PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 8, Pages 2347–2358
S 0002-9939(03)07077-1
Article electronically published on March 18, 2003

THE STRONG OPEN SET CONDITION FOR SELF-CONFORMAL RANDOM FRACTALS

NORBERT PATZSCHKE

(Communicated by David Preiss)

Abstract. We prove that the open set condition and the strong open set condition are equivalent for self-conformal random fractals.

1. Introduction

In the theory of fractal sets and measures, self-similar sets have been the most studied. A set X is called *self-similar* if it is invariant under a family of similarities $S_1, \ldots, S_N$, that is, if $X = \bigcup_{i=1}^N S_i(X)$. If one allows the mappings S_i to be conformal ones, X is called a *self-conformal set*. If the family $S_1, \ldots, S_N$ satisfies the open set condition, that is, if there is an open set O such that $S_i(O) \subseteq O$ for all i and $S_i(O) \cap S_j(O) = \emptyset$ for $i \neq j$, one can compute some characteristics of it. For instance, in the case of similarities the Hausdorff dimension of the self-similar set equals the similarity dimension.

If one considers measures instead of sets, for example a self-similar or self-conformal measure μ with the property $\mu = \sum_{i=1}^N p_i \cdot \mu \circ S_i^{-1}$, it is useful that the strong open set condition is satisfied, that is, in addition to the open set condition $X \cap O \neq \emptyset$ holds. For the case when all mappings are similarities, Schief [S] proved that both conditions are equivalent. For self-conformal sets the problem was open for a long time. Recently, Peres, Rams, Simon, and Solomyak [PRSS] and Lau, Rao, and Ye [LRY] proved the equivalence in the self-conformal case.

By allowing the mappings S_i to be random mappings, one can define random fractals. A random compact set Ξ is a *self-conformal random set* if it has the same distribution as $\bigcup_{i=1}^N S_i(\Xi^i)$, where the Ξ^i are independent copies of Ξ and are independent of $(S_1, \ldots, S_N)$. In other words, a self-conformal random set Ξ consists of N parts, which are conformal copies of independent samples of Ξ. The self-conformal random sets are also known as statistically self-conformal sets.

The equivalence of the open set condition and the strong open set condition for self-similar random sets is proved in [P1] by varying the proof of Schief [S]. In the present paper we will prove that it also holds for self-conformal random sets.

The paper is organized as follows. First, we give a definition of the term self-conformal random set and give a method for constructing such sets. The definition is more or less a combination of the known definitions for self-conformal sets and

Received by the editors March 6, 2001 and, in revised form, August 30, 2001.
2000 *Mathematics Subject Classification.* Primary 28A80; Secondary 60D05, 60G57.
Key words and phrases. Random fractals, (strong) open set condition.

self-similar random sets. For more details concerning some notations the reader is referred to [P2] and [AP].

The main part of this article is the proof of the equivalence of the open set condition and the strong open set condition. At first it is shown that there is a random open set which satisfies the strong open set condition. The proof follows the proof given by Peres, Rams, Simon, and Solomyak [PRSS], but some effort is necessary because of the randomness. Then, using this random open set, a non-random set is constructed such that the strong open set is fulfilled.

2. Self-conformal random fractals

Let $U \subseteq \mathbb{R}^d$ be an open bounded connected subset. For $0 < \gamma \leq 1$ denote by $\mathrm{Con}^{1+\gamma}(U)$ the family of conformal diffeomorphisms $S:\ U \to S(U) \subseteq U$ for which there exists a constant C_S such that

$$|S'(y) - S'(x)| \leq C_S d(x,y)^\gamma$$

holds for all $x, y \in U$, where $S'(x)$ is the differential of S at x and $|S'(x)|$ is the operator norm of this differential. Since S is a conformal mapping, we have $|S'(x)v| = |S'(x)||v|$ for all $v \in T_xU$.

Let $N \geq 2$ be a positive integer. Consider the product space

$$\Omega_0 = (\mathrm{Con}^{1+\gamma}(U))^N \ .$$

It is a separable metrisable space with the Borel σ-algebra $\mathfrak{F}_0$.

We say that a probability measure $\mathbb{P}_0$ on $(\Omega_0, \mathfrak{F}_0)$ is a *conformal random function system* if there are constants $0 < r_{\min} \leq r_{\max} < 1$ and a constant $C_0 > 0$ such that

(I) $C_{S_i} \leq C_0$ for all $i = 1, \ldots, N$ and
(II) $r_{\min} \leq |S_i'(x)| \leq r_{\max}$ for all $x \in U$ and all $i = 1, \ldots, N$

hold for $\mathbb{P}_0$-almost all $(S_1, \ldots, S_N)$.

A random compact set $\Xi \subseteq U$ is called a *self-conformal random set* associated with the conformal random function system $\mathbb{P}_0$ if Ξ has the same distribution as

$$\bigcup_{i=1}^N S_i\, \Xi^i \ ,$$

where the $\Xi^i,\ i = 1, \ldots, N$, are independent samples of Ξ and are independent of $(S_1, \ldots, S_N)$, and where $(S_1, \ldots, S_N)$ is distributed according to $\mathbb{P}_0$. That is, if $\mathbb{P}_\Xi$ is a distribution of a random compact set, then it is a distribution of a self-conformal random set, iff $\mathbb{P}_\Xi = (\mathbb{P}_0 \times \mathbb{P}_\Xi^N) \circ u^{-1}$, where

$$u\big((S_1, \ldots, S_N); X^1, \ldots, X^N\big) = \bigcup_{i=1}^N S_i(X^i) \ .$$

In the case when the S_i are similarities we speak of *self-similar random sets*. It is well known (see [F], [G], [MW]) that, given a conformal random function system, there exists a unique distribution of a self-conformal random set Ξ.

The conformal random function system $\mathbb{P}_0$ is said to satisfy the *open set condition* (OSC) if there is an open set O with $\overline{O} \subseteq U$ such that

(I) $S_i(O) \subseteq O$ for all $i = 1, \ldots, N$ and
(II) $S_i(O) \cap S_j(O) = \emptyset$ for all $i \neq j$

hold for $\mathbb{P}_0$-almost all $(S_1, \ldots, S_N)$.

$\mathbb{P}_0$ satisfies the *strong open set condition* (SOSC), if there is an open set O with $\overline{O} \subseteq U$ and

(I) $S_i(O) \subseteq O$ for all $i = 1, \dots, N$ and
(II) $S_i(O) \cap S_j(O) = \emptyset$ for all $i \neq j$

for $\mathbb{P}_0$-almost all $(S_1, \dots, S_N)$, and

(III) $\Xi \cap O \neq \emptyset$ with probability one,

where Ξ is the self-conformal random set.

Since the random mappings S_i are diffeomorphisms we may assume without loss of generality that $\operatorname{int} \overline{O} = O$.

3. Construction of self-conformal sets

For constructing self-conformal sets and measures we need a more complicated probability space.

Let $\Sigma = \{1, \dots, N\}^{\mathbb{N}}$ be the set of infinite sequences of integers between 1 and N, and let $\mathfrak{S}$ be the usual product σ-algebra on Σ. Further, let $\Sigma_n = \{1, \dots, N\}^n$ be the set of sequences of length n, and let $\Sigma_* = \bigcup_{n=0}^{\infty} \Sigma_n$ be the set of all finite sequences (including the empty sequence $\emptyset$). Now define the probability space

$$\Omega = {\Omega_0}^{\Sigma_*},$$

and let $\mathfrak{F}$ be the corresponding product σ-algebra. Finally, let $\mathbb{P}$ be the product measure on Ω with $\mathbb{P}_0$ on each component. The elements of this probability space assign to each finite sequence $\tau \in \Sigma_*$ a random variable

$$\omega(\tau) = \left(S_1^\tau(\omega), \dots, S_N^\tau(\omega)\right) .$$

Write $\omega(\emptyset) = \left(S_1(\omega), \dots, S_N(\omega)\right)$. The family of these random variables is a family of independent random variables with identical distribution $\mathbb{P}_0$. We need this complicated probability space because of the independence property in the definition of self-conformal sets.

Define operators T^η on Ω by

$$(T^\eta \omega)(\tau) = \omega(\eta\tau)$$

for $\eta \in \Sigma_*$ and $\omega \in \Omega$. By definition, $S_i^\tau(\omega) = S_i(T^\tau \omega)$. It is easy to see that $\mathbb{P}$ is invariant under all these operators. If η, $\tau \in \Sigma_*$, then $T^\eta \circ T^\tau = T^{\tau\eta}$.

We say that $\mathbb{P}$ is an *iterated conformal random function system* if $\mathbb{P}_0$ is a conformal random function system.

By definition,

$$\Omega \cong \Omega_0 \times \Omega^N$$

with the mapping

$$\omega \mapsto (\omega(\emptyset); T^1\omega, \dots, T^N\omega) ,$$

and the image of $\mathbb{P}$ under this mapping is $\mathbb{P}_0 \times \mathbb{P}^N$.

Define $\overline{S}_\emptyset^\tau = id_U$ and

$$\overline{S}_{\eta i}^\tau = \overline{S}_\eta^\tau \circ S_i^{\tau\eta}$$

for $\tau, \eta \in \Sigma_*$ and $i = 1, \dots, N$. With the notation $\overline{S}_\eta^\emptyset = \overline{S}_\eta$, then $\overline{S}_\eta^\tau(\omega) = \overline{S}_\eta(T^\tau\omega)$ and $\overline{S}_{\tau\eta} = \overline{S}_\tau \circ \overline{S}_\eta^\tau$.

Now we are able to construct the self-conformal random set Ξ.

Define a mapping $\pi : \Omega \times \Sigma \to U$ by

$$\pi_\omega(\sigma) = \lim_{n\to\infty} \overline{S}_{\sigma|n}(\omega)(x_0) \ ,$$

where $x_0 \in U$ is an arbitrary point and $\sigma|n = \sigma_1\sigma_2\ldots\sigma_n$ is the finite sequence consisting of the first n entries of σ. This limit exists and does not depend on the choice of x_0 for $\mathbb{P}$-almost all $\omega \in \Omega$. Moreover, $\pi_\omega : \Sigma \to U$ is continuous for $\mathbb{P}$-almost all ω. Set $\Xi(\omega) = \pi_\omega(\Sigma)$.

Lemma 1. *Ξ is a self-conformal random set.*

Proof. By definition,

$$\pi_\omega(i\sigma) = \lim_{n\to\infty} \overline{S}_{i\sigma|n}(\omega)(x_0) = S_i(\omega)\big(\lim_{n\to\infty} \overline{S}_{\sigma|n}(T^i\omega)(x_0)\big) = S_i(\omega)(\pi_{T^i\omega}(\sigma))$$

for all $i = 1, \ldots, N$, all $\sigma \in \Sigma$, and $\mathbb{P}$-almost all $\omega \in \Omega$. Hence,

$$\Xi(\omega) = \pi_\omega(\Sigma) = \bigcup_{i=1}^N \pi_\omega(i\Sigma) = \bigcup_{i=1}^N S_i(\omega)\big(\pi_{T^i\omega}(\Sigma)\big) = \bigcup_{i=1}^N S_i(\omega)\big(\Xi(T^i\omega)\big)$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. By definition of Ω, the random sets $\Xi^i(\omega) = \Xi(T^i\omega)$ are independent copies of Ξ and are independent of $(S_1, \ldots, S_N)$. $\square$

Another way for constructing Ξ is the following. Let $K \subseteq U$ be a compact set with $S_i(K) \subseteq K$ for all $i = 1, \ldots, N$ with probability one. (If the open set condition (OSC) or the strong open set condition (SOSC) is satisfied one can choose $K = \overline{O}$.) Define

$$K_\tau(\omega) = \overline{S}_\tau(\omega)(K)$$

for $\tau \in \Sigma_*$ and $\omega \in \Omega$. Then

$$\Xi(\omega) = \bigcap_{n=0}^\infty \bigcup_{\tau\in\Sigma_n} K_\tau(\omega)$$

for $\mathbb{P}$-almost all $\omega \in \Omega$.

4. The equivalence of the open set conditions

Before proving the main theorem we need some lemmas. Let the open set condition (OSC) be satisfied with the open set O and let $K = \overline{O}$ be compact.

For a conformal mapping S write $\|S'\| = \sup\{|S'(x)| : x \in U\}$. The Hölder continuity of the derivatives of the mappings $S_1, \ldots, S_N$ carries over to the mappings S_τ for $\tau \in \Sigma_*$.

Lemma 2. *Let $\mathbb{P}_0$ be an iterated conformal function system. Then there is an open set $U' \subseteq U$ with the following properties.*

(i) *There is a constant $c_1 \geq 0$ such that*

$$\big|\overline{S}'_\tau(\omega)(x)\big| \leq \big|\overline{S}'_\tau(\omega)(y)\big|\, e^{c_1 d(x,y)^\gamma}$$

holds for all $x, y \in U'$, all $\tau \in \Sigma_*$, and $\mathbb{P}$-almost all $\omega \in \Omega$.

(ii) *There is a constant $c_2 \geq 1$ such that*

$$c_2^{-1}\big\|\overline{S}'_\tau(\omega)\big\| d(x,y) \leq d\big(\overline{S}_\tau(\omega)(x), \overline{S}_\tau(\omega)(y)\big) \leq c_2\big\|\overline{S}'_\tau(\omega)\big\| d(x,y)$$

holds for all $x, y \in U'$, all $\tau \in \Sigma_*$, and all $\omega \in \Omega$.

Proof. We omit the proof here. In [P2] the non-random case is considered. The proof there works also in our random setting. Part (i) is a by-product of the proof of [P2, 2.1] and part (ii) is the same as [P2, 2.2]. □

In the sequel let $U = U'$ from the lemma above. A useful tool for the investigation of the random fractal is the notion of a section. We call a random subset $\Gamma \subseteq \Sigma_*$ a *section* if for each $\sigma \in \Sigma$ and each $\omega \in \Omega$ there is a unique $n \in \mathbb{N}$ with $\sigma|n \in \Gamma(\omega)$.

A simple example is the random set

$$\Gamma_r = \left\{\tau \in \Sigma_* : \ \|\overline{S}'_\tau\| \le r < \|\overline{S}'_{\tau||\tau|-1}\|\right\}$$

for $r \in (0,1)$.

It is easy to see that if Γ is a section and the open set condition is satisfied, then $\left\{\overline{S}_\tau(\omega)(O) : \ \tau \in \Gamma(\omega)\right\}$ is a family of mutually disjoint sets for almost all $\omega \in \Omega$.

Define for $a \in [0,1]$, $b \ge 1$ and $\eta \in \Sigma_*$ the random set

$$\begin{aligned} &I_{a,b}(\eta)(\omega) \\ &= \left\{\tau \in \Sigma_* : \frac{1}{b} \le \frac{\|\overline{S}'_\eta(\omega)\|}{\|\overline{S}'_\tau(\omega)\|} \le b, \ \operatorname{dist}\big(K_\tau(\omega), K_\eta(\omega)\big) \le a\|\overline{S}'_\eta(\omega)\|\right\}. \end{aligned}$$

Remark that, in general, it is not a section.

Lemma 3. *If the open set condition (OSC) is satisfied, then for all $b_0 \ge 1$ there is a constant $c_3 = c_3(b_0)$such that*

$$\#I_{a,b}(\eta)(\omega) \le c_3$$

for all $\eta \in \Sigma_*$ and all $a \in [0,1]$ and $b \in [b_0, 2b_0]$ with probability one.

Proof. It is easy to see that for fixed $r > 0$

$$\left\{\tau \in \Sigma_* : r_{\max} r \le \|\overline{S}_\tau(\omega)\| < r\right\} \subseteq \Gamma_r(\omega)$$

holds, where Γ_r is a section and, in particular, the corresponding sets $\overline{S}_\tau(O)$ are disjoint.

Fix $a \in [0,1]$ and $b \in [b_0, 2b_0]$. Let m be the smallest integer such that $r_{\max}^m < 1/(2b_0^2)$. Then

$$I_{a,b}(\eta)(\omega) \subseteq \bigcup_{k=0}^{m-1} J_k(\eta)(\omega)$$

with

$$\begin{aligned} J_k(\eta)(\omega) \quad = \quad &\left\{\tau \in \Sigma_* : \frac{1}{b_0 {r_{\max}}^k} \le \frac{\|\overline{S}'_\eta(\omega)\|}{\|\overline{S}'_\tau(\omega)\|} \le \frac{1}{b_0 {r_{\max}}^{k+1}}\ , \right. \\ &\qquad \left. \operatorname{dist}\big(K_\tau(\omega), K_\eta(\omega)\big) \le a\|\overline{S}'_\eta(\omega)\|\right\} \end{aligned}$$

for $k = 0, \dots, m-1$.

Hence, $J_k(\eta)(\omega) \subseteq \Gamma_{b_0 {r_{\max}}^k \|\overline{S}'_\eta(\omega)\|}$ with probability one. By the open set condition, all sets $\overline{S}_\tau(\omega)(O)$ with $\tau \in J_k(\eta)(\omega)$ are disjoint. Since $\operatorname{int} K \ne \emptyset$ and K is bounded, there is a point $x_0 \in K$ and there are numbers $r_1, r_2 > 0$ such that

$$B(x_0, r_1) \subseteq \operatorname{int} K \subseteq K \subseteq B(x_0, r_2)\ .$$

By Lemma 2(ii) we get

$$B\big(\overline{S}_\tau(\omega)(x_0), c_2^{-1}\|\overline{S}'_\tau(\omega)\| r_1\big) \subseteq \overline{S}_\tau(\omega)(\operatorname{int} K)$$

and

$$\overline{S}_\tau(\omega)(K) \subseteq B\big(\overline{S}_\tau(\omega)(x_0), c_2\big\|\overline{S}'_\tau(\omega)\big\|r_2\big)$$

for all $\tau \in \Sigma_*$. Hence, the sets $\overline{S}_\tau(\omega)(\operatorname{int} K)$ have a Lebesgue measure of at least

$$c_2^{-d}\big\|\overline{S}'_\tau(\omega)\big\|^d r_1^d \mathcal{L}^d B(0,1) \geq c_2^{-d} b_0^d {r_{\max}}^{md} r_1^d \mathcal{L}^d B(0,1)\big\|\overline{S}'_\eta(\omega)\big\|^d$$

for $\tau \in J_k(\eta)(\omega)$. On the other hand, all these sets are disjoint and contained in a ball of radius

$$c_2\big\|\overline{S}'_\eta(\omega)\big\|r_2 + a\big\|\overline{S}'_\eta(\omega)\big\| + 2c_2\big\|\overline{S}'_\tau(\omega)\big\|r_2 \leq \big(c_2 r_2 + 1 + 2c_2 b_0 r_2\big)\big\|\overline{S}'_\eta(\omega)\big\|$$

and center $\overline{S}_\eta(\omega)(x_0)$. Therefore, by volume estimating,

$$\#J_k(\eta)(\omega) \leq \frac{(c_2 r_2 + 1 + 2c_2 b_0 r_2)^d}{(c_2^{-1} b_0 {r_{\max}}^m r_1)^d}$$

for all k and hence

$$\#I_{a,b}(\eta)(\omega) \leq m\frac{(c_2 r_2 + 1 + 2c_2 b_0 r_2)^d}{(c_2^{-1} b_0 {r_{\max}}^m r_1)^d}\ ,$$

concluding the proof. □

Lemma 4. *Let $b_0 \geq 1$ and $\varepsilon > 0$. If the open set condition is satisfied, there exists a constant $\delta > 0$ with the following property. Let $a \in [0,1]$ and $b \in [b_0, 2b_0]$. For $\mathbb{P}$-almost all $\omega \in \Omega$, if $\tau, \eta \in \Sigma_*$ are such that $\big\|(\overline{S}^\tau_\eta)'(\omega)\big\| \leq \delta$, then $\tau\xi \in I_{a(1+\varepsilon),b(1+\varepsilon)}(\tau\eta)(\omega)$ for all $\xi \in I_{a,b}(\eta)(T^\tau\omega)$.*

Proof. Choose $\delta > 0$ such that $B(x,\delta) \subseteq U$ for each $x \in K$ and

$$c_1\delta^\gamma\big((a + c_2 \operatorname{diam} U)^\gamma + (2b_0 c_2 \operatorname{diam} U)^\gamma\big) \leq \log(1+\varepsilon)$$

hold.

Let the assumptions of the lemma be satisfied, and let $\tau, \eta \in \Sigma_*$ with $\big\|(\overline{S}^\tau_\eta)'(\omega)\big\| \leq \delta$ and $\xi \in I_{a,b}(\eta)(T^\tau\omega)$. Since $\operatorname{dist}\big(K^\tau_\eta(\omega), K^\tau_\xi(\omega)\big) \leq a\big\|(\overline{S}^\tau_\eta)'(\omega)\big\| \leq a\delta$ there exists a point $z_0 \in K^\tau_\xi(\omega)$ with

$$\operatorname{dist}\big(z_0, \overline{S}^\tau_\eta(\omega)(U)\big) \leq \operatorname{dist}\big(z_0, K^\tau_\eta(\omega)\big) \leq a\big\|(\overline{S}^\tau_\eta)'(\omega)\big\| \leq a\delta\ .$$

Hence

$$d\big(z_0, \overline{S}^\tau_\eta(\omega)(x_0)\big) \leq a\delta + c_2 \operatorname{diam} U\big\|(\overline{S}^\tau_\eta)'(\omega)\big\| \leq (a + c_2 \operatorname{diam} U)\delta$$

for all $x_0 \in U$ by Lemma 2(ii) implying

$$\begin{aligned}\big|\overline{S}'_{\tau\eta}(\omega)(x_0)\big| &= \big|\overline{S}'_\tau(\omega)\big(\overline{S}^\tau_\eta(\omega)(x_0)\big)\big|\,\big|\big(\overline{S}^\tau_\eta\big)'(\omega)(x_0)\big| \\ &\leq \big|\overline{S}'_\tau(\omega)(z_0)\big|\,\big\|\big(\overline{S}^\tau_\eta\big)'(\omega)\big\|e^{c_1(a+c_2 \operatorname{diam} U)^\gamma\delta^\gamma}\end{aligned}$$

for all $x_0 \in U$ by Lemma 2(i) and, therefore,

$$\big\|\overline{S}'_{\tau\eta}(\omega)\big\| \leq \big|\overline{S}'_\tau(\omega)(z_0)\big|\,\big\|\big(\overline{S}^\tau_\eta\big)'(\omega)\big\|e^{c_1(a+c_2 \operatorname{diam} U)^\gamma\delta^\gamma}\ .$$

On the other hand,

$$\begin{aligned}\big\|\overline{S}'_{\tau\eta}(\omega)\big\| &\geq \big|\overline{S}'_{\tau\eta}(\omega)(x_0)\big| \\ &= \big|\overline{S}'_\tau(\omega)\big(\overline{S}^\tau_\eta(\omega)(x_0)\big)\big|\,\big|\big(\overline{S}^\tau_\eta\big)'(\omega)(x_0)\big| \\ &\geq \big|\overline{S}'_\tau(\omega)(z_0)\big|\,\big|\big(\overline{S}^\tau_\eta\big)'(\omega)(x_0)\big|e^{-c_1(a+c_2 \operatorname{diam} U)^\gamma\delta^\gamma}\end{aligned}$$

for all $x_0 \in U$ implying

$$\|\overline{S}'_{\tau\eta}(\omega)\| \geq |\overline{S}'_\tau(\omega)(z_0)| \, \|(\overline{S}^\tau_\eta)'(\omega)\| e^{-c_1(a+c_2 \operatorname{diam} U)^\gamma \delta^\gamma} \ .$$

Since

$$\begin{aligned} \operatorname{diam}\big(S^\tau_\xi(\omega)(U)\big) &\leq c_2 \operatorname{diam} U \, \|(\overline{S}^\tau_\xi)'(\omega)\| \\ &\leq c_2 \operatorname{diam} U \cdot b \, \|(\overline{S}^\tau_\eta)'(\omega)\| \\ &\leq c_2 \operatorname{diam} U \cdot 2b_0\delta \end{aligned}$$

and $z_0 \in K^\tau_\xi(\omega) \subseteq S^\tau_\xi(\omega)(U)$, the inequalities

$$\begin{aligned} &|\overline{S}'_\tau(\omega)(z_0)| \, \|(\overline{S}^\tau_\xi)'(\omega)\| e^{-c_1(2b_0c_2 \operatorname{diam} U)^\gamma \delta^\gamma} \\ &\quad \leq \|\overline{S}'_{\tau\xi}(\omega)\| \\ &\quad \leq |\overline{S}'_\tau(\omega)(z_0)| \|(\overline{S}^\tau_\xi)'(\omega)\| \, e^{c_1(2b_0c_2 \operatorname{diam} U)^\gamma \delta^\gamma} \end{aligned}$$

hold. By the assumption that $\xi \in I_{a,b}(\eta)(T^\tau\omega)$, the inequalities

$$b^{-1} \leq \frac{\|(\overline{S}^\tau_\eta)'(\omega)\|}{\|(\overline{S}^\tau_\xi)'(\omega)\|} \leq b$$

are satisfied implying

$$\begin{aligned} \frac{\|\overline{S}'_{\tau\eta}(\omega)\|}{\|\overline{S}'_{\tau\xi}(\omega)\|} &\leq \frac{|\overline{S}'_\tau(\omega)(z_0)| \, \|(\overline{S}^\tau_\eta)'(\omega)\| e^{c_1(a+c_2 \operatorname{diam} U)^\gamma \delta^\gamma}}{|\overline{S}'_\tau(\omega)(z_0)| \, \|(\overline{S}^\tau_\xi)'(\omega)\| e^{-c_1(2b_0c_2 \operatorname{diam} U)^\gamma \delta^\gamma}} \\ &\leq b(1+\varepsilon) \end{aligned}$$

and

$$\frac{\|\overline{S}'_{\tau\eta}(\omega)\|}{\|\overline{S}'_{\tau\xi}(\omega)\|} \geq \frac{1}{b(1+\varepsilon)}$$

by definition of δ, which is the first property to be checked for the definition of $I_{a(1+\varepsilon),b(1+\varepsilon)}(\tau\eta)(\omega)$.

Since $\xi \in I_{a,b}(\eta)(T^\tau\omega)$ there are $x, y \in K$ with

$$d\big(\overline{S}^\tau_\eta(\omega)(x), \overline{S}^\tau_\xi(\omega)(y)\big) \leq a\|(\overline{S}^\tau_\eta)'(\omega)\| \leq \delta \ .$$

Therefore, the line joining $\overline{S}^\tau_\eta(\omega)(x)$ and $\overline{S}^\tau_\xi(\omega)(y)$ is contained in U. So there is a point $z_1 \in U$ with $\operatorname{dist}\big(z_1, K^\tau_\eta(\omega)\big) \leq a\|(\overline{S}^\tau_\eta)'(\omega)\|$ such that

$$\begin{aligned} d\big(\overline{S}_{\tau\eta}(\omega)(x), \overline{S}_{\tau\xi}(\omega)(y)\big) &= |\overline{S}'_\tau(z_1)(\omega)| \, d\big(\overline{S}^\tau_\eta(\omega)(x), \overline{S}^\tau_\xi(\omega)(y)\big) \\ &\leq a|\overline{S}'_\tau(\omega)(z_1)| \, \|(\overline{S}^\tau_\eta)'(\omega)\| \ . \end{aligned}$$

Let $x_0 \in U$. Then

$$\begin{aligned} d\big(z_1, \overline{S}^\tau_\eta(\omega)(x_0)\big) &\leq a\,\|(\overline{S}^\tau_\eta)'(\omega)\| + c_2 \operatorname{diam} U \, \|(\overline{S}^\tau_\eta)'(\omega)\| \\ &\leq (a + c_2 \operatorname{diam} U)\delta \ . \end{aligned}$$

This implies

$$\begin{aligned} &|\overline{S}'_\tau(\omega)(z_1)| \, |(\overline{S}^\tau_\eta)'(\omega)(x_0)| \\ &\quad \leq |\overline{S}'_\tau(\omega)(\overline{S}^\tau_\eta(\omega)(x_0))| \, |(\overline{S}^\tau_\eta)'(\omega)(x_0)| \, e^{c_1(a+c_2 \operatorname{diam} U)^\gamma \delta^\gamma} \\ &\quad \leq \|\overline{S}'_{\tau\eta}(\omega)\| \, e^{c_1(a+c_2 \operatorname{diam} U)^\gamma \delta^\gamma} \end{aligned}$$

for all $x_0 \in U$ and, hence,

$$\left|\overline{S}'_\tau(\omega)(z_1)\right| \left\|\left(\overline{S}^\tau_\eta\right)'(\omega)\right\| \le \left\|\overline{S}'_{\tau\eta}(\omega)\right\| e^{c_1(a+c_2 \operatorname{diam} U)^\gamma \delta^\gamma}$$

and

$$\begin{aligned} \operatorname{dist}\big(K_{\tau\eta}(\omega), K_{\tau\xi}(\omega)\big) &\le d\big(\overline{S}_{\tau\eta}(\omega)(x), \overline{S}_{\tau\xi}(\omega)(y)\big) \\ &\le a \left\|\overline{S}'_{\tau\eta}(\omega)\right\| e^{c_1(a+c_2 \operatorname{diam} U)^\gamma \delta^\gamma} \\ &\le a(1+\varepsilon)\left\|\overline{S}'_{\tau\eta}(\omega)\right\| \end{aligned}$$

which is the second property in the definition of $I_{a(1+\varepsilon),b(1+\varepsilon)}(\tau\eta)(\omega)$, completing the proof. □

Now, we prove a random version of the strong open set condition, as can be found for the self-similar case in [P1, Theorem 3].

Lemma 5. *If the open set condition is satisfied, there exists a random open set G such that the following hold.*

(i) *$S_i(G^i) \subseteq G$ for all $i = 1, \dots, N$ with probability one (where $G^i(\omega) = G(T^i\omega)$).*
(ii) *$S_i(G^i) \cap S_j(G^j) = \emptyset$ for all $i \ne j$ with probability one.*
(iii) *$\Xi \cap G \ne \emptyset$ with probability one.*

Proof. 1. Fix

$$b_0 = \max\Big\{\sqrt{C r_{\min}{}^{-1}}, r_{\min}{}^{-1}\Big\}\,.$$

Then $\left\|\overline{S}'_\xi\right\| \le b_0^2 \left\|\overline{S}'_{\xi j}\right\|$ and $b_0 \left\|S'_j\right\| \ge 1$ hold for all $j = 1, \dots, N$ with probability one. Moreover, if $r \le 1$ and $\tau = \tau_1 \dots \tau_n \in \Sigma_*$ with $\left\|\overline{S}'_\tau\right\| \le r$, then there exists an integer $k \le n$ such that

$$r\, b_0^{-1} \le \left\|\overline{S}'_{\tau|k}\right\| \le r\, b_0$$

holds.

Let $M_a(\eta) = \#I_{a,(1+a)b_0}(\eta)$. By Lemma 3 there is a constant c_3 such that $M_a(\eta) \le c_3$ for all $\eta \in \Sigma_*$ and all $a \in [0,1]$ with probability one. The function $a \mapsto M_a(\eta)(\omega)$ is non-decreasing for all $\omega \in \Omega$ and all $\eta \in \Sigma_*$. Define

$$L_a(r) = \operatorname{ess\,sup}\sup\Big\{M_a(\eta) : \eta \in \Sigma_*,\ \left\|\overline{S}'_\eta\right\| \le r\Big\}$$

for $r \le 1$. By Lemma 3 this value is well defined and finite. Let $\varepsilon = 1/(2c_3)$ and let δ be such that Lemma 4 holds with these numbers. Without loss of generality we may assume that $\delta \le 1$. The function $a \mapsto L_a(\delta)$ on $[0,1]$ is non-decreasing, integer valued, and bounded by c_3. Hence, there exists an interval $[a_1, a_2] \subseteq [0,1]$ with $a_2 - a_1 \ge c_3^{-1}$ such that $L_{a_1}(\delta) = L_{a_2}(\delta)$. Since the random value $M_{a_1}(\eta)$ is integer valued, there is an $\eta \in \Sigma_*$ with

$$\mathbb{P}\big(\left\|\overline{S}'_\eta\right\| \le \delta,\ M_{a_1}(\eta) = L_{a_1}(\delta)\big) > 0\,.$$

Define

$$\Upsilon = \big\{(\omega,\eta) \in \Omega \times \Sigma_* : \left\|\overline{S}'_\eta(\omega)\right\| \le \delta, M_{a_1}(\eta)(\omega) = L_{a_1}(\delta)\big\}$$

and let Ω' be the projection onto the first component. Then $\mathbb{P}(\Omega') > 0$. If $(\omega,\eta) \in \Upsilon$, then

$$M_{a_1}(\eta)(\omega) = M_{a_2}(\eta)(\omega) = L_{a_1}(\delta) = L_{a_2}(\delta)$$

holds.

2. By definition,

$$a_2 \geq a_1 + c_3^{-1} \geq a_1(1 + 1/(2c_3)) = a_1(1+\varepsilon)$$

and

$$\begin{aligned} 1 + a_2 &\geq 1 + a_1 + (1+a_1)/(2c_3) \\ &= (1+a_1)(1+1/(2c_3)) = (1+a_1)(1+\varepsilon) \end{aligned}$$

are satisfied. Hence, if $\xi \in I_{a_1,(1+a_1)b_0}(\eta)(T^\tau\omega)$, then

$$\tau\xi \in I_{a_1(1+\varepsilon),(1+a_1)b_0(1+\varepsilon)}(\tau\eta)(\omega) \subseteq I_{a_2,(1+a_2)b_0}(\tau\eta)(\omega)$$

by Lemma 4. Consequently, $M_{a_2}(\tau\eta)(\omega) \geq M_{a_1}(\eta)(T^\tau\omega)$. If $(\eta, T^\tau\omega) \in \Upsilon$, then

$$M_{a_2}(\tau\eta)(\omega) \leq L_{a_2}(\delta) = L_{a_1}(\delta) = M_{a_1}(\eta)(T^\tau\omega)\ ,$$

which implies

$$M_{a_2}(\eta)(T^\tau\omega) = M_{a_1}(\eta)(T^\tau\omega) = M_{a_2}(\tau\eta)(\omega)\ .$$

Therefore

$$\begin{aligned} I_{a_2,(1+a_2)b_0}(\tau\eta)(\omega) &= \big\{\tau\xi : \xi \in I_{a_1,(1+a_1)b_0}(\eta)(T^\tau\omega)\big\} \\ &= \tau I_{a_1,(1+a_1)b_0}(\eta)(T^\tau\omega) \end{aligned}$$

and

$$M_{a_2}(\tau\eta)(\omega) = M_{a_2}(\eta)(T^\tau\omega) = L_{a_1}(\delta) = M_{a_1}(\eta)(\omega)\ .$$

3. Recall

$$\Omega' = \big\{\omega \in \Omega : \exists \eta \in \Sigma_* \text{ with } \big\|\overline{S}'_\eta(\omega)\big\| \leq \delta,\ M_{a_1}(\eta)(\omega) = L_{a_1}(\delta)\big\}$$

and define

$$\Omega'' = \big\{\omega \in \Omega : \exists \eta \in \Sigma_* \text{ with } \big\|\overline{S}'_\eta(\omega)\big\| \leq \delta,\ M_{a_2}(\eta)(\omega) = L_{a_1}(\delta)\big\}\ .$$

By step 2, $(T^\tau)^{-1}\Omega' \subseteq \Omega''$ for all $\tau \in \Sigma_*$. For integer n the family

$$\Big\{(T^\tau)^{-1}\Omega' : \tau \in \Sigma_n\Big\}$$

is independent. Let $p_1 = \mathbb{P}(\Omega')$ and $p_2 = \mathbb{P}(\Omega'')$. Then $0 < p_1 \leq p_2$. Hence,

$$\begin{aligned} p_2 &= \mathbb{P}(\Omega'') \\ &\geq \mathbb{P}\Big(\bigcup_{\tau\in\Sigma_n} (T^\tau)^{-1}\Omega'\Big) \\ &= 1 - \mathbb{P}\Big(\bigcap_{\tau\in\Sigma_n} (T^\tau)^{-1}(\Omega')^c\Big) \\ &= 1 - (1-p_1)^{N^n} \end{aligned}$$

for all $n \geq 1$. This implies $1 - p_2 \leq (1-p_1)^{N^n}$ for all $n \geq 1$. But $p_1 > 0$, therefore $p_2 = 1$. Set

$$\bar{\Omega} = \big\{\omega \in \Omega : T^\tau\omega \in \Omega'' \text{ for all } \tau \in \Sigma_*\big\}\ .$$

Since Σ_* is countable, we have $\mathbb{P}(\bar{\Omega}) = 1$. By definition, $\bar{\Omega}$ is invariant under all T^τ, $\tau \in \Sigma_*$, and for each $\omega \in \bar{\Omega}$ there is an $\eta \in \Sigma_*$ such that $\big\|\overline{S}'_\eta(\omega)\big\| \leq \delta$ and $M_{a_2}(\eta)(\omega) = L_{a_1}(\delta)$. We may assume that the mapping $\omega \mapsto \eta(\omega)$ is measurable.

4. Set $\varepsilon' = \min\big(a_2/(2c_2), \operatorname{dist}(K, U^c)\big)$ and define

$$G(\omega) = \bigcup_{\tau\in\Sigma_*} \overline{S}_\tau(\omega) \circ \overline{S}_{\eta(T^\tau\omega)}(T^\tau\omega)\big(B(K,\varepsilon')\big)$$

for $\omega \in \bar{\Omega}$, where $B(K,\varepsilon') = \{x \in U : \operatorname{dist}(x,K) < \varepsilon'\}$ (remark: $\overline{S}^{\tau}_{\eta}(\omega) = \overline{S}_{\eta}(T^{\tau}\omega)$). It remains to show that G satisfies the assertions.

(i) If $\omega \in \bar{\Omega}$, then

$$\begin{aligned} S_i(\omega)\big(G^i(\omega)\big) &= S_i(\omega)\Big(\bigcup_{\tau\in\Sigma_*} \overline{S}_{\tau}(T^i\omega)\circ \overline{S}_{\eta(T^{i\tau}\omega)}(T^{i\tau}\omega)\big(B(K,\varepsilon')\big)\Big) \\ &= \bigcup_{\tau\in\Sigma_*} \overline{S}_{i\tau}(\omega)\circ \overline{S}^{i\tau}_{\eta^{i\tau}(\omega)}(\omega)\big(B(K,\varepsilon')\big) \qquad \subseteq \quad G(\omega) \end{aligned}$$

for all $i = 1,\dots,N$ since $T^{\tau}\circ T^i = T^{i\tau}$.

(ii) Assume there is an $\omega \in \bar{\Omega}$ and a pair $i \neq j$ with $S_i\big(G^i(\omega)\big) \cap S_j\big(G^j(\omega)\big) \neq \emptyset$. By definition of G there are $\tau_1,\tau_2 \in \Sigma_*$ and $\eta_1 = \eta(T^{i\tau_1}\omega)$ and $\eta_2 = \eta(T^{j\tau_2}\omega)$ such that there is a $y \in \overline{S}_{i\tau_1}(\omega)\circ \overline{S}^{i\tau_1}_{\eta_1}(\omega)\big(B(K,\varepsilon')\big) \cap \overline{S}_{j\tau_2}(\omega)\circ \overline{S}^{j\tau_2}_{\eta_2}(\omega)\big(B(K,\varepsilon')\big)$. Without loss of generality we assume $\big\|\overline{S}_{j\tau_2\eta_2}(\omega)\big\| \le \big\|\overline{S}_{i\tau_1\eta_1}(\omega)\big\|$. By our choice of b_0 we can write $\tau_2\eta_2 = \tau'\tau''$ such that

$$b_0^{-1} \le \frac{\big\|\overline{S}_{i\tau_1\eta_1}(\omega)\big\|}{\big\|\overline{S}_{j\tau'}(\omega)\big\|} \le b_0 \ .$$

Hence, $j\tau'$ satisfies the first condition of the definition of $I_{a_2,(1+a_2)b_0}(i\tau_1\eta_1)(\omega)$. By definition there are $y_1 \in K_{i\tau_1\eta}(\omega)$ with

$$d(y_1,y) \le c_2\varepsilon'\big\|\overline{S}'_{i\tau_1\eta_1}(\omega)\big\|$$

and $y_2 \in K_{j\tau'}(\omega)$ with

$$d(y_2,y) \le c_2\varepsilon'\big\|\overline{S}'_{j\tau_2\eta_2}(\omega)\big\| \le c_2\varepsilon'\big\|\overline{S}'_{i\tau_1\eta_1}(\omega)\big\| \ .$$

Hence,

$$\operatorname{dist}\big(K_{j\tau'}(\omega), K_{i\tau_1\eta_1}(\omega)\big) \le 2c_2\varepsilon'\big\|\overline{S}'_{i\tau_1\eta_1}(\omega)\big\| \le a_2\big\|\overline{S}'_{i\tau_1\eta_1}(\omega)\big\| \ ,$$

which is the second condition in the definition of $I_{a_2,(1+a_2)b_0}(i\tau_1\eta_1)(\omega)$. Hence, $j\tau' \in I_{a_2,(1+a_2)b_0}(i\tau_1\eta_1)(\omega)$, a contradiction to step 3.

(iii) Clearly, $K_{\eta(\omega)}(\omega) \subseteq G(\omega)$ for all $\omega \in \overline{\Omega}$ and, hence, $\Xi(\omega) \cap G(\omega) \neq \emptyset$. $\square$

By means of the lemma above we can prove the existence of a non-random open set satisfying the strong open set condition.

Theorem 6. *Let $\mathbb{P}_0$ be a conformal random function system satisfying the open set condition. Then the strong open set condition is satisfied.*

Proof. Let $\mathbb{P}$ be the corresponding iterated conformal random function system, and let G be the random set from Lemma 5. Choose a countable basis $\mathcal{B}$ for the topology and define

$$\mathcal{B}_1 = \big\{B \in \mathcal{B} : \ \mathbb{P}(B \subseteq G) > 0\big\} \ .$$

Set

$$O_1 = \bigcup_{B\in\mathcal{B}_1} B$$

and $O = \operatorname{int}\overline{O}_1$. It is clear that O is a non-empty open set and $O = \operatorname{int}\overline{O}$. We have to show that the conditions of the strong open set condition are satisfied. Instead of proving the assertions for O it suffices to show that O_1 satisfies the conditions of the strong open set condition. By standard arguments it follows that O also satisfies these conditions. Remark that, by definition, $G \subseteq O_1 \subseteq O$ with probability one.

Let $B \in \mathcal{B}_1$. By independence of S_i and G^i,

$$\begin{aligned}\mathbb{P}\big(S_i(B) \subseteq O_1\big)\,\mathbb{P}\big(B \subseteq G^i\big) &= \mathbb{P}\big(S_i(B) \subseteq O_1,\ B \subseteq G^i\big)\\ &\geq \mathbb{P}\big(S_i(B) \subseteq G,\ B \subseteq G^i\big)\\ &\geq \mathbb{P}\big(S_i(G^i) \subseteq G,\ B \subseteq G^i\big)\\ &= \mathbb{P}\big(B \subseteq G^i\big)\ ,\end{aligned}$$

because $\mathbb{P}\big(S_i(G^i) \subseteq G\big) = 1$. Since $\mathbb{P}(B \subseteq G^i) = \mathbb{P}(B \subseteq G) > 0$, this implies $\mathbb{P}(S_i(B) \subseteq O_1) = 1$ and hence $\mathbb{P}_0(S_i(B) \subseteq O_1) = 1$. Since $\mathcal{B}$ is countable, we get

$$\begin{aligned}\mathbb{P}_0\big(S_i(O_1) \subseteq O_1\big) &= \mathbb{P}_0\Big(\bigcup_{B\in\mathcal{B}_1} S_i(B) \subseteq O_1\Big)\\ &= \mathbb{P}_0\Big(\bigcap_{B\in\mathcal{B}_1} \{S_i(B) \subseteq O_1\}\Big) = 1\end{aligned}$$

for the open set O_1.

Let $B, B' \in \mathcal{B}_1$. By independence and Lemma 5,

$$\begin{aligned}&\mathbb{P}\big(S_i(B) \cap S_j(B') = \emptyset\big)\,\mathbb{P}\big(B \subseteq G^i\big)\,\mathbb{P}\big(B' \subseteq G^j\big)\\ &= \mathbb{P}\big(S_i(B) \cap S_j(B') = \emptyset,\ B \subseteq G^i,\ B' \subseteq G^j\big)\\ &\geq \mathbb{P}\big(S_i(G^i) \cap S_j(G^j) = \emptyset,\ B \subseteq G^i,\ B' \subseteq G^j\big)\\ &= \mathbb{P}\big(B \subseteq G^i,\ B' \subseteq G^j\big)\\ &= \mathbb{P}\big(B \subseteq G^i\big)\,\mathbb{P}\big(B' \subseteq G^j\big)\ ,\end{aligned}$$

which, again, implies $\mathbb{P}_0\big(S_i(B) \cap S_j(B') = \emptyset\big) = 1$ and $\mathbb{P}_0\big(S_i(O_1) \cap S_j(O_1) = \emptyset\big) = 1$ by definition of O_1, proving (ii).

It remains to show that $\Xi \cap O_1 \neq \emptyset$ with probability one. Let

$$\Omega' = \big\{\omega \in \Omega :\ \Xi(\omega) \cap G(\omega) \neq \emptyset\big\}\ .$$

By Lemma 5, $\mathbb{P}(\Omega') = 1$. Since $\mathcal{B}$ forms a basis for the topology, for each $\omega \in \Omega'$ there is a set $B(\omega) \in \mathcal{B}$ with $\Xi(\omega) \cap B(\omega) \neq \emptyset$ and $B(\omega) \subseteq G(\omega)$. Since G is measurable, we may assume that the mapping $\omega \mapsto B(\omega)$ is measurable. Let $E \in \mathcal{B}$. Then $\mathbb{P}\{\omega \in \Omega :\ B(\omega) = E\} \leq \mathbb{P}\{\omega \in \Omega :\ E \subseteq G(\omega)\}$. Hence, $\mathbb{P}\{\omega \in \Omega :\ B(\omega) = E\} = 0$ for all $E \in \mathcal{B}\backslash\mathcal{B}_1$. Therefore, $\mathbb{P}\{\omega \in \Omega :\ B(\omega) \notin \mathcal{B}_1\} = 0$ implying $\Xi \cap O_1 \neq \emptyset$ with probability one. $\square$

References

[AP] M. Arbeiter and N. Patzschke, *Random self-similar multifractals*, Math. Nachr. **181** (1996), 5 – 42. MR **97j:**28016

[F] K. J. Falconer, *Random fractals*, Math. Proc. Cambridge Philos. Soc. **100** (1986), 559 – 582. MR **88e:**28005

[G] S. Graf, *Statistically self-similar fractals*, Probab. Theory Related Fields **74** (1987), 357 – 392. MR **88c:**60038

[LRY] K.-S. Lau, H. Rao, and Y.-L. Ye, *Corrigendum to: Iterated function systems and Ruelle Operator*, J. Math. Anal. Appl. **262**, (2001), 446 – 451. MR **2002g:**37010

[MW] R. D. Mauldin and S. C. Williams, *Random recursive constructions: asymptotic geometric and topological properties*, Trans. Amer. Math. Soc. **295** (1986), 325 – 346. MR **87j:**60027

[P1] N. Patzschke, *The strong open set condition in the random case*, Proc. Amer. Math. Soc. **125** (1997), 2119 – 2125. MR **97i:**28005

[P2] N. Patzschke, *Self-conformal multifractal measures*, Adv. in Appl. Math. **19** (1997), 486 – 513. MR **99c:**28020

[PRSS] Y. Peres, M. Rams, K. Simon, and B. Solomyak, *Equivalence of positive Hausdorff measure and the open set condition for self-conformal sets*, Proc. Amer. Math. Soc. **129** (2001), 2689 – 2699. MR **2002d:**28004

[S] A. Schief, *Separation properties for self-similar sets*, Proc. Amer. Math. Soc. **122** (1994), 111 – 115. MR **94k:**28012

Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität Jena, D-07740 Jena, Germany

E-mail address: patzschke@mathematik.uni-jena.de