsilon$, then M is diffeomorphic to a nilmanifold up to finite cover.*

Using the smoothness of small balls as we see in Corollary 1.4, we obtain Theorem 1.1.

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2. SPLITTING THEOREM AND SOLVABILITY OF $\pi_1(M)$

In [FY], the authors proved the following generalized Bieberbach theorem, which is useful to find a nilpotent subgroup and a bound on the index of the nilpotent subgroup:

Theorem 2.1 ([FY]). *Let G be a closed subgroup of the group of isometries of \mathbb{R}^n . Then $\pi_0(G)$ contains a finite index, free Abelian subgroup whose rank is not greater than $\dim(\mathbb{R}^n/G)$.*

Corollary 2.2 ([FY]). *Suppose, in addition, that the quotient space \mathbb{R}^n/G is compact. Then there exists a normal subgroup G' of G such that*

- (1) $[G : G'] < w_n$, where w_n is a number depending only on n ,
- (2) \mathbb{R}^n/G' is isometric to a flat torus.

Let M_i be a sequence of compact Riemannian manifolds with $\text{Ric}_{M_i} \geq -(n-1)\epsilon^2 \rightarrow 0$, $\text{conj}_M \geq c_0$ and $\text{diam}(M) \leq 1$. We denote $\pi_1(M_i)$ by Π_i . From [CC], we have the following convergence with respect to the pointed Gromov-Hausdorff distance:

$$(\tilde{M}_i, \Pi_i) \rightarrow (\mathbb{R}^k \times X, \Pi)$$

for some compact length space X and an isometry group Π of X . Also $\gamma \in \Pi$ is of the form $(\gamma_1, \gamma_2)(x, y) = (\gamma_1(x), \gamma_2(y))$, where γ_1 and γ_2 are isometries on \mathbb{R}^k and X , respectively [CG].

Now we review the construction of solvable subgroups with a bounded index. From Corollary 2.2, if we consider a finite ($\leq w(k)$) covering space of M_i , we may assume that M_i converges to a fiber bundle X_1 over T^{k_1} , where $k_1 \leq k$ and k_1 is the rank of the maximal discrete Abelian subgroup of Π . So we may assume that $M_i \rightarrow X_1$. Then γ_1 can be decomposed to $(\gamma_{11}, \gamma_{12})$, where γ_{11} and γ_{12} are isometries on \mathbb{R}^{k_1} and \mathbb{R}^{k-k_1} , respectively. We may consider γ_{11} as an element of $\pi_1(T^{k_1})$. Let $p_1(\gamma)$ be γ_{11} and $p_2(\gamma)$ be (γ_{12}, γ_2) . We denote $\text{Ker}(p_1)$ by K . Then K is normal subgroup of Π and $\Pi/K \simeq \mathbb{Z}^{k_1}$. Using Theorem 3.10 in [FY], we can find a normal subgroup K_i of Π_i such that

$$(\tilde{M}_i, K_i) \rightarrow (\mathbb{R}^k \times X, K)$$

and

$$\Pi_i/K_i \simeq \Pi/K = \mathbb{Z}^{k_1}.$$

Using Theorem 1.3 or Corollary 1.4 from the previous section, we can find $\delta(n, c_0) > 0$ such that $\pi_1(\phi_i(B(0, \delta)))$ is almost nilpotent, so there exists a nilpotent subgroup \tilde{H}_i such that

$$[\pi_1(\phi_i(B(0, \delta))) : \tilde{H}_i] \leq \kappa(n)$$

for some constant κ depending only on n , where ϕ_i is a weak coordinate of M_i (see [G] and [Pa]). Let $i : \phi_i(B(0, \delta)) \rightarrow M_i$ be the inclusion map. We denote $i_*(\tilde{H}_i)$ by H_i . Let

$$(\tilde{M}_i, H_i) \rightarrow (\mathbb{R}^k \times X, H)$$

for some isometry group H of $\mathbb{R}^k \times X$. If we take $\delta > 0$ sufficiently small, we may assume that $p_1(H) = \text{Id}$, so $H \subset K$ and $H_i \subset i_*(\pi_1(\phi_i(B(0, \delta)))) \subset K_i$. From Definition 1.2, $\phi_i(B(0, \delta))$ contains an ϵ -ball for a constant $\epsilon(n, c_0) > 0$ depending on n and c_0 .

Using the Bishop-Gromov comparison theorem, we obtain that for any $x \in \tilde{M}_i$,

$$\frac{\text{vol}(B(x, 20))}{\text{vol}(B(x, \epsilon/100))} \leq C_1(n, c_0)$$

for some constant $C_1(n, c_0) > 0$ depending only on n and c_0 . (In fact, C_1 depends on ϵ but ϵ is determined by c_0 .) For $y \in B(x, 10)$, we have

$$\frac{\text{vol}(B(x, 10))}{\text{vol}(B(y, \epsilon/100))} \leq \frac{\text{vol}(B(y, 20))}{\text{vol}(B(y, \epsilon/100))} \leq C_1(n, c_0),$$

so the maximal number of ϵ -balls disjointly contained in a 10-ball is bounded by C_1 . For convenience, if we assume that $\text{diam}(X) \leq 20$,

$$X = \bigcup_{i=1}^N B(y_i, \epsilon/10)$$

for $N \leq C_1$. Then we obtain that

$$(2.1) \quad [K_i : H_i] = [K_i : i_*(\pi_1(\phi_i(B(0, \delta))))][i_*(\pi_1(\phi_i(B(0, \delta)))) : H_i] \leq \kappa(n)C_1(n, c_0).$$

We choose $\{\alpha_1^{(i)}, \dots, \alpha_{k_1}^{(i)}\} \subset \Pi_i$ such that $\alpha_j^{(i)} \rightarrow \alpha_j \in \Pi$ as $i \rightarrow \infty$ and $\{p_1(\alpha_j)\}$ is a basis of $\pi_1(T^{k_1})$, where T^{k_1} is mentioned above. Let $A(\alpha)$ be an automorphism on K_i defined by

$$A(\alpha)(\gamma) = \alpha\gamma\alpha^{-1}.$$

From (2.1), we can find $l \leq (\kappa(n)C_1(n, c_0))^{k_1}$ such that

$$(2.2) \quad A(\alpha_j^{(i)})^l(H_i) = (\alpha_j^{(i)})^l H_i (\alpha_j^{(i)})^{-l} = H_i.$$

We denote $(\alpha_j^{(i)})^l$ by $\beta_j^{(i)}$. Let Γ_i be the subgroup generated by K_i and $\{\beta_j^{(i)} \mid j = 1, \dots, k_1\}$. From (2.2), H_i is a normal subgroup of Γ_i and

$$[\Pi_i : \Gamma_i] = [K_i : H_i][\langle \alpha_1^{(i)}, \dots, \alpha_{k_1}^{(i)} \rangle : \langle \beta_1^{(i)}, \dots, \beta_{k_1}^{(i)} \rangle] \leq l^{k_1+1} \leq (\kappa(n)C_1(n, c_0))^{2n^2}.$$

We have

$$1 \rightarrow K_i \rightarrow \Gamma_i \rightarrow \mathbb{Z}^{k_1} \rightarrow 1$$

and

$$1 \rightarrow K_i/H_i \rightarrow \Gamma_i/H_i \rightarrow \mathbb{Z}^{k_1} \rightarrow 1.$$

Applying Lemma 4.4 in [FY], we obtain that

$$1 \rightarrow \mathbb{Z}^{k_1} \rightarrow \Gamma_i/H_i \rightarrow \Omega \rightarrow 1$$

for some finite group Ω such that $|\Omega| \leq C_2(n)$ for some constant C_2 depending only on n . Hence we can find a solvable subgroup Γ'_i such that

$$\Gamma'_i/H_i \simeq \mathbb{Z}^{k_1}$$

and

$$[\Pi_i : \Gamma'_i] = [\Pi_i : \Gamma_i][\Gamma_i : \Gamma'_i] \leq C_3(n) := (\kappa(n)C_1(n, c_0))^{2n^2} C_2(n) < \infty.$$

From this, the boundedness of indices of solvable subgroups of $\pi_1(M)$ is obtained.

3. PROOF OF THEOREM 1.1

We will find nilpotent subgroups $\Lambda_i \subset \Gamma'_i$ such that $[\Pi_i : \Lambda_i] < \infty$. Since $\tilde{M}_i \rightarrow \mathbb{R}^k \times X$, we have a pointed Hausdorff approximation

$$f_i : B(o_i, 1/\epsilon_i) \subset \tilde{M}_i \rightarrow \mathbb{R}^k \times X,$$

as $\epsilon_i \rightarrow 0$. For the precise definition of Hausdorff approximation, see [FY]. Assume that $f_i(o_i)$ converges to $o = (o_1, o_2) \in \mathbb{R}^k \times X$. Let N be $\{(x, o_2) \mid x \in \mathbb{R}^k\} \subset \mathbb{R}^k \times X$ and N_i be $f_i^{-1}(N)$.

As in the previous section, we have a quotient map $q_i : \Gamma'_i \rightarrow \mathbb{Z}^{k_1}$. We take a subset $B_i = \{\gamma_1^{(i)}, \dots, \gamma_{k_1}^{(i)}\}$ of Γ'_i such that $q_i(B_i)$ is a basis of \mathbb{Z}^{k_1} . Since $\gamma_j^{(i)}$

converges to an isometry $\gamma_j = (p_1(\gamma_j), p_2(\gamma_j))$ of the $\mathbb{R}^k \times X$, there exist $l_j \leq C_1$ such that

$$d(\gamma_j^{l_j}(o), N) \leq \epsilon,$$

so

$$d((\gamma_j^{(i)})^{l_j}(o_i), N_i) \leq 2\epsilon$$

for C_1 and ϵ in section 2. We denote $\gamma_j^{l_j}$ and $(\gamma_j^{(i)})^{l_j}$ by γ'_j and $\gamma'^{(i)}_j$, respectively. Let x_j be $p_1(\gamma'_j(o))$. We take a point $x_j^{(i)}$ in $f_i^{-1}(x_j)$. (It is possible that $f_i^{-1}(x_j) = \emptyset$. But we can find a sequence $x_{j,h} \rightarrow x_j$ as $h \rightarrow \infty$ in $\mathbb{R}^k \times X$ such that $x_{j,h} \in \text{Im}(f_i)$. So we may assume its existence.)

In Definition 1.2, a weak coordinate for an r -ball is used. Now we use a weak coordinate-like map for the tubular neighborhood of geodesic segment from o_i to $x_j^{(i)}$. We consider a weak coordinate-like map by the same method as in Definition 1.2: We denote $d(\gamma_j(o), o)$ by s_j . Let $c_j^{(i)}(t)$ be the shortest geodesic segment from o_i to $x_j^{(i)}$. We may assume that $c_j^{(i)}(t)$ is a unit speed curve. On $T_\delta = (-10\delta, s_j + 10\delta) \times B(0, 10\delta)$, we define ψ_i as follows:

$$\begin{aligned} \psi_i : T_\delta &\rightarrow \tilde{M}_i, \\ (t, y) &\mapsto \exp_{c_j^{(i)}(t)} y, \end{aligned}$$

where $B(0, R)$ is the R -ball centered at 0 in \mathbb{R}^{n-1} and δ is the positive constant in section 2. Then ψ_i is a nonsingular map. We consider a metric $\psi_i^* g_i$ on T_δ , where g_i is the metric of M_i . From [AC], $\{(T_\delta, \psi_i^* g_i)\}$ is $C^{0,\alpha}$ -compact. So T_δ has a $C^{0,\alpha}$ -bounded norm. Then there exists a weak coordinate-like map

$$\phi_i : T_\delta \rightarrow \tilde{M}_i$$

satisfying the conditions in Definition 1.2. Using the compactness result of [AC], we know that $(T_\delta, \phi_i^* g_i)$ converges to a $C^{0,\alpha}$ -Riemannian manifold. By the fibration theorem in [F] and [W], there exists an almost Riemannian submersion

$$\Psi_i : T_\delta \rightarrow I,$$

where $I = (-10\delta, s_j + 10\delta)$. Then we have the following convergence:

$$(T_\delta, \phi_i^* g_i) \rightarrow (I, g_0) \subset (I \times Y, g_0),$$

where $Y \subset \mathbb{R}^{n-1}$ has a $C^{0,\alpha}$ -Riemannian metric and g_0 is an almost product metric of the metric on $I \subset \mathbb{R}$ and the metric on Y . There exists a diffeomorphism

$$\Phi_i : T \rightarrow T_\delta$$

such that

$$\|\Phi_i^*(\phi_i^* g_i) - g_0\| \rightarrow 0$$

in $C^{0,\alpha}$ -topology. From [CH], we know that isometries converge in $C^{1,\alpha}$ -topology. Isometries from a small ball $B(o_i, \delta) \subset \tilde{M}_i$ to $\text{Im}(\phi_i) \subset \tilde{M}_i$ can be lifted to isometries of a δ -ball in T_δ . We denote the lifting of $\alpha_i \in \text{Isom}(M_i)$ by $\tilde{\alpha}_i \in \text{Isom}(T_\delta)$. Then any elements $\gamma^{(i)} \in \pi_1(\phi_i(B(0, \delta)))$ can be lifted to an isometry $\tilde{\gamma}^{(i)}$ on a δ -ball in T_δ , where $\phi_i(0) = o_i$.

If $\gamma^{(i)} \in \tilde{H}_i$, then an almost isometric map $\Phi_i^{-1}\gamma^{(i)}\Phi_i : T \subset \mathbb{R}^n \rightarrow T \subset \mathbb{R}^n$ can be written in the following form: We denote by $\tau(x)$ a map f such that $f \in C^1$ and $f(0) = Df(0) = 0$. Then we have

$$G(x) = \Phi_i^{-1}\gamma^{(i)}\Phi_i(x) = b + Bx + \tau(d(x, \Psi_i(x))),$$

where $b = G(0)$ and $\|B - I\| \leq \epsilon_1\|b\|$ for some small $\epsilon_1 > 0$ since it is an isometry on $B(o_i, \delta)$ which is almost flat by [Pa] and [PWY].

We define F_j as follows:

$$F_j = \Phi_i^{-1}\gamma_j^{(i)}\Phi_i$$

and write F_j as follows:

$$F_j(x) = a_j + A_jx + \tau(x),$$

where $a_j = F_j(0)$. Similarly as section 2, we denote $F_jGF_j^{-1}$ by $A(F_j)(G)$. Then we also obtain that for $\|x\| \leq 10b$,

$$A(F_j)(G)(0) = (I - A_jBA_j^{-1})a_j + A_jb + \tau(b).$$

Since A_j is an almost isometry and $\|B - I\| \leq \epsilon_1\|b\|$, we have $d(A(F_j)(G)(o), A_jb) \leq 10\epsilon_1\|a_j\|\|b\|$. The cardinality of $\{\gamma \in H_i | b - \delta \leq d(\gamma(o), o) \leq b + \delta\}$ is bounded by 2^{k_1} by the choice of a nilpotent group H_i . Considering the orbit $\{A(F_j^l)(G)(0) \mid l = 1, 2, \dots\}$, we have $\|A_j^s - I\| \leq \epsilon_1\|a_j\|$ for $s \leq 2^{k_1} \leq 2^n$. So we obtain that

$$\|A(F_j^s)(G)(0) - G(0)\| \leq 10\epsilon_1s\|a\|\|b\|$$

and

$$d((\gamma_j')^s\gamma_j'^{-s}(o), \gamma(o)) \leq 10\epsilon_1\|\gamma^s\|\|\gamma\|,$$

where $\|\alpha\| = d(\alpha(o), o)$ for $\alpha \in \Pi$. From this, $\Lambda_i = \langle \{H_i, (\gamma_j^{(i)})^s \mid j = 1, 2, \dots, k_1\} \rangle$ is a nilpotent group since

$$\|[\alpha, \beta]\| = d([\alpha, \beta](o_i), o_i) \leq 10\epsilon_1\|\alpha\|\|\beta\|$$

for $\alpha, \beta \in \Lambda_i$ [Pa]. Then $[\Pi_i : \Lambda_i] \leq (\kappa(n)C_1(n, c_0))^{2n^2}C_2(n)C_1(n)^{2n^2} < \infty$. Until now, we proved that for any sequence of manifolds $\{M_i\}$ which satisfies the condition of Theorem 1.1, there exists a subsequence $\{M_{i_j}\}$ such that $\pi_1(M_{i_j})$'s have nilpotent subgroups whose indices are bounded.

If we assume that Theorem 1.1 is false, then we can find a sequence of manifolds $\{M_i\}$ such that $\lim_{i \rightarrow \infty} [\pi_1(M_i) : \Lambda_i] = \infty$ for any nilpotent subgroups Λ_i , which is a contradiction to the fact we have proved. This completes the proof.

REFERENCES

- [AC] M. T. Anderson, J. Cheeger, *C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded below*, J. Diff. Geom. **35** (1992), 265–281. MR **93c**:53028
- [BK] P. Buser, H. Karcher, *Gromov's almost flat manifolds*, Astérisque, 1981. MR **83m**:53070
- [CC] J. Cheeger, T. B. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. Math. **144** (1996), 189–237. MR **97h**:53038
- [CG] J. Cheeger, D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom. **6** (1971), 119–128. MR **46**:2597
- [CH] E. Calabi, P. Hartman, *On the smoothness of isometries*, Duke Math. J. **37** (1970), 741–750. MR **44**:957
- [F] K. Fukaya, *Collapsing Riemannian manifolds to ones of lower dimensions*, J. Diff. Geom. **25** (1987), 139–156. MR **88b**:53050
- [FY] K. Fukaya, T. Yamaguchi, *The fundamental groups of almost nonnegatively curved manifolds*, Ann. Math. **136** (1992), 253–333. MR **93h**:53041
- [G] M. Gromov, *Almost flat manifolds*, J. Diff. Geom. **13** (1978), 231–241. MR **80h**:53041

- [Pa] S.-H. Paeng, *A generalized almost flat manifolds under a bounded $C^{0,\alpha}$ -weak norm*, Arch. Math. (Basel) **77** (2001), 423–429. MR **2002g**:53050
- [PWY] P. Petersen, G. Wei, R. Ye, *Controlled geometry via smoothing*, Comment. Math. Helv. **74** (1999), 345–363. MR **2000h**:53040
- [W] G. Wei, *Ricci curvature and Betti number*, J. Geom. Anal **7** (1997), 377–386. MR **2000d**:53062

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