$k$-HYPONORMALITY OF POWERS OF WEIGHTED SHIFTS VIA SCHUR PRODUCTS

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(Communicated by Joseph A. Ball)

Abstract. We characterize $k$-hyponormality and quadratic hyponormality of powers of weighted shifts using Schur product techniques.

1. Introduction

Let $H$ be a separable, infinite dimensional complex Hilbert space and let $B(H)$ be the algebra of bounded linear operators on $H$. An operator $T \in B(H)$ is said to be normal if $T^*T = TT^*$, subnormal if $T$ is the restriction of a normal operator (acting on a Hilbert space $K \supseteq H$) to an invariant subspace, and hyponormal if $T^*T \geq TT^*$.

The Bram-Halmos criterion for subnormality states that an operator is subnormal if and only if

$$
\sum_{i,j} (T^i x_j, T^j x_i) \geq 0
$$

for all finite collections $x_0, x_1, x_2, \ldots, x_k \in H$ ([Bra], [Con]). Using Choleski’s Algorithm for operator matrices, it is easy to see that this is equivalent to the following positivity test:

$$
\begin{pmatrix}
I & T^* & \cdots & T^{k*} \\
T & T^*T & \cdots & T^{k*}T \\
\vdots & \vdots & \ddots & \vdots \\
T^k & T^{k*} & \cdots & T^{k*}T^k
\end{pmatrix} \geq 0 \quad \text{(for all } k \geq 1).$
$$

Condition (1.1) provides a measure of the gap between hyponormality and subnormality. The notion of $k$-hyponormality has been introduced and studied in an attempt to bridge that gap ([Ath], [BEJ], [Ch2], [CMX], [IL], [McC]). In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of $T$, while subnormality requires the validity of (1.1) for all $k$.

If we denote by $[A, B] := AB - BA$ the commutator of two operators $A$ and $B$, and if we define $T$ to be $k$-hyponormal whenever the $k \times k$ operator matrix

$$
\begin{pmatrix}
I & T^* & \cdots & T^{k*} \\
T & T^*T & \cdots & T^{k*}T \\
\vdots & \vdots & \ddots & \vdots \\
T^k & T^{k*} & \cdots & T^{k*}T^k
\end{pmatrix} \geq 0 \quad \text{(for all } k \geq 1).$
$$

received by the editors October 5, 2001 and, in revised form, March 27, 2002.

1991 Mathematics Subject Classification. Primary 47B37, 47B20; Secondary 47-04, 47A13.

Key words and phrases. $k$-hyponormality, powers of weighted shifts, Schur products.

The research of the first author was partially supported by NSF grant DMS-9800931.

The research of the second author was partially supported by KOSEF research project no. R01-2000-00003.

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$M_k(T) := ([T^*, T])^k_{i,j=1}$ is positive, or equivalently, the $(k + 1) \times (k + 1)$ operator matrix $(1.1)$ is positive, then the Bram-Halmos criterion can be rephrased as saying that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ (CAIX).

Given a bounded sequence of positive numbers (called weights) $\alpha : \alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$, the (unilateral) weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $l^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $l^2(\mathbb{Z}_+)$. It is straightforward to check that $W_\alpha$ can never be normal, and that it is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The moments of $\alpha$ are usually defined by $\beta_0 := 1$, $\beta_{n+1} := \alpha_n \beta_n$ $(n \geq 0)$ (Shi); however, we will reserve this term for the sequence $\gamma_n := \beta_n^2$ $(n \geq 0)$. Berger’s Theorem, which follows, states that $W_\alpha$ is subnormal if and only if the moments of $\alpha$ are the moments of a probability measure on $[0, ||W_\alpha||^2]$.

**Theorem 1.1** (Berger’s Theorem [Con]). $W_\alpha$ is subnormal if and only if there exists a Borel probability measure $\mu$ supported in $[0, ||W_\alpha||^2]$, with $||W_\alpha||^2 \in \text{supp } \mu$ and such that

$$
\gamma_n = \int t^n d\mu(t) \quad (\text{for all } n \geq 0).
$$

In terms of $k$-hyponormality for weighted shifts, we will often use the following basic result.

**Lemma 1.2** ([Cul] Theorem 4]). $W_\alpha$ is $k$-hyponormal if and only if the $(k + 1) \times (k + 1)$ Hankel matrices

$$(1.2) \quad A_{n,k}(\alpha) := \begin{pmatrix}
\gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\
\gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k}
\end{pmatrix} \quad (n \geq 0)
$$

are all nonnegative.

In this article we study $k$-hyponormality and quadratic hyponormality of powers of weighted shifts, using Schur product techniques. We characterize the $k$-hyponormality of powers of $W_\alpha$ in terms of the $k$-hyponormality of a finite collection of weighted shifts whose weight sequences are naturally derived from $\alpha$. Similar techniques, when combined with the results in [BEJ], [Cul], [CF2], [JP1] and [JP2], allow us to deal with back-step extensions of weighted shifts, and with weak $k$-hyponormality, including quadratic hyponormality.

2. $k$-HYPONORMALITY OF POWERS OF WEIGHTED SHIFTS

For matrices $A, B \in M_n(\mathbb{C})$, we let $A \ast B$ denote their Schur product. The following result is well known.

**Lemma 2.1** ([Pau]). If $A \geq 0$ and $B \geq 0$, then $A \ast B \geq 0$.

**Definition 2.2.** Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ and $\beta \equiv \{\beta_n\}_{n=0}^\infty$. The Schur product of $\alpha$ and $\beta$ is defined by $\alpha \beta := \{\alpha_n \beta_n\}_{n=0}^\infty$.

**Theorem 2.3.** Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ and $\beta \equiv \{\beta_n\}_{n=0}^\infty$ be two weight sequences, and assume that both $W_\alpha$ and $W_\beta$ are $k$-hyponormal. Then $W_{\alpha \beta}$ is $k$-hyponormal.
Let \( A \equiv \text{subspace for } W \alpha \) for a weight sequence \( W \).

Proof. Let \( \{ \epsilon_n \} \) and \( \{ \eta_n \} \) be the moments of \( \alpha \) and \( \beta \), respectively. By hypothesis, \( A_{n,k}(\alpha) \geq 0 \) and \( A_{n,k}(\beta) \geq 0 \) (for all \( n \geq 0 \)). Since the corresponding moments \( \gamma_n \) of \( \alpha \beta \) satisfy \( \gamma_n = \epsilon_n \eta_n \) (for all \( n \geq 0 \)), it follows that \( A_{n,k}(\alpha \beta) = A_{n,k}(\alpha) * A_{n,k}(\beta) \) (for all \( n \geq 0 \)). By Lemma 2.7, \( A_{n,k}(\alpha \beta) \geq 0 \) (for all \( n \geq 0 \)), so Lemma 1.2 now implies that \( W_{n,\beta} \) is \( k \)-hyponormal.

Corollary 2.4. Let \( W_\alpha \) and \( W_\beta \) be two weighted shifts, and assume that each is subnormal. Then \( W_{\alpha,\beta} \) is also subnormal.

Proof. This is a straightforward application of the Bram-Halmos Criterion.

Definition 2.5. Given integers \( i \) and \( \ell \), with \( \ell \geq 1 \) and \( 0 \leq i \leq \ell - 1 \), consider the decomposition \( \mathcal{H} \equiv l^2(\mathbb{Z}) = \bigoplus_{j=0}^{\infty} \{ e_j \} \), and define \( \mathcal{H}_i := \bigoplus_{j=0}^{\infty} e_{\ell j + i} \). Moreover, for a weight sequence \( \alpha \) let \( \alpha(\ell : i) := \{ \sum_{m=0}^{\ell-1} \alpha_{\ell j + i + m} \}_{j=0}^{\infty} \). \( \alpha(\ell : i) \) is the sequence of products of weights in adjacent packets of size \( \ell \), beginning with \( \alpha_i \cdots \alpha_{i+\ell-1} \).

Example 2.6. Let \( \alpha \equiv \{ \alpha_n \}_{n=0}^{\infty} \) be a weight sequence. Then

1. \( \alpha(2 : 0) : \alpha_0 \alpha_1, \alpha_2 \alpha_3, \alpha_4 \alpha_5, \ldots \);
2. \( \alpha(3 : 1) : \alpha_1 \alpha_2 \alpha_3, \alpha_4 \alpha_5 \alpha_6, \alpha_7 \alpha_8 \alpha_9, \ldots \);
3. \( \alpha(3 : 2) : \alpha_2 \alpha_3 \alpha_4, \alpha_5 \alpha_6 \alpha_7, \alpha_8 \alpha_9 \alpha_{10}, \ldots \).

Proposition 2.7. Let \( \ell \geq 1 \), let \( 0 \leq i \leq \ell - 1 \), and let \( \alpha(\ell : i) \) be as in Definition 2.5. Then \( W_{\alpha(\ell : i)} \) is unitarily equivalent to \( W_\alpha^{\ell \downarrow} | \mathcal{H}_i \). Therefore, \( W_\alpha^{\ell \downarrow} \) is unitarily equivalent to \( \bigoplus_{\ell=0}^{\ell-1} W_{\alpha(\ell : i)} \).

Proof. Since \( W_\alpha^{\ell \downarrow} e_{\ell j + i} = \prod_{m=0}^{\ell-1} \alpha_{\ell j + i + m} e_{\ell j + i + 1} \), it is clear that \( \mathcal{H}_i \) is an invariant subspace for \( W_\alpha^{\ell \downarrow} \). Moreover, \( (W_\alpha^{\ell \downarrow})^* e_{\ell j + i} = \prod_{m=0}^{\ell-1} \alpha_{\ell (j - 1) + i + m} e_{\ell (j - 1) + i + 1} \), so \( \mathcal{H}_i \) is also invariant under \( (W_\alpha^{\ell \downarrow})^* \). It follows that \( \mathcal{H}_i \) is a reducing subspace for \( W_\alpha^{\ell \downarrow} \). If we now define a unitary operator \( U : \mathcal{H} \to \mathcal{H}_i \) by \( U(e_j) = e_{\ell j + i} \), we see at once that \( U^*(W_\alpha^{\ell \downarrow} | \mathcal{H}_i) U = W_{\alpha(\ell : i)} \), as desired.

Corollary 2.8. (a) \( W_\alpha^{\ell \downarrow} \) is \( k \)-hyponormal \( \iff \) \( W_{\alpha(\ell : i)} \) is \( k \)-hyponormal for \( 0 \leq i \leq \ell - 1 \).

(b) \( W_\alpha^{\ell \downarrow} \) is subnormal \( \iff \) \( W_{\alpha(\ell : i)} \) is subnormal for \( 0 \leq i \leq \ell - 1 \).

Throughout the rest of this section, we assume that \( W_\alpha \) is subnormal, with Berger measure \( \mu \). Observe that we can always write \( \mu = \nu + \rho_0 \) where \( \nu(\{0\}) = 0 \), and that \( W_\alpha^{\ell \downarrow} \) is subnormal whenever \( W_\alpha \) is subnormal. By Corollary 2.8, we know that each \( W_{\alpha(\ell : i)} \) is subnormal, for \( 0 \leq i \leq \ell - 1 \). We now seek to identify the Berger measures \( \mu_i \) corresponding to each \( W_{\alpha(\ell : i)} \).

Theorem 2.9. (a) \( d\mu_0(t) = d\mu(t^{\ell - 1}) \).

(b) For \( 1 \leq i \leq \ell - 1 \), \( d\mu_i(t) = t^{\ell - i} d\nu(t^{1 / \ell}) \).

Proof. Let \( \gamma_n \) be the moments of \( \alpha \) (\( n \geq 0 \)). Then

\[
\int t^n d\mu_0(t) = \gamma_{tn} = \int t^n d\mu(t),
\]
so \(d\mu_0(t) = d\mu(t^{1/\ell})\). Similarly, for \(1 \leq i \leq l - 1\),
\[
\int t^n d\mu_i(t) = \frac{\gamma_{tn+i}}{\gamma_i} = \frac{1}{\gamma_i} \int t^{tn+i} d\nu(t),
\]
so \(d\mu_i(t) = \frac{t^{i/\ell}}{\gamma_i} d\nu(t^{1/\ell})\). \(\square\)


For a weight sequence \(\alpha\), we consider the back-step extension \(\alpha(x) : x, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots\) where \(x > 0\).

**Lemma 3.1.** Let \(W_\alpha\) be a subnormal weighted shift with associated Berger measure \(\mu\).

(a) (cf. [Cu1 Proposition 8]) \(W_\alpha(x)\) is subnormal if and only if (i) \(1 \in L^1(\mu)\) and (ii) \(x^2 \leq (\|L^{1/2}(\mu)\|^{-1})\). In particular, \(W_\alpha(x)\) is never subnormal when \(\mu(\{0\}) > 0\).

(b) if \(x < (\|L^{1/2}(\mu)\|^{-1})\), the corresponding measure \(\mu_x\) of \(W_\alpha(x)\) satisfies \(\mu_x(\{0\}) > 0\). In particular, \(T := W_\alpha(1/\|L^{1/2}(\mu)\|)\) is the unique back-step extension of \(W_\alpha\) with no mass at the origin.

**Proof.** (b) Let \(\gamma_n\) be the moments of \(T\). Since \(T\) is subnormal, there exists a Berger measure \(\nu\) such that
\[
\gamma_n = \int t^n d\nu = \begin{cases} 1, & n = 0, \\ \int_{t_0}^{t_1} t^n d\nu, & n \geq 1. \end{cases}
\]
Assume \(x < (\|L^{1/2}(\mu)\|^{-1})\) and write \(x = (1 - \varepsilon)(\|L^{1/2}(\mu)\|^{-1})\) for \(0 < \varepsilon < 1\). The moments \(\eta_n\) of \(W_\alpha(x)\) are such that
\[
\eta_n = \begin{cases} 1, & n = 0, \\ (1 - \varepsilon)\gamma_n = \int t^n (1 - \varepsilon) d\nu, & n \geq 1. \end{cases}
\]
It follows that \(\mu_x = (1 - \varepsilon) d\nu + \varepsilon d\nu_0\). \(\square\)

**Lemma 3.2.** Let \(W_\alpha\) be a subnormal weighted shift, let \(\ell \geq 1\), and let \(k \geq 1\). The following statements are equivalent:

(a) \(W_\alpha^{\ell k}(x)\) is \(k\)-hyponormal.

(b) \(W_\alpha(\ell k)(x)\) is \(k\)-hyponormal.

**Proof.** (a) \(\Rightarrow\) (b). Straightforward from Corollary 2.8. (b) \(\Rightarrow\) (a). By Corollary 2.8, we know that \(W_\alpha(\ell k)(x) = W_\alpha(\ell k)(x)\) is subnormal, and by Proposition 2.7 \(W_\alpha^{\ell} \cong \bigoplus_{i=0}^{\ell-1} W_\alpha(\ell k)(x)\). It follows that for \(1 \leq i \leq \ell - 1\), \(W_\alpha(\ell k)(x)\) is \(k\)-hyponormal, which together with the assumption that \(W_\alpha(\ell k)(x)\) is \(k\)-hyponormal shows that \(W_\alpha^{\ell k}(x)\) is \(k\)-hyponormal. \(\square\)

**Theorem 3.3.** Let \(W_\alpha\) be subnormal, with Berger measure \(\mu \equiv \nu + \rho d\nu_0\), and let \(\ell \geq 1\). Then \(W_\alpha^{\ell k}(x)\) is subnormal if and only if \(x < (\|L^{1/2}(\mu)\|^{-1})\). In particular, if \(\rho = 0\), \(W_\alpha^{\ell}(x)\) is subnormal if and only if \(W_\alpha(x)\) is subnormal.
Proof. It suffices to consider \( W_{\alpha(x)(\ell;0)} \). Observe that \( W_{\alpha(x)(\ell;0)} \) is a back-step extension of \( W_{\alpha(x)(\ell-1)} \). By Lemma 3.1, \( W_{\alpha(x)(\ell;0)} \) is subnormal if and only if

\[
x^2 \gamma_{\ell-1} \leq (\frac{1}{2} \| L^1(\mu_{\ell-1}) \|^{-1})^2 = \gamma_{\ell-1} (\frac{1}{2} \| L^1(\mu) \|^{-1})^2.
\]

Therefore, \( W_{\alpha(x)(\ell;0)} \) is subnormal if and only if \( x \leq (\frac{1}{2} \| L^1(\mu) \|^{-1})^{-1/2} \), as desired. \( \Box \)

Remark 3.4. Although for an operator \( T \) the subnormality of \( T^k \) does not imply the subnormality of \( T \), Theorem 3.3 shows that this is the case for back-step extensions of subnormal weighted shifts with Berger measures having no mass at the origin.

Theorem 3.5. Let \( W_\alpha \) be a subnormal weighted shift, with Berger measure \( \mu \). Then \( W_{\alpha(x_n,x_{n-1},\ldots,x_1)} \) is subnormal if and only if

(a) \( \frac{1}{2} \in L^1(\mu) \) for all \( 1 \leq j \leq n \),

(b) \( x_1 \cdots x_j = (\frac{1}{2} \| L^1(\mu) \|^{-1/2})^2 \) for \( 1 \leq j \leq n-1 \) and \( x_1 \cdots x_n \leq (\frac{1}{2} \| L^1(\mu) \|^{-1/2})^2 \).

Proof. The case \( n = 1 \) was established in [Cu1, Proposition 8]. Here, and without loss of generality, we will only consider the case \( n = 2 \).

(\( \Rightarrow \)) Assume that \( W_{\alpha(x_0,x_1)} \) is subnormal. Since \( W_{\alpha(x_1)} \) is a subnormal weighted shift possessing a subnormal extension (namely \( W_{\alpha(x_0,x_1)} \)), Lemma 3.1 implies that \( x_1 = (\frac{1}{2} \| L^1(\mu) \|^{-1/2})^2 \). Moreover, since \( W_{\alpha(x_0,x_1)} \) is subnormal, we must have \( W_{\alpha(x_2,x_1)} \) subnormal, so Lemma 3.2 implies that \( W_{\alpha(x_2,x_1)(2:0)} \equiv W_{\alpha(2:0)(x_2,x_1)} \) is subnormal and

\[
x_1 x_2 \leq (\frac{1}{2} \| L^1(\mu(\ell)) \|^{-1/2})^2 = (\frac{1}{2} \| L^1(\mu) \|^{-1/2})^2.
\]

(\( \Leftarrow \)) Assume that (a) and (b) hold. Since \( \frac{1}{2} \in L^1(\mu) \) and \( x_1^2 = (\frac{1}{4} \| L^1(\mu) \| \mu(\ell)) \), we know that \( W_{\alpha(x_1)} \) is subnormal with measure \( \nu \) such that \( \nu(\{0\}) = 0 \). To check the subnormality of \( W_{\alpha(x_2,x_1)} = W_{\alpha(x_1)(x_2)} \), by Theorem 3.3 it suffices to check the subnormality of \( W_{\alpha(x_2,x_1)}(2:0) \) and by Lemma 3.2 this reduces to verifying the subnormality of \( W_{\alpha(x_2,x_1)(2:0)} = W_{\alpha(2:0)(x_2,x_1)} \). If \( \mu_1 \) denotes the Berger measure of \( W_{\alpha(2:0)} \), that is, \( d\mu_1(l) \equiv d\mu(\ell l^2) \), we know that

\[
x_2 x_1 \leq (\frac{1}{2} \| L^1(\mu_1) \|^{-1/2})^2 = (\frac{1}{2} \| L^1(\mu) \|^{-1/2})^2.
\]

Therefore, we see that \( W_{\alpha(2:0)(x_2,x_1)} \) is subnormal, using Lemma 3.1. Thus, \( W_{\alpha(x_2,x_1)(2:0)} \) is subnormal, as desired. \( \Box \)

4. SOME REVEALING EXAMPLES

Let \( \alpha : \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots \) be a sequence of weights and let \( \gamma_n \) be the corresponding moments. For \( x > 0 \) let \( \alpha(x) : x, \alpha_0, \alpha_1, \ldots \) be the associated back-step extension of \( \alpha \) and assume that \( W_{\alpha} \) is subnormal. It follows from [Cu1, Theorem 4] that \( W_{\alpha(x)} \) is \( k \)-hyponormal if and only if

\[
D_k := \begin{pmatrix}
\frac{1}{x^2} & \gamma_0 & \gamma_1 & \cdots & \gamma_{k-1} \\
\gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\
\gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{k-1} & \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1}
\end{pmatrix} \geq 0.
\]
Theorem 4.1. For \( \ell \geq 1 \), \( W^\ell_{\alpha(x)} \) is \( k \)-hyponormal if and only if
\[
D_k: \begin{pmatrix}
\frac{1}{x^2} & \gamma_{\ell-1} & \gamma_{2\ell-1} & \cdots & \gamma_{k\ell-1} \\
\gamma_{\ell-1} & \gamma_{2\ell-1} & \gamma_{3\ell-1} & \cdots & \gamma_{(k+1)\ell-1} \\
\gamma_{2\ell-1} & \gamma_{3\ell-1} & \gamma_{4\ell-1} & \cdots & \gamma_{(k+2)\ell-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{k\ell-1} & \gamma_{(k+1)\ell-1} & \gamma_{(k+2)\ell-1} & \cdots & \gamma_{2k\ell-1}
\end{pmatrix} \geq 0.
\]

Proof. It suffices to check that \( W^\ell_{\alpha(x)}(x) \) is \( k \)-hyponormal. Now, the matrix detecting \( k \)-hyponormality for \( W^\ell_{\alpha(x)}(0) \) is
\[
D_k = x^2 \begin{pmatrix}
\frac{1}{x^2} & \gamma_{\ell-1} & \gamma_{2\ell-1} & \cdots & \gamma_{k\ell-1} \\
\gamma_{\ell-1} & \gamma_{2\ell-1} & \gamma_{3\ell-1} & \cdots & \gamma_{(k+1)\ell-1} \\
\gamma_{2\ell-1} & \gamma_{3\ell-1} & \gamma_{4\ell-1} & \cdots & \gamma_{(k+2)\ell-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{k\ell-1} & \gamma_{(k+1)\ell-1} & \gamma_{(k+2)\ell-1} & \cdots & \gamma_{2k\ell-1}
\end{pmatrix} = x^2 D_k^\ell,
\]
so the result follows. \( \square \)

Proposition 4.2. For \( \ell \geq 1 \), let \( \alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots \)
\begin{enumerate}
\item \( W^\ell_{\alpha(\sqrt{\frac{2}{3}})} \) is hyponormal \( \iff \) \( x \leq \frac{(\ell+1)^2}{2(2\ell+1)} \).
\item \( W^\ell_{\alpha(\sqrt{\frac{3}{4}})} \) is 2-hyponormal \( \iff \) \( x \leq \frac{(\ell+1)^2(2\ell+1)^2}{2(2\ell+1)(4\ell^2+3\ell+1)} \).
\item \( W^\ell_{\alpha(\sqrt{\frac{4}{5}})} \) is subnormal \( \iff \) \( x \leq \frac{1}{2} \).
\end{enumerate}

Proof. Observe that \( \gamma_{\ell-1} = \frac{2}{x^\ell + 1} \). Now consider
\[
D_2 = \begin{pmatrix}
\frac{1}{x^2} & \frac{2}{2\ell+1} & \frac{2}{3\ell+1} \\
\frac{2}{2\ell+1} & \frac{2}{3\ell+1} & \frac{2}{4\ell+1} \\
\frac{2}{3\ell+1} & \frac{2}{4\ell+1} & \frac{2}{5\ell+1}
\end{pmatrix}.
\]
By direct calculation we obtain
\[
D_2 \geq 0 \iff x \leq \frac{(\ell+1)^2(2\ell+1)^2}{2(2\ell+1)(4\ell^2+3\ell+1)}.
\]
Moreover, since \( W_\alpha \) is subnormal, with measure \( 2tdt \) (in particular, with no mass at the origin), we see that \( W^\ell_{\alpha(\sqrt{\frac{2}{3}})} \) is subnormal \( \iff \) \( W_\alpha(\sqrt{\frac{2}{3}}) \) is subnormal. \( \square \)

Corollary 4.3 (Proposition 7). Let \( \alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots \)
\begin{enumerate}
\item \( W_{\alpha(\sqrt{\frac{2}{3}})} \) is hyponormal \( \iff \) \( x \leq \frac{2}{3} \).
\item \( W_{\alpha(\sqrt{\frac{3}{4}})} \) is 2-hyponormal \( \iff \) \( x \leq \frac{9}{16} \).
\item \( W_{\alpha(\sqrt{\frac{4}{5}})} \) is subnormal \( \iff \) \( x \leq \frac{1}{2} \).
\end{enumerate}

5. Quadratic Hyponormality

We recall some terminology and notation from \( \text{CuI}, \text{CF2} \) and \( \text{CF3} \). An operator \( T \) is said to be \textit{quadratically hyponormal} if \( T + sT^2 \) is hyponormal for every \( s \in \mathbb{C} \). Let \( W_\alpha \) be a hyponormal weighted shift. For \( s \in \mathbb{C} \), let \( D(s) := \)
Then, let $W := (W_\alpha + sW_\alpha^2)^*$, $W_\alpha + sW_\alpha^2$, let $P_n$ be the orthogonal projection onto $V_{1=0}^n(p_i)$, and let

$$D_n := D_n(s) := P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n,$$

where

$$q_k := u_k + |s|^2v_k, r_k := s\sqrt{w_k}, u_k := \alpha_k^2 - \alpha_{k-1}^2, v_k := \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_k^2, w_k := \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2 (k \geq 0)$$

and that $q_k$ is actually a polynomial in $t := |s|^2$ of degree $n+1$, with Maclaurin expansion $d_n(t) = \sum_{i=0}^{n+1} c(n, i)t^i$. This gives at once the relations

$$c(0, 0) = u_0, c(0, 1) = v_0,$$
$$c(1, 0) = u_1 + u_0, c(1, 1) = u_1v_0 + u_0v_1 - w_0, c(1, 2) = v_1v_0,$$

and

$$c(n + 2, i) = u_{n+2}c(n + 1, i) + v_{n+2}c(n + 1, i - 1) - w_{n+1}c(n, i - 1)$$

$$(n \geq 0, 0 \leq i \leq n + 1).$$

To detect the positivity of $d_n$, the following notion was introduced in [CF3].

**Definition 5.1.** We say that $W_\alpha$ is *positively quadratically hyponormal* if $c(n, i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n + 1$.

It is obvious that positive quadratic hyponormality implies quadratic hyponormality; moreover, quadratic hyponormality does not necessarily imply positive quadratic hyponormality [JPT].

**Proposition 5.2.** With the above notation, assume that $u_{n+1}v_n \geq w_n (n \geq 3)$. Then $W_\alpha$ is positively quadratically hyponormal if and only if $c(3, 2) \geq 0$ and $c(4, 3) \geq 0$.

**Proof.** Immediate from [BEJ] Corollary 3.3 and Theorem 3.9. □

**Lemma 5.3.** Assume that $W_\alpha$ is subnormal and let $\ell \geq 1, k \geq 1$. The following statements are equivalent:

1. $W_\alpha^{\ell} \alpha(z)$ is weakly $k$-hyponormal.
2. $W_\alpha(z, \ell, 0)$ is weakly $k$-hyponormal.

**Proof.** Imitate the proof of Lemma 3.2 □
Theorem 5.4. Let \( \alpha_n := \sqrt{\frac{n^2}{n+3}} \) (\( n \geq 0 \)), let \( \alpha = \{\alpha_n\}_{n=0}^{\infty} \), and let \( \ell \geq 1 \). Then \( W_{\alpha(\sqrt{\beta})}(\ell, 0) \) is positively quadratically hyponormal if and only if

\[
\begin{cases}
  x \leq \frac{(\ell+1)^2}{2(2\ell+1)}, & \ell = 1, 2, \\
  x \leq \frac{(\ell+1)^2(1+7\sqrt{\ell}+34\ell^2+44\ell^3)}{2(1+9\sqrt{\ell}+45\ell^2+99\ell^3+94\ell^4)}, & \ell \geq 3.
\end{cases}
\]

Proof. Let \( \beta_0 := \sqrt{\frac{2}{\ell+1}} \) and \( \beta_n := \sqrt{\frac{n+1}{(n+1)!}} \) (\( n \geq 1 \)). Then \( W_{\alpha(\sqrt{\beta})}(\ell, 0) = W_\beta \).

By direct calculation we see that

\[
u_n = \beta_n^2 - \beta_{n-1}^2 = \frac{\ell^2}{((n+1)!)(n+1)} \quad (n \geq 2),
\]

\[
v_n = \beta_n^2 - \beta_{n+1}^2 - \beta_{n-1}^2 = \frac{4\ell^2}{((n+2)!)(n+1)} \quad (n \geq 3),
\]

and

\[
w_n = \beta_n^2 \beta_{n+1}^2 - \beta_{n-1}^2 \beta_n^2 = \frac{4\ell^2}{((n+1)!)(n+1)(n+2)!} \quad (n \geq 2).
\]

Since \( W_\beta \) has the property \( u_{n+1}v_n \geq w_n \) (\( n \geq 3 \)), by Proposition 5.2 it suffices to verify the nonnegativity of \( c(3, 2) \) and \( c(4, 3) \). By direct calculation,

\[
c(3, 2) \geq 0 \iff \frac{(\ell+1)^2(7+11\ell)}{4(3+10\ell+11\ell^2)}.
\]

and

\[
c(4, 3) \geq 0 \iff \frac{(\ell+1)^2(1+7\sqrt{\ell}+34\ell^2+44\ell^3)}{2(1+9\sqrt{\ell}+45\ell^2+99\ell^3+94\ell^4)}.
\]

On the other hand, the hyponormality condition for \( W_\beta \) is \( x \leq \frac{(\ell+1)^2}{2(2\ell+1)} \). Finally, observe that

\[
\frac{(\ell+1)^2(7+11\ell)}{4(3+10\ell+11\ell^2)} \geq \frac{(\ell+1)^2}{2(2\ell+1)} \quad (\text{for all } \ell \geq 1),
\]

\[
\frac{(\ell+1)^2(1+7\sqrt{\ell}+34\ell^2+44\ell^3)}{2(1+9\sqrt{\ell}+45\ell^2+99\ell^3+94\ell^4)} \geq \frac{(\ell+1)^2}{2(2\ell+1)} \quad (\text{if } \ell = 1, 2),
\]

and

\[
\frac{(\ell+1)^2(1+7\sqrt{\ell}+34\ell^2+44\ell^3)}{2(1+9\sqrt{\ell}+45\ell^2+99\ell^3+94\ell^4)} \leq \frac{(\ell+1)^2}{2(2\ell+1)} \quad (\text{if } \ell \geq 3).
\]

This proves (5.1). \( \Box \)

Corollary 5.5. Let \( \alpha_n := \sqrt{\frac{n^2}{n+3}} \) (\( n \geq 0 \)) and \( \alpha \equiv \{\alpha_n\}_{n=0}^{\infty} \).

(a) \( W_{\alpha(\sqrt{\beta})}^2 \) is quadratically hyponormal \( \iff x \leq \frac{9}{16} \).

(b) If \( \ell \geq 3 \) and \( x \leq \frac{(\ell+1)^2(1+7\sqrt{\ell}+34\ell^2+44\ell^3)}{2(1+9\sqrt{\ell}+45\ell^2+99\ell^3+94\ell^4)} \), then \( W_{\alpha(\sqrt{\beta})}^\ell \) is quadratically hyponormal.

Remark 5.6. Let \( \alpha \) be as in Corollary 5.5. Then \( W_{\alpha(\sqrt{\beta})}^2 \) is positively quadratically hyponormal if and only if \( W_{\alpha(\sqrt{\beta})} \) is quadratically hyponormal if and only if \( x \leq \frac{9}{16} \). Moreover, for \( x = \frac{9}{16} \), \( W_{\alpha(\sqrt{\beta})}^2 \) has the first two weights equal, namely \( \beta_0 = \beta_1 = \sqrt{\frac{2}{3}} \); this example resembles [Cu1], Proposition 7, where the first nontrivial quadratically hyponormal weighted shift with two equal weights appears. (For additional results along these lines, see [Cu].) Here we notice that for \( x = \frac{9}{16} \), not
only is \( W_{\alpha(\sqrt{x})} \), quadratically hyponormal with two equal weights but also \( W_{\alpha(\sqrt{x})}^2 \) is quadratically hyponormal!

**Acknowledgment**

The authors are indebted to the referee for several helpful suggestions.

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