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CANCELLATION OF DIRECT SUMS OF COUNTABLE ABELIAN p-GROUPS

RÜDIGER GÖBEL AND WARREN MAY

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ABSTRACT. Let $B \oplus A_1 = C \oplus A_2$ be abelian groups where $B \cong C$ is a direct sum of countable p-groups. A condition is given on the Ulm-Kaplansky p-invariants of B, A_1 and A_2 such that $A_1 \cong A_2$.

Let p denote a fixed prime number. In [3], the following result is shown for two isomorphic modular abelian group algebras: if one group is an $\aleph_1$-separable abelian p-group of cardinality $\aleph_1$, then the two groups are isomorphic under the assumption of MA and $\neg$ CH. A question which arises in the proof is a variation of the "substitution property" in Problem 58 in [2], namely, can a direct sum of cyclic p-groups be cancelled from isomorphic direct sums if the Ulm-Kaplansky invariants of the direct sum of cyclics are "disjoint" from those of the complementary groups. In [1], Crawley proved a cancellation theorem for totally projective groups with all Ulm-Kaplansky invariants finite. Such groups are of necessity countable. We shall prove a cancellation theorem which has both Crawley's theorem and a positive answer to the question above as corollaries. Specifically, we prove the

Theorem. *Let $G = B \oplus A_1 = C \oplus A_2$, where $B \cong C$ is a direct sum of countable abelian p-primary groups, and A_1 and A_2 are arbitrary abelian groups. For every Ulm-Kaplansky p-invariant of B, assume that it is either finite or else the corresponding Ulm-Kaplansky invariants of A_1 and A_2 are zero. Then there exists a subgroup D of G such that $G = D \oplus A_1 = D \oplus A_2$. In particular, A_1 and A_2 are isomorphic.*

Conjecture. *The Theorem is true if B is allowed to be a totally projective p-primary group.*

In fact, we know of no counterexample if B is allowed to be an arbitrary p-group.

All Ulm invariants will be understood to be Ulm-Kaplansky invariants for the prime p. We recall the definition. For an ordinal α, the p-socle elements of p-height $\geq \alpha$ form a vector space over the integers modulo p. The dimension of the quotient space modulo the subspace of elements of p-height $> \alpha$ is the Ulm invariant at α. The Ulm invariant at ∞ is the dimension of the p-socle of the maximal divisible p-subgroup. It will be useful to be able to assume that B is a reduced p-group, so we prove a brief lemma to that effect.

Lemma 1. *To prove the Theorem, we may assume that B is reduced.*

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Proof. Assume the hypothesis of the Theorem and first suppose that the Theorem is true if B is reduced or divisible. For arbitrary B, we may choose a reduced complement B' (respectively, C') for the maximal divisible subgroup of B (respectively, C). Then the Theorem can be applied to B' to obtain D' such that $G = D' \oplus B' \oplus A_1 = D' \oplus C' \oplus A_2$. Passing to G/D', applying the reduced case of the Theorem, and taking inverse image, one obtains $D \supseteq D'$ such that $G = D \oplus A_1 = D \oplus A_2$. Thus the lemma will be shown if we prove the Theorem for the case that B is divisible.

Assume now that B is divisible. If the Ulm invariant of B at ∞ is infinite, then A_1 and A_2 are reduced, hence $B = C$ and we may take $D = B$. If the Ulm invariant of B is finite, then by induction and grouping appropriate summands of B and C with A_1 and A_2, we may assume that $B \cong \mathbb{Z}(p^\infty)$. Let b and c be generators for the socles of B and C, respectively. Then $b = c' + a_2$ for some $c' \in C, a_2 \in A_2$. If c' has order p, then $G = B \oplus A_2$, thus we may take $D = B$. Therefore we may assume that $b = a_2$ and, by symmetry, $c = a_1$ for some $a_1 \in A_1$. We may choose $D \cong \mathbb{Z}(p^\infty)$ with socle generated by $b + a_1$. Since $b + a_1 = c + a_2$, we have $G = D \oplus A_1 = D \oplus A_2$, as desired. □

The proof will need several lemmas, first treating the bounded case and then the countable case. In Lemma 5 we shall use Crawley's idea of induction on the Ulm length, which will be feasible since we may assume that B is reduced. We say that two groups have *disjoint Ulm invariants* if corresponding Ulm invariants are never both nonzero.

Lemma 2. *Let* $G = B \oplus B' \oplus A_1 = C \oplus C' \oplus A_2, C \subseteq B \oplus A_1$, *and let* $\pi : G \to C$ *be the projection with kernel* $C' \oplus A_2$. *Assume that* C *is a* p*-group and that* C *and* A_1 *have disjoint Ulm invariants. Then* $C/\pi(B)$ *is divisible. In particular, if* C *is bounded, then* $C = \pi(B)$ *and we can conclude that* $G = B + (C' \oplus A_2)$.

Proof. It will suffice to show that $C \subseteq \pi(B) + pC$. Let $c \in C$ and write $c = b + a$ $(b \in B, a \in A_1)$. Denote the p-height of an element $g \in G$ by $|g|$. We first claim that if c has order p, then $|c| < |a|$. We may assume that $a \neq 0$, thus a has order p and $|c| \leq |a|$. If $|c| = |a|$, this would contradict the assumption on Ulm invariants, thus the claim is shown. Let the order of c be $p^k, k \geq 1$. We will show by induction on k that $c \in \pi(B) + pC$. We have $c = \pi(b) + \pi(a)$, so we must show that $\pi(a) \in \pi(B) + pC$. If $k = 1$, then by our claim, $|a| \geq 1$, hence $\pi(a) \in pC$. Now assume $k > 1$. The order of a is $\leq p^k$, so by induction we may assume it is p^k. Consider $p^{k-1}c = p^{k-1}b + p^{k-1}a$. Applying the claim again, $k - 1 \leq |p^{k-1}c| < |p^{k-1}a|$, thus $k \leq |p^{k-1}a|$. But then $a = a' + a''$ $(a', a'' \in A_1)$, such that a' has order p^{k-1} and $|a''| \geq 1$. Thus $\pi(a) \in \pi(B) + pC$ and the induction is complete. □

Let us say that the *Ulm invariant conditions* apply to B, A_1 and A_2 if each Ulm invariant of B is either finite or else the corresponding Ulm invariants of A_1 and A_2 are zero. The next lemma allows us to replace two isomorphic bounded direct summands by a common summand.

Lemma 3. *Let* $G = B_1 \oplus B_2 \oplus B' \oplus A_1 = C_1 \oplus C_2 \oplus C' \oplus A_2$ *such that* $B_i \cong C_i$ $(i = 1, 2), B_1 \oplus B_2 \subseteq C_1 \oplus C_2 \oplus A_2$, *and* $C_1 \oplus C_2 \subseteq B_1 \oplus B_2 \oplus A_1$. *Assume that* B_1 *is a bounded* p*-group, that we are given an element* u *of the socle of* B_1, *that the Ulm* *invariants of* B_1 *and* B_2 *are disjoint, and that the Ulm invariant conditions hold for* B_1, A_1 *and* A_2*. Then there exists* D *such that:*

(i) $G = D \oplus B_2 \oplus B' \oplus A_1 = D \oplus C_2 \oplus C' \oplus A_2$;
(ii) $B_1 \oplus B_2 \oplus A_1 = D \oplus B_2 \oplus A_1$ *and* $C_1 \oplus C_2 \oplus A_2 = D \oplus C_2 \oplus A_2$; *and*
(iii) $u \in D \oplus A_1$.

Proof. Note that (ii) will follow from (i) and the hypothesis of the lemma if we have $D \subseteq (B_1 \oplus B_2 \oplus A_1) \cap (C_1 \oplus C_2 \oplus A_2)$.

We shall induct on the sum of the finite Ulm invariants of B_1 plus the number of infinite Ulm invariants. We shall consider decompositions $B_1 = B_1' \oplus B_1''$ and $C_1 = C_1' \oplus C_1''$ such that $B_1' \cong C_1'$ and $B_1'' \cong C_1''$. We shall obtain D as $D' \oplus D''$.

First suppose that B_1 has an infinite Ulm invariant. Then we may take B_1' and B_1'' such that B_1' is nontrivial and has Ulm invariants disjoint from those of B_1'' (thus C_1''), A_1 and A_2. Write $u = u' + u''$ ($u' \in B_1', u'' \in B_1''$). Grouping B_1'' and B_2 with A_1 and C_1'' and C_2 with A_2, we may apply Lemma 2 with $B = B_1'$ and $C = C_1'$ since $C_1' \subseteq B_1 \oplus B_2 \oplus A_1 = B_1' \oplus (B_1'' \oplus B_2 \oplus A_1)$. Thus, $G = B_1' + (C_1'' \oplus C_2 \oplus C' \oplus A_2)$. If $g \in B_1' \cap (C_1'' \oplus C_2 \oplus C' \oplus A_2)$, then $g \in C_1'' \oplus C_2 \oplus A_2$ since $B_1 \subseteq C_1 \oplus C_2 \oplus A_2$. If $g \neq 0$, then there is an element of order p in $B_1' \cap (C_1'' \oplus C_2 \oplus A_2)$. The p-height of such an element must occur at an ordinal for which the Ulm invariants of both B_1' and $C_1'' \oplus C_2 \oplus A_2$ are nonzero. This contradicts the choice of B_1' and the assumption on Ulm invariants, thus the sum for G is a direct sum. Therefore, we can take $D' = B_1'$, replacing both B_1' and C_1'. Note that $u' \in D'$. Now group D' with A_1 and A_2 and apply induction to B_1'', replacing both B_1'' and C_1'' by D'', with $u'' \in D'' + (D' + A_1)$. Put $D = D' \oplus D''$.

If B_1 has no infinite Ulm invariant, then it is a finite group, so we may take $B_1' = \langle b \rangle$ such that $u \in B_1'$. Let $C_1' = \langle c \rangle$. Then we have $b = mc + c'' + c_2 + a_2 (m \in \mathbb{Z}, c'' \in C_1'', c_2 \in C_2, a_2 \in A_2)$. If $p \nmid m$, we can take $D' = B_1'$. Therefore, assume that $p \mid m$. Further, $c = nb + b'' + b_2 + a_1$ $(n \in \mathbb{Z}, b'' \in B_1'', b_2 \in B_2, a_1 \in A_1)$ and $b_2 = kc + \overline{c}'' + \overline{c}_2 + \overline{a}_2$ $(k \in \mathbb{Z}, \overline{c}'' \in C_1'', \overline{c}_2 \in C_2, \overline{a}_2 \in A_2)$. By the first equation, the order of b_2 cannot exceed the order of c. In the second, if $p \nmid k$, then b_2 generates a cyclic summand of B_2 of the same order as c, contradicting the Ulm invariants of B_1 and B_2 being disjoint. Thus $p \mid k$. Put $D' = \langle b + b'' + a_1 \rangle$. Clearly, $G = D' \oplus B_1'' \oplus B_2 \oplus B' \oplus A_1$. If we can show that $G = D' + (C_1'' \oplus C_2 \oplus C' \oplus A_2)$, then the sum will be direct since D' and C_1' have the same order, which is the index of $C_1'' \oplus C_2 \oplus C' \oplus A_2$ in G. It suffices to show that c lies in this sum. Reading the above equations modulo $C_1'' \oplus C_2 \oplus C' \oplus A_2$, we have $b \equiv mc, c \equiv nb + b'' + b_2 + a_1$, and $b_2 \equiv kc$, hence $c \equiv (b + b'' + a_1) + (n-1)mc + kc$. Since p divides both m and k, c lies in the sum above. Now group D' with A_1 and A_2 and apply induction to B_1'' and a generator of the socle of $\langle b'' \rangle$. Thus we get D'' which replaces B_1'' and C_1''. Taking $D = D' \oplus D''$, and noting that $u \in D \oplus A_1$, the induction is finished. □

Before considering countable B, we prove a simple extension lemma. For α an ordinal, we let G^α denote the α-th Ulm subgroup of G.

Lemma 4. *Let* $G = V \oplus H$ *and* $G^\alpha = Z \oplus H^\alpha$*. Assume that* V *is a* p*-group such that* V/V^α *is totally projective. Then there exists* X *such that* $G = X \oplus H$ *and* $X^\alpha = Z$*.*

Proof. We have $G^\alpha = V^\alpha \oplus H^\alpha = Z \oplus H^\alpha$. Let π be the projection $Z \oplus H^\alpha \to H^\alpha$ with kernel Z. The homomorphism $\phi : V^\alpha \oplus H \to H$ given by $\phi(v, h) = \pi(v) + h$

does not decrease p-heights relative to $V \oplus H = G$. Since V^α is a nice subgroup of V with totally projective quotient, [2, Corollary 81.4] implies that ϕ extends to a homomorphism $\overline{\phi} : V \oplus H \to H$. Let X be the kernel of $\overline{\phi}$. Clearly, $G = X \oplus H$. If $z \in Z$, then $z = v + h$ ($v \in V^\alpha, h \in H^\alpha$). Thus, $\overline{\phi}(z) = \pi(v) + h = \pi(v) + \pi(h) = \pi(z) = 0$, and we have $Z \subseteq X$. This implies that $Z \subseteq X^\alpha$, and $G^\alpha = X^\alpha \oplus H^\alpha = Z \oplus H^\alpha$ shows that $Z = X^\alpha$. □

Lemma 5. *Assume that $G = B \oplus B' \oplus A_1 = C \oplus C' \oplus A_2, B \subseteq C \oplus A_2, C \subseteq B \oplus A_1$, and that $B \cong C$ is a countable p-group. Assume the Ulm invariant conditions for B, A_1 and A_2. Then there exists D such that $G = D \oplus B' \oplus A_1 = D \oplus C' \oplus A_2, B \oplus A_1 = D \oplus A_1$, and $C \oplus A_2 = D \oplus A_2$.*

Proof. We induct on the Ulm length λ of B. For $\lambda = 0$, we have $B = 0$ and take $D = 0$. Therefore assume $\lambda > 0$. As we noted in the proof of Lemma 3, we only need to show that $G = D \oplus B' \oplus A_1 = D \oplus C' \oplus A_2$ and $D \subseteq (C \oplus A_2) \cap (B \oplus A_1)$. We will obtain D by a second induction. We shall construct B_n, C_n and D_n for $n < \omega$ such that:

(a) $G = D_n \oplus B_n \oplus B' \oplus A_1 = D_n \oplus C_n \oplus C' \oplus A_2$;
(b) B_n and C_n are direct summands of B and C, respectively, $B_n \cong C_n$, and $D_n \subseteq D_{n+1}$;
(c) $B_n \subseteq D_n \oplus C_n \oplus A_2, C_n \subseteq D_n \oplus B_n \oplus A_1$, and $D_n \subseteq (B \oplus A_1) \cap (C \oplus A_2)$;
(d) B_n and D_n have disjoint Ulm invariants; and
(e) putting $D = \bigcup_{n<\omega} D_n$, we have $B[p] \subseteq D \oplus B' \oplus A_1$ and $C[p] \subseteq D \oplus C' \oplus A_2$.

Each $D_n \oplus B' \oplus A_1$ is a direct summand of G, hence pure in G. Thus $D \oplus B' \oplus A_1 = \bigcup_{n<\omega}(D_n \oplus B' \oplus A_1)$ is a pure subgroup of G containing the socles of B, B' and A_1, hence will equal G. Similarly, $G = D \oplus C' \oplus A_2$. Moreover, condition (c) gives $D \subseteq (B \oplus A_1) \cap (C \oplus A_2)$, thus we will be done if we carry out the construction.

We start with $D_0 = 0, B_0 = B$ and $C_0 = C$. Enumerate the elements of the socles of B and C and alternate the construction so that each element of $B[p]$ or $C[p]$ lies in some $D_n \oplus B' \oplus A_1$ or $D_n \oplus C' \oplus A_2$, respectively. This will take care of condition (e). By symmetry, we may assume that $s \in B[p]$, that (a)–(d) hold (except for $D_n \subseteq D_{n+1}$), and we shall construct appropriate B_{n+1}, C_{n+1}, and $D_{n+1} \supseteq D_n$ such that $s \in D_{n+1} \oplus B' \oplus A_1$.

Let u be the coordinate of s in B_n in the decomposition (a). We shall achieve $s \in D_{n+1} \oplus B' \oplus A_1$ if $u \in D_{n+1} \oplus B' \oplus A_1$. If $u = 0$, we can take $D_{n+1} = D_n, B_{n+1} = B_n$ and $C_{n+1} = C_n$, so assume that $u \neq 0$. Thus there is an ordinal $\alpha < \lambda$ such that $u \in B_n^\alpha \setminus B_n^{\alpha+1}$. We may decompose $B_n^\alpha = U \oplus U'$, where $u \in U, p^r U = 0$, and U' has no cyclic summand of order $\leq p^r$. We claim that there is a decomposition $B_n = V \oplus B_{n+1}$ such that $V^\alpha = U, B_{n+1}^\alpha = U'$, and for which V and B_{n+1} have disjoint Ulm invariants. First we use [2, Corollary 76.2] to produce countable groups $\overline{V}$ and $\overline{B}_{n+1}$ by specifying Ulm factors. By (b), B_n is countable, thus for $\sigma < \alpha$ we may decompose the Ulm factor $(B_n)_\sigma = \overline{V}_\sigma \oplus (\overline{B}_{n+1})_\sigma$ so that each summand is an unbounded countable direct sum of cyclic groups and the Ulm invariants are disjoint. For $\sigma \geq \alpha$, we take $\overline{V}_\sigma$ and $(\overline{B}_{n+1})_\sigma$ to be the appropriate Ulm factors of U and U', respectively. Thus we obtain countable groups $\overline{V}$ and $\overline{B}_{n+1}$ with the specified Ulm factors, hence with disjoint Ulm invariants. Moreover, [2, Corollary 77.3] implies that $\overline{V}^\alpha \cong U, \overline{B}_{n+1}^\alpha \cong U'$, and $B_n \cong \overline{V} \oplus \overline{B}_{n+1}$ since we are dealing with countable groups with the same Ulm factors. Thus, we may assume that $\overline{V}^\alpha = U$ and $\overline{B}_{n+1}^\alpha = U'$. By [2, Corollary 77.4], there is an isomorphism

$B_n \cong \overline{V} \oplus \overline{B}_{n+1}$ which is the identity on B_n^α. Using this isomorphism, we pull back $\overline{V}$ and $\overline{B}_{n+1}$ to obtain $B_n = V \oplus B_{n+1}$, as desired.

Since $C_n \cong B_n$, we may write $C_n = W \oplus C_{n+1}$, where $W \cong V$ and $C_{n+1} \cong B_{n+1}$. Thus we have $G = D_n \oplus V \oplus B_{n+1} \oplus B' \oplus A_1 = D_n \oplus W \oplus C_{n+1} \oplus C' \oplus A_2$. Taking α-th Ulm subgroups, $G^\alpha = D_n^\alpha \oplus V^\alpha \oplus B_{n+1}^\alpha \oplus (B')^\alpha \oplus A_1^\alpha = D_n^\alpha \oplus W^\alpha \oplus C_{n+1}^\alpha \oplus (C')^\alpha \oplus A_2^\alpha$. If we group D_n^α with A_1^α and A_2^α, then (c) together with the arrangements we have made with Ulm invariants allow us to apply Lemma 3 to obtain Z such that:

(i) $G^\alpha = Z \oplus D_n^\alpha \oplus B_{n+1}^\alpha \oplus (B')^\alpha \oplus A_1^\alpha = Z \oplus D_n^\alpha \oplus C_{n+1}^\alpha \oplus (C')^\alpha \oplus A_2^\alpha$;
(ii) $V^\alpha \oplus D_n^\alpha \oplus B_{n+1}^\alpha \oplus A_1^\alpha = Z^\alpha \oplus D_n^\alpha \oplus B_{n+1}^\alpha \oplus A_1^\alpha$;
(iii) $W^\alpha \oplus D_n^\alpha \oplus B_{n+1}^\alpha \oplus A_1^\alpha = Z^\alpha \oplus D_n^\alpha \oplus B_{n+1}^\alpha \oplus A_1^\alpha$; and
(iv) $u \in Z \oplus A_1^\alpha$.

We now apply Lemma 4 to $D_n \oplus V \oplus B_{n+1} \oplus A_1$ and $D_n \oplus W \oplus C_{n+1} \oplus A_2$, utilizing (ii) and (iii) to obtain X and Y with $X^\alpha = Y^\alpha = Z, V \oplus D_n \oplus B_{n+1} \oplus A_1 = X \oplus D_n \oplus B_{n+1} \oplus A_1$, and $W \oplus D_n \oplus C_{n+1} \oplus A_2 = Y \oplus D_n \oplus C_{n+1} \oplus A_2$. Since $X \cong Y \cong V$, then $X/Z \cong Y/Z \cong V/V^\alpha$ satisfies the Ulm invariant conditions for $X/Z, D_n \oplus B_{n+1} \oplus A_1$, and $D_n \oplus C_{n+1} \oplus A_2$. Passing to the quotient group G/Z, induction can be used since the Ulm length of X/Z is α. Thus we get $T \supseteq Z$ such that $G = T \oplus D_n \oplus B_{n+1} \oplus B' \oplus A_1 = T \oplus D_n \oplus C_{n+1} \oplus C' \oplus A_2, X \oplus D_n \oplus B_{n+1} \oplus A_1 = T \oplus D_n \oplus B_{n+1} \oplus A_1$, and $Y \oplus D_n \oplus C_{n+1} \oplus A_2 = T \oplus D_n \oplus C_{n+1} \oplus A_2$.

Put $D_{n+1} = T \oplus D_n$. Note that $u \in D_{n+1} \oplus A_1$. All requirements of the induction step from n to $n+1$ are easily verified except possibly (c). For B_{n+1}, we have $B_{n+1} \subseteq B_n \subseteq D_n \oplus C_n \oplus A_2 = D_n \oplus W \oplus C_{n+1} \oplus A_2 = Y \oplus D_n \oplus C_{n+1} \oplus A_2 = T \oplus D_n \oplus C_{n+1} \oplus A_2 = D_{n+1} \oplus C_{n+1} \oplus A_2$. To check that $D_{n+1} \subseteq B \oplus A_1$, it suffices to show that $T \subseteq B \oplus A_1$. But previous equalities yield $T \subseteq X \oplus D_n \oplus B_{n+1} \oplus A_1, X \subseteq V \oplus D_n \oplus B_{n+1} \oplus A_1$, and $D_n \subseteq B \oplus A_1$. This completes the induction. □

Proof of the Theorem. Suppose that $B = \bigoplus_{j \in J} B'_j$ and $C = \bigoplus_{j \in J} C'_j$, where $B'_j \cong C'_j$ is countable for every $j \in J$. By a standard countable back-and-forth argument for combining summands, we may assume that $B = \bigoplus_{i<\gamma} B_i, C = \bigoplus_{i<\gamma} C_i, B_i \cong C_i$, and for every $\alpha < \gamma$, we have $\bigoplus_{i<\alpha} B_i \subseteq \bigoplus_{i<\alpha} C_i \oplus A_2$ and $\bigoplus_{i<\alpha} C_i \subseteq \bigoplus_{i<\alpha} B_i \oplus A_1$.

We shall construct D_α $(\alpha \leq \gamma)$ such that $G = D_\alpha \oplus (\bigoplus_{\alpha \leq i < \gamma} B_i) \oplus A_1 = D_\alpha \oplus (\bigoplus_{\alpha \leq i < \gamma} C_i) \oplus A_2$, and such that for $\beta < \alpha \leq \gamma$, we have $D_\beta \subseteq D_\alpha, B_\beta \subseteq D_\alpha \oplus A_1$, and $C_\beta \subseteq D_\alpha \oplus A_2$. D_γ will then be the required D. For a limit ordinal α, we take $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$. It is clear that this will work, hence we need to consider going from α to $\alpha + 1$. Thus we assume that

$$\begin{aligned} G &= D_\alpha \oplus B_\alpha \oplus (\bigoplus_{\alpha<i<\gamma} B_i) \oplus A_1 \\ &= D_\alpha \oplus C_\alpha \oplus (\bigoplus_{\alpha<i<\gamma} C_i) \oplus A_2, \end{aligned}$$

and for $\beta < \alpha$ that $B_\beta \subseteq D_\alpha \oplus A_1$ and $C_\beta \subseteq D_\alpha \oplus A_2$. We have $B_\alpha \subseteq \bigoplus_{i \leq \alpha} C_i \oplus A_2$ and $C_\alpha \subseteq \bigoplus_{i \leq \alpha} B_i \oplus A_1$, thus $B_\alpha \subseteq D_\alpha \oplus C_\alpha \oplus A_2$ and $C_\alpha \subseteq D_\alpha \oplus B_\alpha \oplus A_1$. Passing to the quotient G/D_α, Lemma 5 applies to yield $D_{\alpha+1} \supseteq D_\alpha$ such that $G = D_{\alpha+1} \oplus (\bigoplus_{\alpha+1 \leq i < \gamma} B_i) \oplus A_1 = D_{\alpha+1} \oplus (\bigoplus_{\alpha+1 \leq i < \gamma} C_i) \oplus A_2, B_\alpha \subseteq D_{\alpha+1} \oplus A_1$, and $C_\alpha \subseteq D_{\alpha+1} \oplus A_2$. The Theorem is proved.

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Fachbereich 6, Mathematik und Informatik, Universität Essen, Universitätsstr. 3, 45117 Essen, Germany
E-mail address: `R.Goebel@Uni-Essen.De`

Department of Mathematics, University of Arizona, Tucson, Arizona 85721
E-mail address: `may@math.arizona.edu`