ON A CHARACTERIZATION
OF THE MAXIMAL IDEAL SPACES OF
ALGEBRAICALLY CLOSED COMMUTATIVE $C^*$-ALGEBRAS

TAKESHI MIURA AND KAZUKI NIJIMA

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Abstract. Let $C(X)$ be the algebra of all complex-valued continuous functions on a compact Hausdorff space $X$. We say that $C(X)$ is algebraically closed if each monic polynomial equation over $C(X)$ has a continuous solution. We give a necessary and sufficient condition for $C(X)$ to be algebraically closed for a locally connected compact Hausdorff space $X$. In this case, it is proved that $C(X)$ is algebraically closed if each element of $C(X)$ is the square of another. We also give a characterization of a first-countable compact Hausdorff space $X$ such that $C(X)$ is algebraically closed.

1. Introduction

Let $X$ be a compact Hausdorff space and $C(X)$ the commutative Banach algebra of all complex-valued continuous functions on $X$. Let $P(x, z)$ be a monic polynomial over $C(X)$. That is, $P(x, z) = z^n + a_{n-1}(x)z^{n-1} + \cdots + a_1(x)z + a_0(x)$ ($x \in X$) for some natural number $n$ and some $a_0, a_1, \ldots, a_{n-1} \in C(X)$. We say that $C(X)$ is algebraically closed if $P(x, z) = 0$ has a root in $C(X)$ for every monic polynomial $P(x, z)$ over $C(X)$; that is, there exists an $f \in C(X)$ such that $P(x, f(x)) = 0$ on $X$. If to every $a_0 \in C(X)$ there corresponds a $g \in C(X)$ so that $a_0(x) = g^2(x)$ for every $x \in X$, then we say that $C(X)$ is square-root closed. By definition, if $C(X)$ is algebraically closed then the algebra is square-root closed.

Deckard and Pearcy [3] prove that $C(X)$ is algebraically closed if $X$ is a Stonian space: a compact Hausdorff space which is also extremely disconnected. In [4] they consider two spaces: a totally disconnected compact Hausdorff space; a linearly ordered and order-complete topological space. In both cases, algebraic closedness is proved. It is also noted that $C(X)$ need not be square-root closed. Indeed, there is no continuous function on $S^1$, the unit circle in the complex plane $\mathbb{C}$, whose square is the identity function on $S^1$ (cf. [2] Lemma 2.1)). On the other hand, $C([0,1])$ is algebraically closed by [3, Theorem 2]. Here $[0,1]$ denotes the closed unit interval.

Countryman [2] gives some necessary and sufficient conditions for a first-countable compact Hausdorff space $X$ in order that $C(X)$ be algebraically closed. In this case, $C(X)$ is algebraically closed if and only if $X$ is hereditarily unicoherent.

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and almost locally connected; intuitively $X$ contains neither $S^1$ nor the closure of $\bigcup_{n \in \mathbb{N}} \{(1/n) \times [0,1]\}$ in $\mathbb{R}^2$ with its usual topology. Here and after, $\mathbb{N}$ and $\mathbb{R}$ denote the space of all natural numbers and that of real numbers, respectively. It is also proved, for such $X$, that $C(X)$ is algebraically closed if the algebra is square-root closed. On the other hand, this is not the case unless $X$ is first-countable. In fact, he gives a compact Hausdorff space $X$ that is not first-countable with the following property: each function of $C(X)$ has a continuous $2^n$th root for every $n \in \mathbb{N}$, while some function has no continuous fifth root ([2, Remark (3)]).

Recall that a uniform algebra on a compact Hausdorff space $X$ is a uniformly closed subalgebra of $C(X)$ which contains the constants and separates the points of $X$. Let $A$ be a uniform algebra on a locally connected compact Hausdorff space $X$. Čirka [1] proved that $A = C(X)$, if each function of $A$ is the square of another. As noted above, $C(X)$ need not be square-root closed even if $X$ is locally connected. In [6], a characterization is given of a locally connected compact Hausdorff space $X$ such that $C(X)$ be square-root closed: A necessary and sufficient condition in order that $C(X)$ be square-root closed is as follows.

(2) The covering dimension of $X$ is less than or equal to 1, and the first Čech cohomology group with integer coefficient is trivial.

The condition (2) need not be necessary nor sufficient for $C(X)$ being square-root closed, unless $X$ is locally connected. On the other hand, if $X$ satisfies (2) then $C(X)$ is “almost” square-root closed, even if $X$ is not locally connected, in the following sense: For every $\varepsilon > 0$ and every $f \in C(X)$ with $\|f\|_\infty \leq 1$, there exist $g, h \in C(X)$ such that $f = gh$ and $\|g - h\|_\infty \leq \varepsilon$. Here $\|\cdot\|_\infty$ denotes the supremum norm on $X$ (cf. [3], Theorem 2.1).

In this paper we consider a locally connected compact Hausdorff space $X$ which need not be first-countable. In a way similar to the arguments in [2], we show that $C(X)$ is algebraically closed if and only if $C(X)$ is square-root closed. They are also equivalent to the condition that $X$ is hereditarily unicoherent. When these conditions are satisfied, then $X$ has a base for the topology whose elements have at most finitely many boundary points. In terms of the covering dimension and the Čech cohomology group, we also give a characterization of a first-countable compact Hausdorff space $X$ for which $C(X)$ is algebraically closed.

2. Preliminaries

We state characterization theorems obtained in [2] and [6]. To do this we need some terminology.

We say that a point $p$ of a topological space $T$ is of finite order, if for every open neighborhood $V$ of $p$ there exists an open set $V_0$ such that $p \in V_0 \subset V$ and that $V_0$ has at most finitely many boundary points. In this paper we say that a compact Hausdorff space $X$ is an $A$-space, if each point of $X$ is of finite order. Note that the definition of an $A$-space is not the same in [2]. However, for compact Hausdorff spaces, the two definitions are shown to be equivalent; see [2, p. 440].

A $C$-space is a compact Hausdorff space $X$ such that $C(X_\lambda)$ is algebraically closed for every connected component $X_\lambda$. For simplicity we say that a compact Hausdorff space $X$ is an $AC$-space, if $X$ is both an $A$-space and a $C$-space.

We say that a topological space $T$ is almost locally connected, if $T$ contains no mutually disjoint connected closed subsets $C_n (n \in \mathbb{N})$, which are open in the
Theorem B (Theorem 2.2, [6]). Let \( X \) be a locally connected compact Hausdorff space. Then the following conditions are equivalent:

(i) \( X \) is square-root closed.

(ii) \( \dim X \leq 1 \) and \( \check{H}^1(X, \mathbb{Z}) = 0 \).

To prove our main result, we need the following lemma.

Lemma C (Lemma 2.2, [3]). Let \( X \) be a compact Hausdorff space, and \( P(x, z) \) a monic polynomial over \( C(X) \). Suppose that \( x_0 \in X \) and that \( z_0 \in \mathbb{C} \) is a root of \( P(x_0, z) = 0 \) of multiplicity \( m \). Let \( \varepsilon > 0 \) so that \( P(x_0, z) = 0 \) has no root in \( \{ z \in \mathbb{C} : 0 < |z - z_0| \leq \varepsilon \} \). Then there exists an open neighborhood \( V_0 \) of \( x_0 \) such that \( P(y, z) = 0 \) has exactly \( m \) roots, counting multiplicities, in \( \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \} \) for every \( y \in V_0 \).
3. Main results

It seems natural to expect that the conditions of Theorem 3.1 have deep connections with those of Theorem 3.2. In fact, the following result holds.

**Lemma 3.1.** Let \( X \) be a compact Hausdorff space with \( \dim X \leq 1 \). If \( \hat{H}^1(X, \mathbb{Z}) = 0 \), then \( X \) is hereditarily unicoherent.

**Proof.** Suppose that \( X \) is not hereditarily unicoherent. We show that \( \hat{H}^1(X, \mathbb{Z}) \neq 0 \).

In a way similar to the proof of [2, Lemma 2.1] we see that there exist a closed subset \( F \) of \( X \) and an \( h \in C(F)^{-1} \), the set of all invertible elements of \( C(F) \), such that \( h \neq g^2 \) for all \( g \in C(F)^{-1} \). Put \( h_1 = h/|h| \in C(F)^{-1} \). Since \( \dim X \leq 1 \), we can find an \( \tilde{h}_1 \in C(X)^{-1} \) such that \( |\tilde{h}_1| = 1 \) and \( \tilde{h}_1|_F = h_1 \). Then there is no function \( f \in C(X) \) such that \( \tilde{h}_1 = f^2 \). In particular, \( \tilde{h}_1 \) does not belong to \( \exp C(X) \). Therefore, we have \( C(X)^{-1} \setminus \exp C(X) \neq \emptyset \). This implies \( \hat{H}^1(X, \mathbb{Z}) \neq 0 \), by a theorem of Arens and Royden. This completes the proof. \( \square \)

Let \( T \) be a connected topological space and \( p \) a point of \( T \). Recall that \( p \) separates the distinct points \( a \) and \( b \) of \( T \setminus \{p\} \) in \( T \), if there exist disjoint open sets \( A \) and \( B \) such that \( a \in A \), \( b \in B \) and \( T \setminus \{p\} = A \cup B \). In this case both \( A \cup \{p\} \) and \( B \cup \{p\} \) are connected. If \( p \) belongs to every connected closed subset that contains both \( a \) and \( b \), then we say that \( p \) cuts \( T \) between \( a \) and \( b \). If \( X \) is a locally connected and connected compact Hausdorff space, then by [7, Theorem 3-6], a point \( p \in X \) separates the distinct points \( a \) and \( b \) of \( X \setminus \{p\} \) in \( X \), if and only if \( p \) cuts \( X \) between \( a \) and \( b \).

Let \( X \) be a connected compact Hausdorff space and \( a, b \in X \). It is well known that there exists a minimal connected closed subset of \( X \), with respect to the set inclusion, that contains both \( a \) and \( b \) (cf. [7, Theorem 2-10]). If \( X \) is also hereditarily unicoherent, then we see that such a set is uniquely determined. In this case, \( E[a, b] \) denotes the smallest connected closed subset containing both \( a \) and \( b \). Therefore, each point of \( E[a, b] \setminus \{a, b\} \) cuts \( X \) between \( a \) and \( b \), and hence separates \( a \) and \( b \) in \( X \) by [7, Theorem 3-6], if \( X \) is a locally connected and connected compact Hausdorff space which is also hereditarily unicoherent.

The separation order \( \leq \) in \( E[a, b] \) is defined as follows: for every distinct points \( p, q \in E[a, b] \) we define \( p < q \) if \( p = a \) or if \( p \) separates \( a \) and \( q \) in \( X \). Then \( x \leq y \) means that \( x = y \) or \( x < y \). It is well known that the separation order in \( E[a, b] \) is a total order [7, Theorem 2-21]. Then the order topology in \( E[a, b] \) is defined by a base for the topology: the full space \( E[a, b] \): for every \( x \in E[a, b] \) the set of all \( y \) with \( y < x \) and the set of \( y \) with \( x < y \): for every \( x < y \) the set of all \( z \) with \( x < z < y \). By [7, Theorem 2-25], the order topology defined by the total order \( \leq \) in \( E[a, b] \) is the same as the relative topology in \( E[a, b] \).

By [7, Theorem 2-26], we see that each non-empty subset of \( E[a, b] \), which is bounded above, has at least upper bound with respect to the separation order. That is, \( E[a, b] \) is order-complete. These facts will be used later.

**Lemma 3.2.** Let \( X \) be a locally connected compact Hausdorff space which is also hereditarily unicoherent. Then \( X \) is an \( A \)-space.

**Proof.** Since \( X \) is locally connected, each connected component of \( X \) is open. Thus we may assume that \( X \) is connected. Let \( x_0 \) be any point of \( X \) and \( V \) any open neighborhood of \( x_0 \). It is enough to consider the case that \( X \setminus V \neq \emptyset \). For every \( x \in X \setminus V \), let \( E[x_0, x] \) be the smallest connected closed subset containing \( x_0 \) and \( x \).
Fix any point of $V \cap (E[x_0, x] \setminus \{x_0, x\})$, say $y(x)$. Then $y(x)$ separates $x_0$ and $x$ in $X$ as noted above. That is, there exist disjoint open sets $A_x, B_x$ such that $x_0 \in A_x, x \in B_x$ and $X \setminus \{y(x)\} = A_x \cup B_x$. Note that $y(x)$ is the only boundary point of $A_x$.

Since $X \setminus V$ is compact, there are finitely many points $x_1, x_2, \ldots, x_n \in X \setminus V$ so that $X \setminus V \subset \bigcup_{j=1}^n B_{x_j}$. Put $V_0 = \bigcap_{j=1}^n A_{x_j}$. Then $V_0$ is an open set with $x_0 \in V_0 \subset V$. Since $V_0$ has at most $n$ boundary points, the point $x_0$ is of finite order. That is, $X$ is an $A$-space and this completes the proof.

**Theorem 3.3.** Let $X$ be a locally connected compact Hausdorff space. Then the following conditions are equivalent.

(i) $X$ is an AC-space.

(ii) $X$ is a $C$-space.

(iii) $C(X)$ is algebraically closed.

(iv) $C(X)$ is square-root closed.

(v) $\dim X \leq 1$ and $H^1(X, \mathbb{Z}) = 0$.

(vi) $X$ is hereditarily unicoherent.

**Proof.** Since $X$ is locally connected, each connected component of $X$ is open. Therefore, $C(X)$ is algebraically closed if $X$ is a $C$-space. That is, (ii) implies (iii). By Theorem B and Lemma 3.1, it is enough to show that (vi) implies (i). By Lemma 3.2, it suffices to prove that $X$ is a $C$-space. To do this, without loss of generality, we may assume that $X$ is a compact Hausdorff space which is locally connected and connected.

Let $P(x, z)$ be any monic polynomial over $C(X)$. Let $\mathcal{D}$ be the set of all pairs $(D, f)$ of $D \subset X$ and a complex-valued continuous function $f$ on $D$ with the following properties:

$$E[a, b] \subset D \text{ for every } a, b \in D \text{ and } P(x, f(x)) = 0 \quad (x \in D).$$

Note that each such set $D$ is connected since $E[a, b] \subset D (a, b \in D)$. For every $(D_1, f_1), (D_2, f_2) \in \mathcal{D}$ we define $(D_1, f_1) \preceq (D_2, f_2)$, if $D_1 \subset D_2$ and $f_2|_{D_1} = f_1$. Then $\preceq$ is a partial order in $\mathcal{D}$. We denote $(D_1, f_1) \prec (D_2, f_2)$, if $(D_1, f_1) \preceq (D_2, f_2)$ and $D_1 \subsetneq D_2$.

We show that $\mathcal{D}$ has a maximal element. To do this, let $\{D_\lambda, f_\lambda\}_{\lambda \in \Lambda}$ be any chain of $\mathcal{D}$. Put $D_0 = \bigcup_{\lambda \in \Lambda} D_\lambda$; then it is elementary that $a, b \in D_0$ implies $E[a, b] \subset D_0$. The function $f_0$ on $D_0$ defined by $x \mapsto f_\lambda(x)$ is well defined, where $\lambda \in \Lambda$ so that $x \in D_\lambda$. By the definition of $D_0$ and $f_0$, we have $P(x, f_0(x)) = 0$ for every $x \in D_0$. We show that $f_0$ is continuous on $D_0$. Assume to the contrary that $f_0$ is not continuous on $D_0$. That is, there exist an $x_0 \in D_0$ and an $\varepsilon_0 > 0$ such that $f_0(D_0 \cap V) \nsubseteq \{z \in \mathbb{C} : |z - f_0(x_0)| < \varepsilon_0\}$ for every open neighborhood $V$ of $x_0$. Let $z_1, z_2, \ldots, z_k$ be all the distinct roots of $P(x_0, z) = 0$. Put $2\varepsilon_1 = \min\{|z_s - z_t| : 1 \leq s < t \leq k\}$ and $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$. If we apply Lemma C to each $z_i$ for $1 \leq i \leq k$, we can find a connected open neighborhood $V(x_0)$ of $x_0$ such that $P(y, w) = 0$ implies $w \in \bigcup_{i=1}^k \{z \in \mathbb{C} : |z - z_i| < \varepsilon\}$ for every $y \in V(x_0)$. Since $f_0(D_0 \cap V(x_0)) \nsubseteq \{z \in \mathbb{C} : |z - f_0(x_0)| < \varepsilon_0\}$, there is a $y_0 \in D_0 \cap V(x_0)$ such that $|f_0(y_0) - f_0(x_0)| \geq \varepsilon_0$. Let $\mu$ be an element of $\Lambda$ such that $x_0, y_0 \in D_\mu$. Since each point of $E[x_0, y_0] \setminus \{x_0, y_0\}$ separates $x_0$ and $y_0$ in $X$, we see that $E[x_0, y_0] \subset D_\mu \cap V(x_0)$. Therefore, we obtain $f_0(E[x_0, y_0]) = f_\mu(E[x_0, y_0]) \subset \bigcup_{i=1}^k \{z \in \mathbb{C} : |z - z_i| < \varepsilon\}$. On the other hand, $f_0(E[x_0, y_0])$ meets at least two disks of $\bigcup_{i=1}^k \{z \in \mathbb{C} : |z - z_i| < \varepsilon\}$, since
\[ |f_0(y_0) - f_0(x_0)| \geq \varepsilon_0. \] This contradicts that the range \( f_\mu(E[x_0, y_0]) \) is connected. Hence, \( f_0 \) is continuous on \( D_0 \). By Zorn’s lemma we see that \( \mathcal{D} \) has a maximal element.

Let \( (D^*, f^*) \) be a maximal element of \( \mathcal{D} \). Then we show that \( D^* = X \). Assume to the contrary that \( X \setminus D^* \neq \emptyset \). Then there exists a \( b \in X \setminus D^* \). Fix any element \( a \in D^* \). Put \( m \) be the least upper bound of \( E[a, b] \cap D^* \) with respect to the separation order in \( E[a, b] \). Clearly, we see that \( E[a, m] \setminus \{m\} \subset D^* \) and \( E[m, b] \setminus \{m\} \subset X \setminus D^* \). We show that \( m \in D^* \). Suppose \( m \in X \setminus D^* \); then we see that there is no continuous extension of \( f^* \) to \( D^* \cup \{m\} \). Suppose that there exists a continuous extension of \( f^* \), say \( \tilde{f}^* \). It follows that \( P(x, \tilde{f}^*(x)) = 0 \) for every \( x \in D^* \cup \{m\} \), since the function \( x \mapsto P(x, \tilde{f}^*(x)) \) is continuous on the connected set \( D^* \cup \{m\} \) and is identically 0 on \( D^* \). Therefore, we have \( (D^*, f^*) \prec (D^* \cup \{m\}, \tilde{f}^*) \). This contradicts the maximality of \((D^*, f^*)\). Hence we have that there is no continuous extension of \( f^* \) to \( D^* \cup \{m\} \). Therefore, there exists a \( \delta_0 > 0 \) with the following property:

To every open neighborhood \( V \) of \( m \) there correspond \( p, q \in V \cap D^* \) so that \( |f^*(p) - f^*(q)| \geq \delta_0 \). Let \( \eta_1, \eta_2, \ldots, \eta_l \) be all the distinct roots of \( P(m, z) = 0 \). Put \( 2\delta_1 = \min\{|\eta_s - \eta_t| : 1 \leq s < t \leq l\} \) and \( \delta = \min\{\delta_0, \delta_1\} \). Then by Lemma \( \text{C} \) there exists a connected open neighborhood \( V(m) \) of \( m \) such that \( P(y, w) = 0 \) implies \( w \in \bigcup_{j=1}^l \{z \in \mathbb{C} : |z - \eta_j| < \delta \} \) for every \( y \in V(m) \). Therefore, \( f^*(D^* \cap V(m)) \) is contained in \( \bigcup_{j=1}^l \{z \in \mathbb{C} : |z - \eta_j| < \delta \} \). Since each point of \( E[x, y] \setminus \{x, y\} \) separates \( x \) and \( y \) in \( X \), we have \( E[x, y] \subset V(m) \) if \( x, y \in V(m) \). Hence \( E[x, y] \subset D^* \cap V(m) \) for every \( x, y \in D^* \cap V(m) \). This implies that \( D^* \cap V(m) \) is connected. Since \( f^* \) is continuous, the range \( f^*(D^* \cap V(m)) \) is also connected. Recall that there is no continuous extension of \( f^* \), thus we can find \( p, q \in D^* \cap V(m) \) with \( |f^*(p) - f^*(q)| \geq \delta_0 \). Therefore, \( f^*(D^* \cap V(m)) \) meets at least two disks of \( \bigcup_{j=1}^l \{z \in \mathbb{C} : |z - \eta_j| < \delta \} \), though \( f^*(D^* \cap V(m)) \) is connected. We arrived at a contradiction, hence \( m \in D^* \) is proved.

Finally, since \( E[m, b] \) is a totally ordered and order-complete space, there exist \( f_1, f_2, \ldots, f_n \in C(E[m, b]) \) such that

\[
P(x, z) = (z - f_1(x))(z - f_2(x)) \cdots (z - f_n(x)) \quad (x \in E[m, b]),
\]

by \( \text{H} \) Theorem 3. Without loss of generality we may assume \( f^*(m) = f_1(m) \). Put \( \tilde{D} = D^* \cup E[m, b] \); then the function

\[
\tilde{f}(x) = \begin{cases} 
  f^*(x), & (x \in D^*) \\
  f_1(x), & (x \in E[m, b])
\end{cases}
\]

is well defined. Note that both \( D^* \setminus \{m\} \) and \( E[m, b] \setminus \{m\} \) are open subsets of \( \tilde{D} \), since \( m \) separates \( a \) and \( b \) in \( X \) if \( m \neq a \). As a result we see that \( \tilde{f} \) is continuous on \( \tilde{D} \). Hence we obtain \( (D^*, f^*) \prec (\tilde{D}, \tilde{f}) \). This contradicts the maximality of \((D^*, f^*)\). Therefore, we have \( D^* = X \) and this completes the proof. \( \square \)

**Theorem 3.4.** Let \( X \) be a first-countable compact Hausdorff space. Then the following conditions are equivalent.

(i) \( X \) is an AC-space.

(ii) \( C(X) \) is algebraically closed.
(iii) $C(X)$ is square-root closed.
(iv) $X$ is almost locally connected and hereditarily unicoherent.
(v) $X$ is almost locally connected and for every connected component $X_\lambda$ of $X$, $X_\lambda$ is locally connected, $\dim X_\lambda \leq 1$ and $\hat{H}^1(X_\lambda, \mathbb{Z}) = 0$.

Proof. It is enough to show that (v) is equivalent to (iv). Suppose that (iv) is true. Let $X_\lambda$ be an arbitrary connected component of $X$. Note that $X_\lambda$ is almost locally connected and hereditarily unicoherent. Therefore, we see that $X_\lambda$ is locally connected by [2, Proof of Lemma 2.5]. Recall that $C(X_\lambda)$ is square-root closed by Theorem A. Therefore, we have $\dim X_\lambda \leq 1$ and $\hat{H}^1(X_\lambda, \mathbb{Z}) = 0$ by Theorem B. This implies that (iv) $\Rightarrow$ (v).

Conversely, suppose that (v) holds. Then each connected component of $X$ is hereditarily unicoherent, by Lemma 3.1. Thus $X$ is also hereditarily unicoherent. Therefore we have that (v) implies (iv). This completes the proof. □

Example 3.1. $C(X)$ being algebraically closed need not imply $\hat{H}^1(X, \mathbb{Z}) = 0$ unless $X$ is locally connected. In fact, let $X$ be the Stone-Čech compactification of the space $\mathbb{N} \times [0,1]$ with its usual topology. Then $\dim X \leq 1$ and $\hat{H}^1(X, \mathbb{Z}) \neq 0$ (cf. [6, p. 1188, (i)]). By a simple calculation, we see that $C(X)$ is algebraically closed.

Note that $X$ is hereditarily unicoherent by Theorem A. Therefore, this example also shows that the converse need not be true in Lemma 3.1.

Example 3.2. Lemma 3.2 need not be true, unless $X$ is locally connected. In fact, let $X$ be the closure of $\bigcup_{n \in \mathbb{N}} \{1/n\} \times [0,1]$ with its usual topology. Since each connected component is hereditarily unicoherent, so is $X$. On the other hand, we see that no point of $\{0\} \times [0,1]$ is of finite order. Hence $X$ is not an $A$-space.

Example 3.3. In Theorem 3.4, the condition (ii) need not imply (v) unless $X$ is first-countable. In fact, let $X$ be the Stone-Čech compactification of $\mathbb{R}$. Then by a simple calculation, we see that $C(X)$ is algebraically closed (cf. [4, Corollary]). By [9, Theorem 5.14] we have $\dim X = 1$. On the other hand, $X$ is connected but is not locally connected (cf. [10, p. 221, Theorem]). Also $\hat{H}^1(X, \mathbb{Z}) \neq 0$ since the function $e^{ix}$ ($x \in \mathbb{R}$) is extended to $X$.

Example 3.4. As stated above, $C(X)$ is algebraically closed, if and only if $C(X)$ is square-root closed for a compact Hausdorff space $X$ that is first-countable or locally connected. On the other hand, by the example in [2, Remark (3)], we see that it is not the case if $X$ is neither first-countable nor locally connected.

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Department of Basic Technology, Applied Mathematics and Physics, Yamagata University, Yonezawa 992-8510, Japan

E-mail address: miura@yz.yamagata-u.ac.jp

Gumma Prefectural Ōta Technical High School, 380 Motegi-chou, Ōta 373-0809, Japan