ON A CHARACTERIZATION
OF THE MAXIMAL IDEAL SPACES OF
ALGEBRAICALLY CLOSED COMMUTATIVE $C^*$-ALGEBRAS

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Abstract. Let $C(X)$ be the algebra of all complex-valued continuous functions on a compact Hausdorff space $X$. We say that $C(X)$ is algebraically closed if each monic polynomial equation over $C(X)$ has a continuous solution. We give a necessary and sufficient condition for $C(X)$ to be algebraically closed for a locally connected compact Hausdorff space $X$. In this case, it is proved that $C(X)$ is algebraically closed if each element of $C(X)$ is the square of another. We also give a characterization of a first-countable compact Hausdorff space $X$ such that $C(X)$ is algebraically closed.

1. Introduction

Let $X$ be a compact Hausdorff space and $C(X)$ the commutative Banach algebra of all complex-valued continuous functions on $X$. Let $P(x, z) = z^n + a_{n-1}(x)z^{n-1} + \cdots + a_1(x)z + a_0(x)$ for some natural number $n$ and some $a_0, a_1, \ldots, a_{n-1} \in C(X)$. We say that $C(X)$ is algebraically closed if $P(x, z) = 0$ has a root in $C(X)$ for every monic polynomial $P(x, z)$ over $C(X)$; that is, there exists an $f \in C(X)$ such that $P(x, f(x)) = 0$ on $X$. If to every $a_0 \in C(X)$ there corresponds a $g \in C(X)$ so that $a_0(x) = g^2(x)$ for every $x \in X$, then we say that $C(X)$ is square-root closed. By definition, if $C(X)$ is algebraically closed then the algebra is square-root closed.

Deckard and Pearcy [3] prove that $C(X)$ is algebraically closed if $X$ is a Stonian space: a compact Hausdorff space which is also extremely disconnected. In [4] they consider two spaces: a totally disconnected compact Hausdorff space; a linearly ordered and order-complete topological space. In both cases, algebraic closedness is proved. It is also noted that $C(X)$ need not be square-root closed. Indeed, there is no continuous function on $S^1$, the unit circle in the complex plane $\mathbb{C}$, whose square is the identity function on $S^1$ (cf. [2, Lemma 2.1]). On the other hand, $C([0,1])$ is algebraically closed by [3, Theorem 2]. Here $[0,1]$ denotes the closed unit interval.

Countryman [2] gives some necessary and sufficient conditions for a first-countable compact Hausdorff space $X$ in order that $C(X)$ be algebraically closed. In this case, $C(X)$ is algebraically closed if and only if $X$ is hereditarily unicoherent.
and almost locally connected; intuitively $X$ contains neither $S^1$ nor the closure of $\bigcup_{n \in \mathbb{N}} \{(1/n) \times [0, 1]\}$ in $\mathbb{R}^2$ with its usual topology. Here and after, $\mathbb{N}$ and $\mathbb{R}$ denote the space of all natural numbers and that of real numbers, respectively. It is also proved, for such $X$, that $C(X)$ is algebraically closed if the algebra is square-root closed. On the other hand, this is not the case unless $X$ is first-countable. In fact, he gives a compact Hausdorff space $X$ that is not first-countable with the following property: each function of $C(X)$ has a continuous $2^n$th root for every $n \in \mathbb{N}$, while some function has no continuous fifth root ([2, Remark (3)]).

Recall that a uniform algebra on a compact Hausdorff space $X$ is a uniformly closed subalgebra of $C(X)$ which contains the constants and separates the points of $X$. Let $A$ be a uniform algebra on a locally connected compact Hausdorff space $X$. Čirka [1] proved that $A = C(X)$, if each function of $A$ is the square of another. As noted above, $C(X)$ need not be square-root closed even if $X$ is locally connected. In [6], a characterization is given of a locally connected compact Hausdorff space $X$ such that $C(X)$ be square-root closed: A necessary and sufficient condition in order that $C(X)$ be square-root closed is as follows.

(2) The covering dimension of $X$ is less than or equal to 1, and the first Čech cohomology group with integer coefficient is trivial.

The condition (2) need not be necessary nor sufficient for $C(X)$ being square-root closed, unless $X$ is locally connected. On the other hand, if $X$ satisfies (2) then $C(X)$ is “almost” square-root closed, even if $X$ is not locally connected, in the following sense: For every $\varepsilon > 0$ and every $f \in C(X)$ with $\|f\|_\infty \leq 1$, there exist $g, h \in C(X)$ such that $f = gh$ and $\|g - h\|_\infty \leq \varepsilon$. Here $\| \cdot \|_\infty$ denotes the supremum norm on $X$ (cf. [8, Theorem 2.1]).

In this paper we consider a locally connected compact Hausdorff space $X$ which need not be first-countable. In a way similar to the arguments in [2], we show that $C(X)$ is algebraically closed if and only if $C(X)$ is square-root closed. They are also equivalent to the condition that $X$ is hereditarily unicoherent. When these conditions are satisfied, then $X$ has a base for the topology whose elements have at most finitely many boundary points. In terms of the covering dimension and the Čech cohomology group, we also give a characterization of a first-countable compact Hausdorff space $X$ for which $C(X)$ is algebraically closed.

2. Preliminaries

We state characterization theorems obtained in [2] and [6]. To do this we need some terminology.

We say that a point $p$ of a topological space $T$ is of finite order, if for every open neighborhood $V$ of $p$ there exists an open set $V_0$ such that $p \in V_0 \subseteq V$ and that $V_0$ has at most finitely many boundary points. In this paper we say that a compact Hausdorff space $X$ is an $A$-space, if each point of $X$ is of finite order. Note that the definition of an $A$-space is not the same in [2]. However, for compact Hausdorff spaces, the two definitions are shown to be equivalent; see [2, p. 440].

A $C$-space is a compact Hausdorff space $X$ such that $C(X_\lambda)$ is algebraically closed for every connected component $X_\lambda$. For simplicity we say that a compact Hausdorff space $X$ is an $AC$-space, if $X$ is both an $A$-space and a $C$-space.

We say that a topological space $T$ is almost locally connected, if $T$ contains no mutually disjoint connected closed subsets $C_n \ (n \in \mathbb{N})$, which are open in the
closure of $\bigcup_{n \in \mathbb{N}} C_n$, with the following property:

$$x_n, y_n \in C_n \text{ and } \{x_n\}_{n \in \mathbb{N}} \text{ and } \{y_n\}_{n \in \mathbb{N}} \text{ converge to distinct points.}$$

A topological space $T$ is hereditarily unicoherent, if $M \cap N$ is connected for all connected closed subsets $M, N$ of $T$.

For example, the unit interval $[0,1]$ is an AC-space, almost locally connected and hereditarily unicoherent. The closure of $\bigcup_{n \in \mathbb{N}} (\{1/n\} \times [0,1])$ is neither an A-space nor almost locally connected. Note that a locally connected topological space need not be almost locally connected. On the other hand, any topological space that contains a homeomorphic image of the unit circle $S^1$ is neither a $C$-space nor hereditarily unicoherent.

Let $Y$ be a normal space, and $n$ a non-negative integer. We say that the covering dimension $\dim Y$ of $Y$ is less than or equal to $n$, if for every finite open covering $\mathcal{A}$ of $Y$ there exists a refinement $\mathcal{B}$ of $\mathcal{A}$ such that each $y \in Y$ is in at most $(n + 1)$ elements of $\mathcal{B}$. It is well known that $\dim Y \leq n$ if and only if for every closed subset $F$ of $Y$ and every $S^n$-valued continuous function $f$ on $F$, there exists an $S^n$-valued continuous function $\tilde{f}$ on $Y$ such that $\tilde{f}|_F = f$ (cf. [9]). Here $S^n$ denotes the $n$-sphere. Let $m$ be a natural number. We say that $\dim Y = m$, if $\dim Y \leq m$ and if $\dim Y \leq m - 1$. For example, the covering dimension of $\mathbb{R}^m$ is $m$.

Let $X$ be a compact Hausdorff space. Then $\check{H}^1(X, \mathbb{Z})$ denotes the first Čech cohomology group of $X$ with integer coefficients. Here we are not concerned with the definition of $\check{H}^1(X, \mathbb{Z})$. Let $C(X)^{-1}$ be the multiplicative group of all invertible elements of $C(X)$ and $\exp C(X) = \{e^f : f \in C(X)\}$. It is well known that $\check{H}^1(X, \mathbb{Z})$ is isomorphic to the quotient group $C(X)^{-1}/\exp C(X)$, by a theorem of Arens and Royden (cf. [5] Theorem 7.2 of Chapter III).

This completes our preparation for stating characterization theorems for a compact Hausdorff space $X$ in order that $C(X)$ be algebraically closed or square-root closed.

**Theorem A** ([2]). Let $X$ be a compact Hausdorff space. For the following conditions, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) holds:

(i) $X$ is an AC-space.
(ii) $C(X)$ is algebraically closed.
(iii) $C(X)$ is square-root closed.
(iv) $X$ is almost locally connected and hereditarily unicoherent.

Moreover if $X$ is first-countable, then the condition (iv) implies (i); that is, the conditions from (i) to (iv) are all equivalent.

**Theorem B** (Theorem 2.2, [6]). Let $X$ be a locally connected compact Hausdorff space. Then the following conditions are equivalent:

(i) $C(X)$ is square-root closed.
(ii) $\dim X \leq 1$ and $\check{H}^1(X, \mathbb{Z}) = 0$.

To prove our main result, we need the following lemma.

**Lemma C** (Lemma 2.2, [3]). Let $X$ be a compact Hausdorff space, and $P(x, z)$ a monic polynomial over $C(X)$. Suppose that $x_0 \in X$ and that $z_0 \in \mathbb{C}$ is a root of $P(x_0, z) = 0$ of multiplicity $m$. Let $\varepsilon > 0$ so that $P(x_0, z) = 0$ has no root in $\{z \in \mathbb{C} : 0 < |z - z_0| \leq \varepsilon\}$. Then there exists an open neighborhood $V_0$ of $x_0$ such that $P(y, z) = 0$ has exactly $m$ roots, counting multiplicities, in $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ for every $y \in V_0$.  

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3. Main results

It seems natural to expect that the conditions of Theorem A have deep connections with those of Theorem B. In fact, the following result holds.

Lemma 3.1. Let $X$ be a compact Hausdorff space with $\dim X \leq 1$. If $\hat{H}^1(X, \mathbb{Z}) = 0$ then $X$ is hereditarily unicoherent.

Proof. Suppose that $X$ is not hereditarily unicoherent. We show that $\hat{H}^1(X, \mathbb{Z}) \neq 0$. In a way similar to the proof of [3, Lemma 2.1] we see that there exist a closed subset $F$ of $X$ and an $h \in C(F)^{-1}$, the set of all invertible elements of $C(F)$, such that $h \neq g^2$ for all $g \in C(F)^{-1}$. Put $h_1 = h/|h| \in C(F)^{-1}$. Since $\dim X \leq 1$, we can find an $\tilde{h}_1 \in C(X)^{-1}$ such that $|\tilde{h}_1| = 1$ and $\tilde{h}_1|_F = h_1$. Then there is no function $f \in C(X)$ such that $\tilde{h}_1 = f^2$. In particular, $\tilde{h}_1$ does not belong to $\text{exp}(C(X))$. Therefore, we have $C(X)^{-1} \setminus \text{exp}(C(X)) \neq \emptyset$. This implies $\hat{H}^1(X, \mathbb{Z}) \neq 0$, by a theorem of Arens and Royden. This completes the proof.

Let $T$ be a connected topological space and $p$ a point of $T$. Recall that $p$ separates the distinct points $a$ and $b$ of $T \setminus \{p\}$ in $T$, if there exist disjoint open sets $A$ and $B$ such that $a \in A, b \in B$ and $T \setminus \{p\} = A \cup B$. In this case both $A \cup \{p\}$ and $B \cup \{p\}$ are connected. If $p$ belongs to every connected closed subset that contains both $a$ and $b$, then we say that $p$ cuts $T$ between $a$ and $b$. If $X$ is a locally connected and connected compact Hausdorff space, then by [7, Theorem 3-6], a point $p \in X$ separates the distinct points $a$ and $b$ of $X \setminus \{p\}$ in $X$, if and only if $p$ cuts $X$ between $a$ and $b$.

Let $X$ be a connected compact Hausdorff space and $a, b \in X$. It is well known that there exists a minimal connected closed subset of $X$, with respect to the set inclusion, that contains both $a$ and $b$ (cf. [7, Theorem 2-10]). If $X$ is also hereditarily unicoherent, then we see that such a set is uniquely determined. In this case, $E[a, b]$ denotes the smallest connected closed subset containing both $a$ and $b$. Therefore, each point of $E[a, b] \setminus \{a, b\}$ cuts $X$ between $a$ and $b$, and hence separates $a$ and $b$ in $X$ by [7, Theorem 3-6], if $X$ is a locally connected and connected compact Hausdorff space which is also hereditarily unicoherent.

The separation order $\preceq$ in $E[a, b]$ is defined as follows: for every distinct points $p, q \in E[a, b]$ we define $p \prec q$ if $p = a$ or if $p$ separates $a$ and $q$ in $X$. Then $x \preceq y$ means that $x = y$ or $x < y$. It is well known that the separation order in $E[a, b]$ is a total order [7, Theorem 2-21]. Then the order topology in $E[a, b]$ is defined by a base for the topology: the full space $E[a, b]$; for every $x \in E[a, b]$ the set of all $y$ with $y < x$ and the set of $y$ with $x < y$; for every $x < y$ the set of all $z$ with $x < z < y$. By [7, Theorem 2-25], the order topology defined by the total order $\preceq$ in $E[a, b]$ is the same as the relative topology in $E[a, b]$.

By [7, Theorem 2-26], we see that each non-empty subset of $E[a, b]$, which is bounded above, has a least upper bound with respect to the separation order. That is, $E[a, b]$ is order-complete. These facts will be used later.

Lemma 3.2. Let $X$ be a locally connected compact Hausdorff space which is also hereditarily unicoherent. Then $X$ is an $A$-space.

Proof. Since $X$ is locally connected, each connected component of $X$ is open. Thus we may assume that $X$ is connected. Let $x_0$ be any point of $X$ and $V$ any open neighborhood of $x_0$. It is enough to consider the case that $X \setminus V \neq \emptyset$. For every $x \in X \setminus V$ let $E[x_0, x]$ be the smallest connected closed subset containing $x_0$ and $x$. 

Fix any point of $V \cap (E[x_0, x] \setminus \{x_0, x\})$, say $y(x)$. Then $y(x)$ separates $x_0$ and $x$ in $X$ as noted above. That is, there exist disjoint open sets $A_x, B_x$ such that $x_0 \in A_x$, $x \in B_x$ and $X \setminus \{y(x)\} = A_x \cup B_x$. Note that $y(x)$ is the only boundary point of $A_x$.

Since $X \setminus V$ is compact, there are finitely many points $x_1, x_2, \ldots, x_n \in X \setminus V$ so that $X \setminus V \subseteq \bigcup_{j=1}^n B_{x_j}$. Put $V_0 = \bigcap_{j=1}^n A_{x_j}$. Then $V_0$ is an open set with $x_0 \in V_0 \subseteq V$. Since $V_0$ has at most $n$ boundary points, the point $x_0$ is of finite order. That is, $X$ is an $A$-space and this completes the proof. □

**Theorem 3.3.** Let $X$ be a locally connected compact Hausdorff space. Then the following conditions are equivalent.

(i) $X$ is an AC-space.

(ii) $X$ is a C-space.

(iii) $C(X)$ is algebraically closed.

(iv) $C(X)$ is square-root closed.

(v) $\dim X \leq 1$ and $\hat{H}^1(X, \mathbb{Z}) = 0$.

(vi) $X$ is hereditarily unicoherent.

**Proof.** Since $X$ is locally connected, each connected component of $X$ is open. Therefore, $C(X)$ is algebraically closed if $X$ is a C-space. That is, (ii) implies (iii). By Theorem 3.1 and Lemma 3.1, it is enough to show that (vi) implies (i). By Lemma 3.2 it suffices to prove that $X$ is a C-space. To do this, without loss of generality we may assume that $X$ is a compact Hausdorff space which is locally connected and connected.

Let $P(x, z)$ be any monic polynomial over $C(X)$. Let $\mathcal{D}$ be the set of all pairs $(D, f)$ of $D \subseteq X$ and a complex-valued continuous function $f$ on $D$ with the following properties:

$E[a, b] \subset D$ for every $a, b \in D$ and $P(x, f(x)) = 0$ ($x \in D$).

Note that each such set $D$ is connected since $E[a, b] \subset D (a, b \in D)$. For every $(D_1, f_1), (D_2, f_2) \in \mathcal{D}$ we define $(D_1, f_1) \preceq (D_2, f_2)$, if $D_1 \subset D_2$ and $f_2|_{D_1} = f_1$. Then $\preceq$ is a partial order in $\mathcal{D}$. We denote $(D_1, f_1) \prec (D_2, f_2)$, if $(D_1, f_1) \preceq (D_2, f_2)$ and $D_1 \nsubseteq D_2$.

We show that $\mathcal{D}$ has a maximal element. To do this, let $\{(D_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$ be any chain of $\mathcal{D}$. Put $D_\emptyset = \bigcup_{\lambda \in \Lambda} D_\lambda$; then it is elementary that $a, b \in D_0$ implies $E[a, b] \subset D_0$. The function $f_0$ on $D_0$ defined by $x \mapsto f_\lambda(x)$ is well defined, where $\lambda \in \Lambda$ so that $x \in D_\lambda$. By the definition of $D_0$ and $f_0$, we have $P(x, f_0(x)) = 0$ for every $x \in D_0$. We show that $f_0$ is continuous on $D_0$. Assume to the contrary that $f_0$ is not continuous on $D_0$. That is, there exist an $x_0 \in D_0$ and an $\varepsilon_0 > 0$ such that $f_0(D_0 \cap V) \not\subseteq \{z \in \mathbb{C} : |z - f_0(x_0)| < \varepsilon_0\}$ for every open neighborhood $V$ of $x_0$. Let $z_1, z_2, \ldots, z_k$ be all the distinct roots of $P(x_0, z) = 0$. Put $2 \varepsilon_1 = \min\{|z_s - z_t| : 1 \leq s < t \leq k\}$ and $\varepsilon = \min\{|\varepsilon_0, \varepsilon_1\}$. If we apply Lemma 3.3 to each $z_i$ for $1 \leq i \leq k$, we can find a connected open neighborhood $V(x_0)$ of $x_0$ such that $P(y, w) = 0$ implies $w \in \bigcup_{i=1}^k \{z \in \mathbb{C} : |z - z_i| < \varepsilon\}$ for every $y \in V(x_0)$. Since $f_0(D_0 \cap V(x_0)) \not\subseteq \{z \in \mathbb{C} : |z - f_0(x_0)| < \varepsilon_0\}$, there is a $y_0 \in D_0 \cap V(x_0)$ such that $|f_0(y_0) - f_0(x_0)| \geq \varepsilon_0$. Let $\mu$ be an element of $\Lambda$ such that $x_0, y_0 \in D_\mu$. Since each point of $E[x_0, y_0] \setminus \{x_0, y_0\}$ separates $x_0$ and $y_0$ in $X$, we see that $E[x_0, y_0] \subset D_\mu \cap V(x_0)$. Therefore, we obtain $f_\mu(E[x_0, y_0]) = f_\mu(E[x_0, y_0]) \subset \bigcup_{i=1}^k \{z \in \mathbb{C} : |z - z_i| < \varepsilon\}$. On the other hand, $f_0(E[x_0, y_0])$ meets at least two disks of $\bigcup_{i=1}^k \{z \in \mathbb{C} : |z - z_i| < \varepsilon\}$, since
Therefore, we see that there is no continuous extension of \( f \) with the following property:

\[
\text{though } f \exists x, y \in \text{open neighborhood of } X \text{ such that } f(x) < f(y).
\]

Hence, there is no continuous extension of \( f \) to \( D^* \cup \{m\} \). Suppose that there exists a continuous extension of \( f^* \) to \( D^* \cup \{m\} \). Therefore, we have \( (D^*, f^*) \prec (D^* \cup \{m\}, f^*) \). This contradicts the maximality of \( (D^*, f^*) \). Hence we have that there is no continuous extension of \( f^* \) to \( D^* \cup \{m\} \). Therefore, there exists a \( \delta_0 > 0 \) with the following property:

To every open neighborhood \( V \) of \( m \) there correspond \( p, q \in V \cap D^* \) so that \( |f^*(p) - f^*(q)| \geq \delta_0 \).

Let \( \eta_1, \eta_2, \ldots, \eta_k \) be all the distinct roots of \( P(m, z) = 0 \). Put \( 2\delta_1 = \min\{|\eta_s - \eta_t| : 1 \leq s < t \leq k\} \) and \( \delta = \min\{\delta_0, \delta_1\} \). Then by Lemma\[\text{XXX}\] there exists a connected open neighborhood \( V(m) \) of \( m \) such that \( P(y, w) = 0 \) implies \( w \in \bigcup_{s=1}^j \{z \in \mathbb{C} : |z - \eta_j| < \delta\} \) for every \( y \in V(m) \). Therefore, \( f^*(D^* \cap V(m)) \) is contained in \( \bigcup_{s=1}^j \{z \in \mathbb{C} : |z - \eta_j| < \delta\} \). Since each point of \( E[x, y] \setminus \{x, y\} \) separates \( x \) and \( y \) in \( X \), we have \( E[x, y] \subset V(m) \) if \( x, y \in V(m) \). Hence \( E[x, y] \subset D^* \cap V(m) \) for every \( x, y \in D^* \cap V(m) \). This implies that \( D^* \cap V(m) \) is connected. Since \( f^* \) is continuous, the range \( f^*(D^* \cap V(m)) \) is also connected. Recall that there is no continuous extension of \( f^* \), thus we can find \( p, q \in D^* \cap V(m) \) with \( |f^*(p) - f^*(q)| \geq \delta_0 \). Therefore, \( f^*(D^* \cap V(m)) \) meets at least two disks of \( \bigcup_{s=1}^j \{z \in \mathbb{C} : |z - \eta_j| < \delta\} \), though \( f^*(D^* \cap V(m)) \) is connected. We arrived at a contradiction, hence \( m \in D^* \) is proved.

Finally, since \( E[m, b] \) is a totally ordered and order-complete space, there exist \( f_1, f_2, \ldots, f_n \in C(E[m, b]) \) such that

\[
P(x, z) = (z - f_1(x))(z - f_2(x)) \cdots (z - f_n(x)) \quad (x \in E[m, b]),
\]

by \[\text{XXX}\] Theorem 3. Without loss of generality we may assume \( f^*(m) = f_1(m) \). Put \( \tilde{D} = D^* \cup E[m, b] \); then the function

\[
\tilde{f}(x) = \begin{cases} 
  f^*(x), & (x \in D^*) \\
  f_1(x), & (x \in E[m, b])
\end{cases}
\]

is well defined. Note that both \( D^* \setminus \{m\} \) and \( E[m, b] \setminus \{m\} \) are open subsets of \( \tilde{D} \), since \( m \) separates \( a \) and \( b \) in \( X \) if \( m \neq a \). As a result we see that \( \tilde{f} \) is continuous on \( \tilde{D} \). Hence we obtain \( (D^*, f^*) \prec (\tilde{D}, \tilde{f}) \). This contradicts the maximality of \( (D^*, f^*) \). Therefore, we have \( D^* = X \) and this completes the proof. \( \square \)

**Theorem 3.4.** Let \( X \) be a first-countable compact Hausdorff space. Then the following conditions are equivalent.

1. \( X \) is an AC-space.
2. \( C(X) \) is algebraically closed.
(iii) $C(X)$ is square-root closed.
(iv) $X$ is almost locally connected and hereditarily unicoherent.
(v) $X$ is almost locally connected and for every connected component $X_\lambda$ of $X$, $X_\lambda$ is locally connected, $\text{dim } X_\lambda \leq 1$ and $\tilde{H}^1(X_\lambda, \mathbb{Z}) = 0$.

**Proof.** It is enough to show that (v) is equivalent to (iv). Suppose that (iv) is true. Let $X_\lambda$ be an arbitrary connected component of $X$. Note that $X_\lambda$ is almost locally connected and hereditarily unicoherent. Therefore, we see that $X_\lambda$ is locally connected by [2, Proof of Lemma 2.5]. Recall that $C(X_\lambda)$ is square-root closed by Theorem A. Therefore, we have $\text{dim } X_\lambda \leq 1$ and $\tilde{H}^1(X_\lambda, \mathbb{Z}) = 0$ by Theorem B. This implies that (iv) $\Rightarrow$ (v).

Conversely, suppose that (v) holds. Then each connected component of $X$ is hereditarily unicoherent, by Lemma 3.1. Thus $X$ is also hereditarily unicoherent. Therefore we have that (v) implies (iv). This completes the proof. □

**Example 3.1.** $C(X)$ being algebraically closed need not imply $\tilde{H}^1(X, \mathbb{Z}) = 0$ unless $X$ is locally connected. In fact, let $X$ be the Stone-Čech compactification of the space $\mathbb{N} \times [0, 1]$ with its usual topology. Then $\text{dim } X \leq 1$ and $\tilde{H}^1(X, \mathbb{Z}) \neq 0$ (cf. [6, p. 1188, (i)]). By a simple calculation, we see that $C(X)$ is algebraically closed.

Note that $X$ is hereditarily unicoherent by Theorem A. Therefore, this example also shows that the converse need not be true in Lemma 3.1.

**Example 3.2.** Lemma 3.2 need not be true, unless $X$ is locally connected. In fact, let $X$ be the closure of $\bigcup_{n \in \mathbb{N}} \{(1/n) \times [0, 1]\}$ with its usual topology. Since each connected component is hereditarily unicoherent, so is $X$. On the other hand, we see that no point of $\{0\} \times [0, 1]$ is of finite order. Hence $X$ is not an $A$-space.

**Example 3.3.** In Theorem 3.3 the condition (ii) need not imply (v) unless $X$ is first-countable. In fact, let $X$ be the Stone-Čech compactification of $\mathbb{R}$. Then by a simple calculation, we see that $C(X)$ is algebraically closed (cf. [4, Corollary]). By [9, Theorem 5.14] we have $\text{dim } X = 1$. On the other hand, $X$ is connected but is not locally connected (cf. [10, p. 221, Theorem]). Also $\tilde{H}^1(X, \mathbb{Z}) \neq 0$ since the function $e^{ix}$ ($x \in \mathbb{R}$) is extended to $X$.

**Example 3.4.** As stated above, $C(X)$ is algebraically closed, if and only if $C(X)$ is square-root closed for a compact Hausdorff space $X$ that is first-countable or locally connected. On the other hand, by the example in [2, Remark (3)], we see that it is not the case if $X$ is neither first-countable nor locally connected.

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