CANCELLATION OF DIRECT SUMS
OF COUNTABLE ABELIAN $p$-GROUPS

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Abstract. Let $B \oplus A_1 = C \oplus A_2$ be abelian groups where $B \cong C$ is a direct sum of countable $p$-groups. A condition is given on the Ulm-Kaplansky $p$-invariants of $B, A_1$ and $A_2$ such that $A_1 \cong A_2$.

Let $p$ denote a fixed prime number. In [3], the following result is shown for two isomorphic modular abelian group algebras: if one group is an $\aleph_1$-separable abelian $p$-group of cardinality $\aleph_1$, then the two groups are isomorphic under the assumption of MA and $\neg$ CH. A question which arises in the proof is a variation of the “substitution property” in Problem 58 in [2], namely, can a direct sum of cyclic $p$-groups be cancelled from isomorphic direct sums if the Ulm-Kaplansky invariants of the direct sum of cyclics are “disjoint” from those of the complementary groups.

In [1], Crawley proved a cancellation theorem for totally projective groups with all Ulm-Kaplansky invariants finite. Such groups are of necessity countable. We shall prove a cancellation theorem which has both Crawley’s theorem and a positive answer to the question above as corollaries. Specifically, we prove the

Theorem. Let $G = B \oplus A_1 = C \oplus A_2$, where $B \cong C$ is a direct sum of countable abelian $p$-primary groups, and $A_1$ and $A_2$ are arbitrary abelian groups. For every Ulm-Kaplansky $p$-invariant of $B$, assume that it is either finite or else the corresponding Ulm-Kaplansky invariants of $A_1$ and $A_2$ are zero. Then there exists a subgroup $D$ of $G$ such that $G = D \oplus A_1 = D \oplus A_2$. In particular, $A_1$ and $A_2$ are isomorphic.

Conjecture. The Theorem is true if $B$ is allowed to be a totally projective $p$-primary group.

In fact, we know of no counterexample if $B$ is allowed to be an arbitrary $p$-group.

All Ulm invariants will be understood to be Ulm-Kaplansky invariants for the prime $p$. We recall the definition. For an ordinal $\alpha$, the $p$-socle elements of $p$-height $\geq \alpha$ form a vector space over the integers modulo $p$. The dimension of the quotient space modulo the subspace of elements of $p$-height $> \alpha$ is the Ulm invariant at $\alpha$. The Ulm invariant at $\infty$ is the dimension of the $p$-socle of the maximal divisible $p$-subgroup. It will be useful to be able to assume that $B$ is a reduced $p$-group, so we prove a brief lemma to that effect.

Lemma 1. To prove the Theorem, we may assume that $B$ is reduced.
Proof. Assume the hypothesis of the theorem and first suppose that the theorem is true if $B$ is reduced or divisible. For arbitrary $B$, we may choose a reduced complement $B'$ (respectively, $C'$) for the maximal divisible subgroup of $B$ (respectively, $C$). Then the theorem can be applied to $B'$ to obtain $D'$ such that $G = D' \oplus B' \oplus A_1 = D' \oplus C' \oplus A_2$. Passing to $G/D'$, applying the reduced case of the theorem, and taking inverse image, one obtains $D \supseteq D'$ such that $G = D \oplus A_1 = D \oplus A_2$. Thus the lemma will be shown if we prove the theorem for the case that $B$ is divisible.

Assume now that $B$ is divisible. If the Ulm invariant of $B$ at $\infty$ is infinite, then $A_1$ and $A_2$ are reduced, hence $B = C$ and we may take $D = B$. If the Ulm invariant of $B$ is finite, then by induction and grouping appropriate summands of $B$ and $C$ with $A_1$ and $A_2$, we may assume that $B \cong \mathbb{Z}(p^\infty)$. Let $b$ and $c$ be generators for the socles of $B$ and $C$, respectively. Then $b = c' + a_2$ for some $c' \in C, a_2 \in A_2$. If $c'$ has order $p$, then $G = B \oplus A_2$, thus we may take $D = B$. Therefore we may assume that $b = a_2$ and, by symmetry, $c = a_1$ for some $a_1 \in A_1$. We may choose $D \cong \mathbb{Z}(p^\infty)$ with socle generated by $b + a_1$. Since $b + a_1 = c + a_2$, we have $G = D \oplus A_1 = D \oplus A_2$, as desired. \hfill $\Box$

The proof will need several lemmas, first treating the bounded case and then the countable case. In Lemma 4 we shall use Crawley's idea of induction on the Ulm length, which will be feasible since we may assume that $B$ is reduced. We say that two groups have disjoint Ulm invariants if corresponding Ulm invariants are never both nonzero.

Lemma 2. Let $G = B \oplus B' \oplus A_1 = C \oplus C' \oplus A_2, C \subseteq B \oplus A_1, and let $\pi : G \to C$ be the projection with kernel $C' \oplus A_2$. Assume that $C$ is a $p$-group and that $C$ and $A_1$ have disjoint Ulm invariants. Then $C/\pi(B)$ is divisible. In particular, if $C$ is bounded, then $C = \pi(B)$ and we can conclude that $G = B + (C' \oplus A_2)$.

Proof. It will suffice to show that $C \subseteq \pi(B) + pC$. Let $c \in C$ and write $c = b + a$ ($b \in B, a \in A_1$). Denote the $p$-height of an element $g \in G$ by $|g|$. We first claim that if $c$ has order $p$, then $|c| < |a|$. We may assume that $a \neq 0$, thus $a$ has order $p$ and $|c| \leq |a|$. If $|c| = |a|$, this would contradict the assumption on Ulm invariants, thus the claim is shown. Let the order of $c$ be $p^k, k \geq 1$. We will show by induction on $k$ that $c \in \pi(B) + pC$. We have $c = \pi(b) + \pi(a)$, so we must show that $\pi(a) \in \pi(B) + pC$. If $k = 1$, then by our claim, $|a| \geq 1$, hence $\pi(a) \in pC$. Now assume $k > 1$. The order of $a$ is $\leq p^k$, so by induction we may assume it is $p^k$. Consider $p^{k-1}c = p^{k-1}b + p^{k-1}a$. Applying the claim again, $k - 1 \leq |p^{k-1}c| < |p^{k-1}a|$, thus $k \leq |p^{k-1}a|$. But then $a = a' + a'' (a', a'' \in A_1)$, such that $a'$ has order $p^{k-1}$ and $|a''| \geq 1$. Thus $\pi(a) \in \pi(B) + pC$ and the induction is complete. \hfill $\Box$

Let us say that the Ulm invariant conditions apply to $B, A_1$ and $A_2$ if each Ulm invariant of $B$ is either finite or else the corresponding Ulm invariants of $A_1$ and $A_2$ are zero. The next lemma allows us to replace two isomorphic bounded direct summands by a common summand.

Lemma 3. Let $G = B_1 \oplus B_2 \oplus B' \oplus A_1 = C_1 \oplus C_2 \oplus C' \oplus A_2$ such that $B_i \cong C_i$ ($i = 1, 2), B_1 \oplus B_2 \subseteq C_1 \oplus C_2 \oplus A_2,$ and $C_1 \oplus C_2 \subseteq B_1 \oplus B_2 \oplus A_1$. Assume that $B_1$ is a bounded $p$-group, that we are given an element $u$ of the socle of $B_1$, that the Ulm
invariants of $B_1$ and $B_2$ are disjoint, and that the Ulm invariant conditions hold for $B_1, A_1$ and $A_2$. Then there exists $D$ such that:

(i) $G = D \oplus B_2 \oplus B' \oplus A_1 = D \oplus C_2 \oplus C' \oplus A_2$;
(ii) $B_1 \oplus B_2 \oplus A_1 = D \oplus B_2 \oplus A_1$ and $C_1 \oplus C_2 \oplus A_2 = D \oplus C_2 \oplus A_2$; and
(iii) $u \in D \oplus A_1$.

Proof. Note that (ii) will follow from (i) and the hypothesis of the lemma if we have $D \subseteq (B_1 \oplus B_2 \oplus A_1) \cap (C_1 \oplus C_2 \oplus A_2)$.

We shall induct on the sum of the finite Ulm invariants of $B_1$ plus the number of infinite Ulm invariants. We shall consider decompositions $B_1 = B_1' \oplus B_1''$ and $C_1 = C_1' \oplus C_1''$ such that $B_1' \cong C_1'$ and $B_1'' \cong C_1''$. We shall obtain $D$ as $D' \oplus D''$.

First suppose that $B_1$ has an infinite Ulm invariant. Then we may take $B_1'$ and $B_1''$ such that $B_1'$ is nontrivial and has Ulm invariants disjoint from those of $B_1''$ (thus $C_1''$), $A_1$ and $A_2$. Write $u = u' + u''$ ($u' \in B_1'$, $u'' \in B_1''$). Grouping $B_1''$ and $B_2$ with $A_1$ and $C_1''$ and $C_2$ with $A_2$, we may apply Lemma 4 with $B = B_1'$ and $C = C_1'$ since $C_1'' \subseteq B_1' \oplus B_2 \oplus A_1 = B_1' \oplus (B_1'' \oplus B_2 \oplus A_1)$. Thus, $G = B_1' + (C_1'' \oplus C_2 \oplus C' \oplus A_2)$.

If $g \in B_1' \cap (C_1'' \oplus C_2 \oplus C' \oplus A_2)$, then $g \in C_1'' \oplus C_2 \oplus A_2$ since $B_1 \subseteq C_1 \oplus C_2 \oplus A_2$. If $g \neq 0$, then there is an element of order $p$ in $B_1' \cap (C_1'' \oplus C_2 \oplus A_2)$. The $p$-height of such an element must occur at an ordinal for which the Ulm invariants of both $B_1'$ and $C_1'' \oplus C_2 \oplus A_2$ are nonzero. This contradicts the choice of $B_1'$ and the assumption on Ulm invariants, thus the sum for $G$ is a direct sum. Therefore, we can take $D' = B_1'$, replacing both $B_1'$ and $C_1'$. Note that $u' \in D'$. Now group $D'$ with $A_1$ and $A_2$ and apply induction to $B_1''$, replacing both $B_1''$ and $C_1''$ by $D''$, with $u'' \in D'' + (D' + A_1)$. Put $D = D' \oplus D''$.

If $B_1$ has no infinite Ulm invariant, then it is a finite group, so we may take $B_1' = (b)$ such that $u \in B_1'$. Let $C_1' = \langle c \rangle$. Then we have $b = mc + c'' + c_2 + a_2 (m \in \mathbb{Z}, c'' \in C_1', c_2 \in C_2, a_2 \in A_2)$. If $p \nmid m$, we can take $D' = B_1'$. Therefore, assume that $p \mid m$. Further, $c = nb + b'' + b_2 + a_1 (n \in \mathbb{Z}, b'' \in B_1'', b_2 \in B_2, a_1 \in A_1)$ and $b_2 = kc + p'' + \bar{c}_2 + \bar{c}_2 (k \in \mathbb{Z}, p'' \in C_1'', \bar{c}_2 \in C_2, \bar{c}_2 \in A_2)$. By the first equation, the order of $b_2$ cannot exceed the order of $c$. In the second, if $p \nmid k$, then $b_2$ generates a cyclic summand of $B_2$ of the same order as $c$, contradicting the Ulm invariants of $B_1$ and $B_2$ being disjoint. Thus $p \mid k$. Put $D' = (b + b'' + a_1)$. Clearly, $G = D' \oplus B_1'' \oplus B_2 \oplus B' \oplus A_1$. If we can show that $G = D' + (C_1'' \oplus C_2 \oplus C' \oplus A_2)$, then the sum will be direct since $D'$ and $C_1''$ have the same order, which is the index of $C_1'' \oplus C_2 \oplus C' \oplus A_2$ in $G$. It suffices to show that $c$ lies in this sum. Reading the above equations modulo $C_1'' \oplus C_2 \oplus C' \oplus A_2$, we have $b \equiv mc, c \equiv nb + b'' + b_2 + a_1$, and $b_2 \equiv kc$, hence $c \equiv (b + b'' + a_1) + (n - 1)mc + kc$. Since $p$ divides both $m$ and $k$, $c$ lies in the sum above. Now group $D'$ with $A_1$ and $A_2$ and apply induction to $B_1''$ and a generator of the socle of $\langle b'' \rangle$. Thus we get $D''$ which replaces $B_1''$ and $C_1''$. Taking $D = D' \oplus D''$, and noting that $u \in D \oplus A_1$, the induction is finished. \[\square\]

Before considering countable $B$, we prove a simple extension lemma. For $\alpha$ an ordinal, we let $G^{\alpha}$ denote the $\alpha$-th Ulm subgroup of $G$.

Lemma 4. Let $G = V \oplus H$ and $G^{\alpha} = Z \oplus H^{\alpha}$. Assume that $V$ is a $p$-group such that $V/V^{\alpha}$ is totally projective. Then there exists $X$ such that $G = X \oplus H$ and $X^{\alpha} = Z$.

Proof. We have $G^{\alpha} = V^{\alpha} \oplus H^{\alpha} = Z \oplus H^{\alpha}$. Let $\pi$ be the projection $Z \oplus H^{\alpha} \rightarrow H^{\alpha}$ with kernel $Z$. The homomorphism $\phi : V^{\alpha} \oplus H \rightarrow H$ given by $\phi(v, h) = \pi(v) + h$
does not decrease $p$-heights relative to $V \oplus H = G$. Since $V^\alpha$ is a nice subgroup of $V$ with totally projective quotient, \cite{2} Corollary 81.4] implies that $\phi$ extends to a homomorphism $\phi: V \oplus H \to H$. Let $X$ be the kernel of $\phi$. Clearly, $G = X \oplus H$. If $z \in Z$, then $z = v + h$ ($v \in V^\alpha, h \in H^\alpha$). Thus, $\phi(z) = \pi(v) + h = \pi(v) + \pi(h) = \pi(z) = 0$, and we have $Z \subseteq X$. This implies that $Z \subseteq X^\alpha$, and $G^\alpha = V^\alpha \oplus H^\alpha = Z \oplus H^\alpha$ shows that $Z = X^\alpha$.

\begin{lemma}
Assume that $G = B \oplus B' \oplus A_1 = C \oplus C' \oplus A_2, B \subseteq C \oplus A_2, C \subseteq B \oplus A_1$, and that $B \cong C$ is a countable $p$-group. Assume the Ulm invariant conditions for $B, A_1$ and $A_2$. Then there exists $D$ such that $G = D \oplus B' \oplus A_1 = D \oplus C' \oplus A_2, B \oplus A_1 = D \oplus A_1, C \oplus A_2 = D \oplus A_2$.
\end{lemma}

\begin{proof}
We induct on the Ulm length $\lambda$ of $B$. For $\lambda = 0$, we have $B = 0$ and take $D = 0$. Therefore assume $\lambda > 0$. As we noted in the proof of Lemma 5, we only need to show that $G = D \oplus B' \oplus A_1 = D \oplus C' \oplus A_2$ and $D \subseteq (C \oplus A_2) \cap (B \oplus A_1)$. We will obtain $D$ by a second induction. We shall construct $B_n, C_n$ and $D_n$ for $n < \omega$ such that:

- (a) $G = D_n \oplus B_n \oplus B' \oplus A_1 = D_n \oplus C_n \oplus C' \oplus A_2$;
- (b) $B_n$ and $C_n$ are direct summands of $B$ and $C$, respectively, $B_n \cong C_n$, and $D_n \subseteq D_{n+1}$;
- (c) $B_n \subseteq D_n \oplus C_n \oplus A_2, C_n \subseteq D_n \oplus B_n \oplus A_1$, and $D_n \subseteq (B \oplus A_1) \cap (C \oplus A_2)$;
- (d) $B_n$ and $D_n$ have disjoint Ulm invariants; and
- (e) putting $D = \bigcup_{n<\omega} D_n$, we have $B[p] \subseteq D \oplus B' \oplus A_1$ and $C[p] \subseteq D \oplus C' \oplus A_2$.

Each $D_n \oplus B' \oplus A_1$ is a direct summand of $G$, hence pure in $G$. Thus $D \oplus B' \oplus A_1 = \bigcup_{n<\omega} (D_n \oplus B' \oplus A_1)$ is a pure subgroup of $G$ containing the socles of $B, B'$ and $A_1$, hence will equal $G$. Similarly, $G = D \oplus C' \oplus A_2$. Moreover, condition (c) gives $D \subseteq (B \oplus A_1) \cap (C \oplus A_2)$, thus we will be done if we carry out the construction.

We start with $D_0 = 0, B_0 = B$ and $C_0 = C$. Enumerate the elements of the socles of $B$ and $C$ and alternate the construction so that each element of $B[p]$ or $C[p]$ lies in some $D_n \oplus B' \oplus A_1$ or $D_n \oplus C' \oplus A_2$, respectively. This will take care of condition (c). By symmetry, we may assume that $s \in B[p]$, that (a)–(d) hold (except for $D_n \subseteq D_{n+1}$), and we shall construct appropriate $B_{n+1}, C_{n+1}$, and $D_{n+1} \supseteq D_n$ such that $s \in D_{n+1} \oplus B' \oplus A_1$.

Let $u$ be the coordinate of $s$ in $B_n$ in the decomposition (a). We shall achieve $s \in D_{n+1} \oplus B' \oplus A_1$ if $u \in D_{n+1} \oplus B' \oplus A_1$. If $u = 0$, we can take $D_{n+1} = D_n, B_{n+1} = B_n$ and $C_{n+1} = C_n$, so assume that $u \neq 0$. Thus there is an ordinal $\alpha < \lambda$ such that $u \in B^{\alpha}_n \setminus B^{\alpha+1}_n$. We may decompose $B^{\alpha}_n = U \oplus U'$, where $u \in U, p'U = 0$, and $U'$ has no cyclic summand of order $\leq p'$. We claim that there is a decomposition $B_n = V \oplus B_{n+1}$ such that $V^\alpha = U, B^{\alpha}_n = U'$, and for which $V$ and $B_{n+1}$ have disjoint Ulm invariants. First we use \cite{2} Corollary 76.2 to produce countable groups $V$ and $B_{n+1}$ by specifying Ulm factors. By (b), $B_n$ is countable, thus for $\sigma < \alpha$ we may decompose the Ulm factor $(B_n)_{\sigma} = V_{\sigma} \oplus (B_{n+1})_{\sigma}$ so that each summand is an unbounded countable direct sum of cyclic groups and the Ulm invariants are disjoint. For $\sigma \geq \alpha$, we take $V_{\sigma} = (B_{n+1})_{\sigma}$ to be the appropriate Ulm factors of $U$ and $U'$, respectively. Thus we obtain countable groups $V$ and $B_{n+1}$ with the specified Ulm factors, hence with disjoint Ulm invariants. Moreover, \cite{2} Corollary 77.3] implies that $\overline{V}^\alpha \cong U, B^{\alpha}_{n+1} \cong U'$, and $B_n \cong \overline{V} \oplus B_{n+1}$ since we are dealing with countable groups with the same Ulm factors. Thus, we may assume that $\overline{V}^\alpha = U$ and $B^{\alpha}_{n+1} = U'$. By \cite{2} Corollary 77.4], there is an isomorphism
Passing to the quotient group, we have 

\[ \sim X \oplus B \cong X/Z. \]

Thus we have 

\[ \beta \leq \alpha \] and for \( \beta < \alpha \) that \( \beta \leq \alpha \). We have 

\[ B \subseteq \bigoplus_{i} C_i + A_2 \]

and 

\[ C \subseteq \bigoplus_{i} B_i + A_1, \]

thus 

\[ B \subseteq \bigoplus_{i} C_i + A_2 \] and 

\[ C \subseteq \bigoplus_{i} B_i + A_1. \]

Passing to the quotient group \( G/D \), Lemma \( \S \) applies to yield 

\[ D_{\alpha+1} \subseteq D_\beta, \]

such that  

\[ G = D_{\alpha+1} \oplus \bigoplus_{i \leq \alpha} B_i \cong D_{\alpha+1} \oplus \bigoplus_{i \leq \alpha} C_i \oplus A_2, \]

and 

\[ B \subseteq D_{\alpha+1} \oplus A_1 \] and 

\[ C \subseteq D_{\alpha+1} \oplus A_2. \]

Thus we assume that 

\[ G = D_{\alpha+1} \oplus \bigoplus_{i \leq \alpha} B_i \oplus A_1 \]

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\[ C \oplus A_2. \]

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References


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