CANCELLATION OF DIRECT SUMS OF COUNTABLE ABELIAN $p$-GROUPS

RÜDIGER GÖBEL AND WARREN MAY

(Communicated by Stephen D. Smith)

Abstract. Let $B \oplus A_1 = C \oplus A_2$ be abelian groups where $B \cong C$ is a direct sum of countable $p$-groups. A condition is given on the Ulm-Kaplansky $p$-invariants of $B, A_1$ and $A_2$ such that $A_1 \cong A_2$.

Let $p$ denote a fixed prime number. In [3], the following result is shown for two isomorphic modular abelian group algebras: if one group is a $\aleph_1$-separable abelian $p$-group of cardinality $\aleph_1$, then the two groups are isomorphic under the assumption of MA and $\neg$ CH. A question which arises in the proof is a variation of the “substitution property” in Problem 58 in [2], namely, can a direct sum of cyclic $p$-groups be cancelled from isomorphic direct sums if the Ulm-Kaplansky invariants of the direct sum of cyclics are “disjoint” from those of the complementary groups. In [1], Crawley proved a cancellation theorem for totally projective groups with all Ulm-Kaplansky invariants finite. Such groups are of necessity countable. We shall prove a cancellation theorem which has both Crawley’s theorem and a positive answer to the question above as corollaries. Specifically, we prove the

Theorem. Let $G = B \oplus A_1 = C \oplus A_2$, where $B \cong C$ is a direct sum of countable abelian $p$-primary groups, and $A_1$ and $A_2$ are arbitrary abelian groups. For every Ulm-Kaplansky $p$-invariant of $B$, assume that it is either finite or else the corresponding Ulm-Kaplansky invariants of $A_1$ and $A_2$ are zero. Then there exists a subgroup $D$ of $G$ such that $G = D \oplus A_1 = D \oplus A_2$. In particular, $A_1$ and $A_2$ are isomorphic.

Conjecture. The Theorem is true if $B$ is allowed to be a totally projective $p$-primary group.

In fact, we know of no counterexample if $B$ is allowed to be an arbitrary $p$-group.

All Ulm invariants will be understood to be Ulm-Kaplansky invariants for the prime $p$. We recall the definition. For an ordinal $\alpha$, the $p$-socle elements of $p$-height $\geq \alpha$ form a vector space over the integers modulo $p$. The dimension of the quotient space modulo the subspace of elements of $p$-height $> \alpha$ is the Ulm invariant at $\alpha$. The Ulm invariant at $\infty$ is the dimension of the $p$-socle of the maximal divisible $p$-subgroup. It will be useful to be able to assume that $B$ is a reduced $p$-group, so we prove a brief lemma to that effect.

Lemma 1. To prove the Theorem, we may assume that $B$ is reduced.
Proof. Assume the hypothesis of the Theorem and first suppose that the Theorem is true if $B$ is reduced or divisible. For arbitrary $B$, we may choose a reduced complement $B'$ (respectively, $C'$) for the maximal divisible subgroup of $B$ (respectively, $C$). Then the Theorem can be applied to $B'$ to obtain $D'$ such that $G = D' \oplus B' \oplus A_1 = D' \oplus C' \oplus A_2$. Passing to $G/D'$, applying the reduced case of the Theorem, and taking inverse image, one obtains $D \supseteq D'$ such that $G = D \oplus A_1 = D \oplus A_2$. Thus the lemma will be shown if we prove the Theorem for the case that $B$ is divisible.

Assume now that $B$ is divisible. If the Ulm invariant of $B$ at $\infty$ is finite, then $A_1$ and $A_2$ are reduced, hence $B = C$ and we may take $D = B$. If the Ulm invariant of $B$ is finite, then by induction and grouping appropriate summands of $B$ and $C$ with $A_1$ and $A_2$, we may assume that $B \cong \mathbb{Z}(p^{\infty})$. Let $b$ and $c$ be generators for the socles of $B$ and $C$, respectively. Then $b = c' + a_2$ for some $c' \in C, a_2 \in A_2$. If $c'$ has order $p$, then $G = B \oplus A_2$, thus we may take $D = B$. Therefore we may assume that $b = a_2$ and, by symmetry, $c = a_1$ for some $a_1 \in A_1$. We may choose $D \cong \mathbb{Z}(p^{\infty})$ with socle generated by $b + a_1$. Since $b + a_1 = c + a_2$, we have $G = D \oplus A_1 = D \oplus A_2$, as desired.

The proof will need several lemmas, first treating the bounded case and then the countable case. In Lemma 4 we shall use Crawley’s idea of induction on the Ulm length, which will be feasible since we may assume that $B$ is reduced. We say that two groups have disjoint Ulm invariants if corresponding Ulm invariants are never both nonzero.

**Lemma 2.** Let $G = B \oplus B' \oplus A_1 = C \oplus C' \oplus A_2, C \subseteq B \oplus A_1$, and let $\pi : G \rightarrow C$ be the projection with kernel $C' \oplus A_2$. Assume that $C$ is a $p$-group and that $C$ and $A_1$ have disjoint Ulm invariants. Then $C/\pi(B)$ is divisible. In particular, if $C$ is bounded, then $C = \pi(B)$ and we can conclude that $G = B + (C' \oplus A_2)$.

**Proof.** It will suffice to show that $C \subseteq \pi(B) + pC$. Let $c \in C$ and write $c = b + a$ ($b \in B, a \in A_1$). Denote the $p$-height of an element $g \in G$ by $\|g\|$. We first claim that if $c$ has order $p$, then $\|c\| < \|a\|$. We may assume that $a \neq 0$, thus $a$ has order $p$ and $|c| \leq |a|$. If $|c| = |a|$, this would contradict the assumption on Ulm invariants, thus the claim is shown. Let the order of $c$ be $p^k, k \geq 1$. We will show by induction on $k$ that $c \in \pi(B) + pC$. We have $c = \pi(b) + \pi(a)$, so we must show that $\pi(a) \in \pi(B) + pC$. If $k = 1$, then by our claim, $|a| \geq 1$, hence $\pi(a) \in pC$. Now assume $k > 1$. The order of $a$ is $\leq p^k$, so by induction we may assume it is $p^k$. Consider $p^{k-1}c = p^{k-1}b + p^{k-1}a$. Applying the claim again, $k - 1 \leq \|p^{k-1}c\| < \|p^{k-1}a\|$, thus $k \leq \|p^{k-1}a\|$. But then $a = a' + a'' (a', a'' \in A_1)$, such that $a'$ has order $p^{k-1}$ and $|a''| \geq 1$. Thus $\pi(a) \in \pi(B) + pC$ and the induction is complete.

Let us say that the Ulm invariant conditions apply to $B, A_1$ and $A_2$ if each Ulm invariant of $B$ is either finite or else the corresponding Ulm invariants of $A_1$ and $A_2$ are zero. The next lemma allows us to replace two isomorphic bounded direct summands by a common summand.

**Lemma 3.** Let $G = B_1 \oplus B_2 \oplus B' \oplus A_1 = C_1 \oplus C_2 \oplus C' \oplus A_2$ such that $B_i \cong C_i$ ($i = 1, 2$), $B_1 \oplus B_2 \subseteq C_1 \oplus C_2 \oplus A_2$, and $C_1 \oplus C_2 \subseteq B_1 \oplus B_2 \oplus A_1$. Assume that $B_1$ is a bounded $p$-group, that we are given an element $u$ of the socle of $B_1$, that the Ulm
invariants of $B_1$ and $B_2$ are disjoint, and that the Ulm invariant conditions hold for $B_1, A_1$ and $A_2$. Then there exists $D$ such that:

(i) $G = D \oplus B_2 \oplus B' \oplus A_1 = D \oplus C_2 \oplus C' \oplus A_2$;
(ii) $B_1 \oplus B_2 \oplus A_1 = D \oplus B_2 \oplus A_1$ and $C_1 \oplus C_2 \oplus A_2 = D \oplus C_2 \oplus A_2$; and
(iii) $u \in D \oplus A_1$.

Proof. Note that (ii) will follow from (i) and the hypothesis of the lemma if we have $D \subseteq (B_1 \oplus B_2 \oplus A_1) \cap (C_1 \oplus C_2 \oplus A_2)$.

We shall induct on the sum of the finite Ulm invariants of $B_1$ plus the number of infinite Ulm invariants. We shall consider decompositions $B_1 = B'_1 \oplus B''_1$ and $C_1 = C'_1 \oplus C''_1$ such that $B'_1 \cong C'_1$ and $B''_1 \cong C''_1$. We shall obtain $D$ as $D' \oplus D''$.

First suppose that $B_1$ has an infinite Ulm invariant. Then we may take $B'_1$ and $B''_1$ such that $B'_1$ is nontrivial and has Ulm invariants disjoint from those of $B''_1$ (thus $C''_1$), $A_1$ and $A_2$. Write $u = u' + u''$ ($u' \in B'_1, u'' \in B''_1$). Grouping $B''_1$ and $B_2$ with $A_1$ and $C''_1$ and $C_2$ with $A_2$, we may apply Lemma 2 with $B = B'_1$ and $C = C'_1$ since $C'_1 \subseteq B'_1 \oplus B_2 \oplus A_1 = B'_1 \oplus (B'_1 \oplus B_2 \oplus A_1)$. Thus, $G = B'_1 \oplus (C''_1 \oplus C_2 \oplus C' \oplus A_2)$. If $g \in B'_1 \cap (C''_1 \oplus C_2 \oplus C' \oplus A_2)$, then $g \in C''_1 \oplus C_2 \oplus A_2$ since $B'_1 \subseteq C_1 \oplus C_2 \oplus A_2$. If $g \neq 0$, then there is an element of order $p$ in $B'_1 \cap (C''_1 \oplus C_2 \oplus A_2)$. The $p$-height of such an element must occur at an ordinal for which the Ulm invariants of both $B'_1$ and $C''_1 \oplus C_2 \oplus A_2$ are nonzero. This contradicts the choice of $B'_1$ and the assumption on Ulm invariants, thus the sum for $G$ is a direct sum. Therefore, we can take $D' = B'_1$, replacing both $B'_1$ and $C'_1$. Note that $u' \in D'$. Now group $D'$ with $A_1$ and $A_2$ and apply induction to $B''_1$, replacing both $B'_1$ and $C''_1$ by $D''$, with $u'' \in D'' + (D'_1 + A_1)$. Put $D = D' \oplus D''$.

If $B_1$ has no infinite Ulm invariant, then it is a finite group, so we may take $B'_1 = \langle b \rangle$ such that $u \in B'_1$. Let $C'_1 = \langle c \rangle$. Then we have $b = mc + c'' + c_2 + a_2 (m \in \mathbb{Z}, c'' \in C''_1, c_2 \in C_2, a_2 \in A_2)$. If $p \nmid m$, we can take $D' = B'_1$. Therefore, assume that $p \mid m$. Further, $c = nb + b'' + b_2 + a_1 (n \in \mathbb{Z}, b'' \in B''_1, b_2 \in B_2, a_1 \in A_1)$ and $b_2 = kc + c'' + c_2 + a_2 (k \in \mathbb{Z}, c'' \in C''_1, c_2 \in C_2, a_2 \in A_2)$. By the first equation, the order of $b_2$ cannot exceed the order of $c$. In the second, if $p \nmid k$, then $b_2$ generates a cyclic summand of $B_2$ of the same order as $c$, contradicting the Ulm invariants of $B_1$ and $B_2$ being disjoint. Thus $p \mid k$. Put $D' = \langle b + b'' + a_1 \rangle$. Clearly, $G = D' \oplus B''_1 \oplus B_2 \oplus B' \oplus A_1$. If we can show that $G = D' \oplus (C''_1 \oplus C_2 \oplus C' \oplus A_2)$, then the sum will be direct since $D'$ and $C''_1$ have the same order, which is the index of $C''_1 \oplus C_2 \oplus C' \oplus A_2$ in $G$. It suffices to show that $c$ lies in this sum. Reading the above equations modulo $C''_1 \oplus C_2 \oplus C' \oplus A_2$, we have $b = mc, c = nb + b'' + b_2 + a_1$, and $b_2 = kc$, hence $c = (b + b'' + a_1) + (m - 1)mc$. Since $p$ divides both $m$ and $k$, $c$ lies in the sum above. Now group $D'$ with $A_1$ and $A_2$ and apply induction to $B''_1$ and a generator of the socle of $\langle b'' \rangle$. Thus we get $D''$ which replaces $B''_1$ and $C''_1$. Taking $D = D' \oplus D''$, and noting that $u \in D \oplus A_1$, the induction is finished.

Before considering countable $B$, we prove a simple extension lemma. For $\alpha$ an ordinal, we let $G^\alpha$ denote the $\alpha$-th Ulm subgroup of $G$.

**Lemma 4.** Let $G = V \oplus H$ and $G^\alpha = Z \oplus H^\alpha$. Assume that $V$ is a $p$-group such that $V/V^\alpha$ is totally projective. Then there exists $X$ such that $G = X \oplus H$ and $X^\alpha = Z$.

Proof. We have $G^\alpha = V^\alpha \oplus H^\alpha = Z \oplus H^\alpha$. Let $\pi$ be the projection $Z \oplus H^\alpha \to H^\alpha$ with kernel $Z$. The homomorphism $\phi : V^\alpha \oplus H \to H$ given by $\phi(v, h) = \pi(v) + h$
does not decrease \( p \)-heights relative to \( V \oplus H = G \). Since \( V^\alpha \) is a nice subgroup of \( V \) with totally projective quotient, [2, Corollary 81.4] implies that \( \phi \) extends to a homomorphism \( \overline{\phi} : V \oplus H \to H \). Let \( X \) be the kernel of \( \overline{\phi} \). Clearly, \( G = X \oplus H \). If \( z \in Z \), then \( z = v + h \) \((v \in V^\alpha, h \in H^\alpha)\]. Thus, \( \overline{\phi}(z) = \pi(v) + h = \pi(v) + \pi(h) = \pi(z) = 0 \), and we have \( Z \subseteq X \). This implies that \( Z \subseteq X^\alpha \), and \( G^\alpha = X^\alpha \oplus H^\alpha = Z \oplus H^\alpha \) shows that \( Z = X^\alpha \).

\( \square \)

**Lemma 5.** Assume that \( G = B \oplus B' \oplus A_1 = C \oplus C' \oplus A_2, B \subseteq C \oplus A_2, C \subseteq B \oplus A_1 \), and that \( B \cong C \) is a countable \( p \)-group. Assume the Ulm invariant conditions for \( B, A_1 \), and \( A_2 \). Then there exists \( D \) such that \( G = D \oplus B' \oplus A_1 = D \oplus C' \oplus A_2, B \oplus A_1 = D \oplus A_1, \) and \( C \oplus A_2 = D \oplus A_2 \).

**Proof.** We induct on the Ulm length \( \lambda \) of \( B \). For \( \lambda = 0 \), we have \( B = 0 \) and take \( D = 0 \). Therefore assume \( \lambda > 0 \). As we noted in the proof of Lemma 3, we only need to show that \( G = D \oplus B' \oplus A_1 = D \oplus C' \oplus A_2 \) and \( D \subseteq (C \oplus A_2) \cap (B \oplus A_1) \). We will obtain \( D \) by a second induction. We shall construct \( B_n, C_n \) and \( D_n \) for \( n < \omega \) such that:

(a) \( G = D_n \oplus B_n \oplus B' \oplus A_1 = D_n \oplus C_n \oplus C' \oplus A_2; \)

(b) \( B_n \) and \( C_n \) are direct summands of \( B \) and \( C \), respectively, \( B_n \cong C_n \), and \( D_n \subseteq D_{n+1}; \)

(c) \( B_n \subseteq D_n \oplus C_n \oplus A_2, C_n \subseteq D_n \oplus B_n \oplus A_1 \), and \( D_n \subseteq (B \oplus A_1) \cap (C \oplus A_2); \)

(d) \( B_n \) and \( D_n \) have disjoint Ulm invariants; and

(e) putting \( D = \bigcup_{n<\omega} D_n \), we have \( B[p] \subseteq D \oplus B' \oplus A_1 \) and \( C[p] \subseteq D \oplus C' \oplus A_2 \).

Each \( D_n \oplus B' \oplus A_1 \) is a direct summand of \( G \), hence pure in \( G \). Thus \( D \oplus B' \oplus A_1 = \bigcup_{n<\omega} (D_n \oplus B' \oplus A_1) \) is a pure subgroup of \( G \) containing the socles of \( B, B' \) and \( A_1 \), hence will equal \( G \). Similarly, \( G = D \oplus C' \oplus A_2 \). Moreover, condition (c) gives \( D \subseteq (B \oplus A_1) \cap (C \oplus A_2) \), thus we will be done if we carry out the construction.

We start with \( D_0 = 0, B_0 = B \) and \( C_0 = C \). Enumerate the elements of the socles of \( B \) and \( C \) and alternate the construction so that each element of \( B[p] \) or \( C[p] \) lies in some \( D_n \oplus B' \oplus A_1 \) or \( D_n \oplus C' \oplus A_2 \), respectively. This will take care of condition (e). By symmetry, we may assume that \( s \in B[p] \), that (a)-(d) hold (except for \( D_n \subseteq D_{n+1} \)), and we shall construct appropriate \( B_{n+1}, C_{n+1} \), and \( D_{n+1} \) such that \( s \in D_{n+1} \oplus B' \oplus A_1 \).

Let \( u \) be the coordinate of \( s \) in \( B_n \) in the decomposition (a). We shall achieve \( s \in D_{n+1} \oplus B' \oplus A_1 \) if \( u \in D_{n+1} \oplus B' \oplus A_1 \). If \( u = 0 \), we can take \( D_{n+1} = D_n, B_{n+1} = B_n \) and \( C_{n+1} = C_n \), so assume that \( u \neq 0 \). Thus there is an ordinal \( \alpha < \lambda \) such that \( u \in B^\alpha_n \setminus B^{\alpha+1}_n \). We may decompose \( B^\alpha_n = U \oplus U' \), where \( u \in U, p'U = 0 \), and \( U' \) has no cyclic summand of order \( < p' \). We claim that there is a decomposition \( B_n = V \oplus B_{n+1} \) such that \( V^\alpha = U, B^\alpha_{n+1} = U' \), and for which \( V \) and \( B_{n+1} \) have disjoint Ulm invariants. First we use [2, Corollary 76.2] to produce countable groups \( \overline{V} \) and \( \overline{B}_{n+1} \) by specifying Ulm factors. By (b), \( B_n \) is countable, thus for \( \sigma < \alpha \) we may decompose the Ulm factor \( (B_n)_\sigma = \overline{V}_\sigma \oplus (\overline{B}_{n+1})_\sigma \) so that each summand is an unbounded countable direct sum of cyclic groups and the Ulm invariants are disjoint. For \( \sigma \geq \alpha \), we take \( \overline{V}_\sigma \) and \((\overline{B}_{n+1})_\sigma\) to be the appropriate Ulm factors of \( U \) and \( U' \), respectively. Thus we obtain countable groups \( \overline{V} \) and \( \overline{B}_{n+1} \) with the specified Ulm factors, hence with disjoint Ulm invariants. Moreover, [2, Corollary 77.3] implies that \( \overline{V}^\alpha \cong U, \overline{B}^\alpha_{n+1} \cong U' \), and \( B_n \cong \overline{V} \oplus \overline{B}_{n+1} \) since we are dealing with countable groups with the same Ulm factors. Thus, we may assume that \( \overline{V}^\alpha = U \) and \( \overline{B}^\alpha_{n+1} = U' \). By [2, Corollary 77.4], there is an isomorphism
Passing to the quotient $X/Z$, we take $B_i^n = B_{i+1}^n + A_i$, for and for $β < α$ that $B_β ⊆ D_α + A_1$ and $C_β ⊆ D_α + A_2$. We have $B_α ⊆ \bigoplus_{i < α} C_i + A_2$ and $C_α ⊆ \bigoplus_{i < α} B_i + A_1$, thus $B_α ⊆ D_α + C_α + A_2$ and $C_α ⊆ D_α + B_α + A_1$. Passing to the quotient $G/D_α$, Lemma \ref{lem:Ulmsubgroups} applies to yield $D_{α+1} ⊆ D_α$ such that $G = D_{α+1} + \bigoplus_{i < α} B_i + A_1 = D_{α+1} + \bigoplus_{i < α} C_i + A_2, B_α ⊆ D_{α+1} + A_1$, and $C_α ⊆ D_{α+1} + A_2$. The Theorem is proved.
References


Fachbereich 6, Mathematik und Informatik, Universität Essen, Universitätsstr. 3, 45117 Essen, Germany
E-mail address: R.Goebel@Uni-Essen.De

Department of Mathematics, University of Arizona, Tucson, Arizona 85721
E-mail address: may@math.arizona.edu