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ARBITRARILY LARGE SOLUTIONS OF THE CONFORMAL SCALAR CURVATURE PROBLEM AT AN ISOLATED SINGULARITY

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Abstract. We study the conformal scalar curvature problem

$$k(x)u^{\frac{n+2}{n-2}} \leq -\Delta u \leq u^{\frac{n+2}{n-2}} \qquad \text{in} \qquad \mathbf{R}^n,\ n \geq 3,$$

where $k : \mathbf{R}^n \to (0,1]$ is a continuous function. We show that a necessary and sufficient condition on k for this problem to have C^2 positive solutions which are arbitrarily large at ∞ is that k be less than 1 on a sequence of points in $\mathbf{R}^n$ which tends to ∞.

1. Introduction

In this paper we study the conformal scalar curvature problem

$$k(x)u^{n^*} \leq -\Delta u \leq u^{n^*} \qquad \text{in} \qquad \mathbf{R}^n,\ n \geq 3, \tag{1.1}$$

where $n^* = (n+2)/(n-2)$. We also study the problem

$$k(x)u^{n^*} \leq -\Delta u \leq K(x)u^{n^*} \qquad \text{in} \qquad \mathbf{B}^n - \{0\},\ n \geq 3, \tag{1.2}$$

where $\mathbf{B}^n = \{x \in \mathbf{R}^n : |x| < 1\}$.

The following theorem gives conditions on k under which (1.1) has C^2 positive solutions which are arbitrarily large at ∞.

Theorem 1. *Let $k : \mathbf{R}^n \to (0,1]$ be a continuous function such that for some sequence $\{x^j\}_{j=1}^{\infty}$ of distinct points in $\mathbf{R}^n$ satisfying $\lim_{j\to\infty} |x^j| = \infty$ we have*

$$k(x^j) < 1 \qquad \text{for} \qquad j = 1, 2, \ldots.$$

Then, for each continuous function $\varphi : (1,\infty) \to (0,\infty)$, there exists a C^2 positive solution $u(x)$ of (1.1) satisfying

$$u(x) \neq \mathcal{O}(\varphi(|x|)) \qquad \text{as} \qquad |x| \to \infty.$$

The following corollary of Theorem 1 gives necessary and sufficient conditions on the function k for C^2 positive solutions of (1.1) to satisfy an a priori bound at ∞.

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Corollary 1. *Let $k : \mathbf{R}^n \to (0,1]$ be a continuous function. Then there exists a continuous function $\varphi : (1,\infty) \to (0,\infty)$ such that each C^2 positive solution $u(x)$ of* (1.1) *satisfies*

$$u(x) = \mathcal{O}(\varphi(|x|)) \qquad as \qquad |x| \to \infty \tag{1.3}$$

if and only if k is identically equal to 1 in the complement of some compact subset of $\mathbf{R}^n$. In this case, (1.3) *holds with $\varphi(r) = r^{-(n-2)/2}$.*

Proof. The "only if" part of Corollary 1 follows from Theorem 1. The "if" part was proved by Caffarelli, Gidas, and Spruck [1].

The next corollary of Theorem 1 deals with the case the isolated singularity of (1.1) is at the origin instead of at ∞.

Corollary 2. *Let $\kappa : (\mathbf{R}^n - \{0\}) \to (0,1]$ be a continuous function such that for some sequence $\{y^j\}_{j=1}^\infty$ of distinct points in $\mathbf{R}^n - \{0\}$ satisfying $\lim_{j\to\infty} |y^j| = 0$ we have*

$$\kappa(y^j) < 1 \qquad for \qquad j = 1, 2, \ldots.$$

Then, for each continuous function $\varphi : (0,1) \to (0,\infty)$, there exists a C^2 positive solution $v(y)$ of

$$\begin{aligned} &\kappa(y)v^{n^*} \le -\Delta v \le v^{n^*} \qquad in \qquad \mathbf{R}^n - \{0\},\ n \ge 3, \\ &\lim_{|y|\to\infty} |y|^{n-2} v(y) = L \qquad for\ some\ L \in (0,\infty) \end{aligned} \tag{1.4}$$

satisfying

$$v(y) \ne \mathcal{O}(\varphi(|y|)) \qquad as \qquad |y| \to 0^+.$$

Proof. Clearly there exists a continuous function $\hat\kappa : (\mathbf{R}^n - \{0\}) \to (0,1]$ such that $\kappa \le \hat\kappa$ in $\mathbf{R}^n - \{0\}$, $\hat\kappa(y^j) < 1$ for $j = 1, 2, \ldots$, and $\hat\kappa$ is identically equal to 1 in the complement of some compact subset of $\mathbf{R}^n$. It therefore suffices to prove Corollary 2 under the assumption that κ is identically equal to 1 in the complement of some compact subset of $\mathbf{R}^n$. Define $k : \mathbf{R}^n \to (0,1]$ by $k(0) = 1$ and $k(x) = \kappa(x/|x|^2)$ for $x \in \mathbf{R}^n - \{0\}$. Then k is continuous and the Kelvin transform, $v(y) = |x|^{n-2}u(x)$, $x = y/|y|^2$, of each C^2 positive solution $u(x)$ of (1.1) is a C^2 positive solution of (1.4) with $L = u(0)$. Thus Corollary 2 follows from Theorem 1.

The following theorem gives conditions on k and K under which (1.2) has C^2 positive solutions which are arbitrarily large at the origin.

Theorem 2. *Let $k, K : (\mathbf{B}^n - \{0\}) \to \mathbf{R}$ be continuous functions such that for some positive constants a and b we have*

$$a \le k \le K \le b \qquad in \qquad \mathbf{B}^n - \{0\},$$

and for some sequence $\{x^j\}_{j=1}^\infty$ of distinct points in $\mathbf{B}^n - \{0\}$ satisfying $\lim_{j\to\infty} |x^j| = 0$ we have

$$k(x^j) < K(x^j) \qquad for \qquad j = 1, 2, \ldots.$$

Then, for each continuous function $\varphi : (0,1) \to (0,\infty)$, there exists a C^1 positive solution $u(x)$ of (1.2) *satisfying*

$$u(x) \ne \mathcal{O}(\varphi(|x|)) \qquad as \qquad |x| \to 0^+. \tag{1.5}$$

If, in addition, either k or K is locally Hölder continuous with exponent α in $\mathbf{B}^n - \{0\}$ for some $\alpha \in (0,1)$, then, for each continuous function $\varphi : (0,1) \to (0,\infty)$, there exists a C^2 positive solution $u(x)$ of (1.2) *satisfying* (1.5).

In light of Corollary 2 and Theorem 2, it is natural to ask the

Open question. *Is Theorem 2 true in* $\mathbf{R}^n - \{0\}$*?*

By ingeniously piecing together modified Delaunay–Fowler type solutions, Leung [4] very recently proved, for each positive constant ε and each continuous function $\varphi : (0,1) \to (0,\infty)$, that the problem

$$(1-\varepsilon)u^{n^*} \le -\Delta u \le u^{n^*} \qquad \text{in} \qquad \mathbf{R}^n - \{0\},\ n \ge 3,$$

has a C^2 positive solution $u(x)$ satisfying (1.5). Our method of proving Theorem 1, consisting of piecing together the simpler standard bubbles (see Section 2), is very different than his, and our proof is significantly shorter. Leung also shows that there exists a positive *Lipschitz* continuous function $K(x)$ on $\mathbf{R}^n$ such that the equation

$$-\Delta u = K(x)u^{n^*} \qquad \text{in} \qquad \mathbf{R}^n - \{0\},\ n > 4,$$

has a C^2 positive solution $u(x)$ not satisfying

$$u(x) = \mathcal{O}(|x|^{-(n-2)/2}) \qquad \text{as} \qquad |x| \to 0^+. \tag{1.6}$$

In the other direction, Korevaar, Mazzeo, Pacard, and Schoen [3] obtain precise asymptotic estimates at the origin for singular positive solutions of

$$-\Delta u = K(x)u^{n^*} \qquad \text{in} \qquad \mathbf{B}^n - \{0\},\ n \ge 3, \tag{1.7}$$

when $K(x) \equiv 1$. In the case of non-constant K, Chen and Lin [2], [5] and Zhang [9] give conditions on K such that every C^2 positive solution $u(x)$ of (1.7) satisfies (1.6).

Finally, Taliaferro [7], [8] has studied arbitrarily large solutions of (1.1) with the exponent n^* replaced by a constant $\lambda < n^*$.

2. Proofs

In this section we prove Theorems 1 and 2. But first we introduce some notation and state some easily verified facts that will be used in these proofs.

Let

$$w(r,\sigma) = \frac{[n(n-2)]^{\frac{n-2}{4}}\sigma^{\frac{n-2}{2}}}{(\sigma^2+r^2)^{\frac{n-2}{2}}}.$$

It is well-known that the function $U(x) := w(|x|,\sigma)$, which is sometimes called a bubble, satisfies $-\Delta U = U^{n^*}$ in $\mathbf{R}^n$ for each positive constant σ. Thus letting

$$\rho(x) = w(|x|,1)/2^{n/2}$$

we have

$$-\Delta\rho = 2^{n^*+1}\rho^{n^*} \qquad \text{in} \qquad \mathbf{R}^n. \tag{2.1}$$

As $\sigma \to 0^+$, $w(|x|,\sigma)$ and each of its partial derivatives with respect to x converges uniformly to zero on each closed subset of $\mathbf{R}^n - \{0\}$.

Define $f : [0,\infty) \times (0,1) \times (0,\infty) \to \mathbf{R}$ and $M : (0,1) \times (0,\infty) \to \mathbf{R}$ by

$$f(z,\psi,\zeta) = \psi(\zeta+z)^{n^*} - z^{n^*} \qquad \text{and} \qquad M(\psi,\zeta) = \frac{\psi\zeta^{n^*}}{\left(1-\psi^{\frac{1}{n^*-1}}\right)^{n^*-1}}.$$

For each fixed $(\psi,\zeta)\in(0,1)\times(0,\infty)$, the function $f(\cdot,\psi,\zeta):\ [0,\infty)\to\mathbf{R}$ assumes its maximum at

$$z_{(\psi,\zeta)}=\frac{\zeta\psi^{\frac{1}{n^*-1}}}{1-\psi^{\frac{1}{n^*-1}}}\qquad\text{and}\qquad f(z_{(\psi,\zeta)},\psi,\zeta)=M(\psi,\zeta).$$

Let N be the Newtonian potential operator over $\mathbf{R}^n$ defined by

$$(Ng)(x)=\frac{1}{(n-2)n\omega_n}\int_{\mathbf{R}^n}\frac{1}{|x-y|^{n-2}}g(y)\,dy$$

where ω_n is the volume of $\mathbf{B}^n$.

Proof of Theorem 1. Clearly there exists a C^∞ function $\hat{k}:\mathbf{R}^n\to(0,1]$ such that $k\le\hat{k}$ in $\mathbf{R}^n$ and $\hat{k}(x^j)<1$ for $j=1,2,\dots$. It therefore suffices to prove Theorem 1 under the assumption that k is C^∞ in $\mathbf{R}^n$.

Choose a sequence $\{r_j\}_{j=1}^\infty\subset(0,1)$ such that

$$B_{2r_i}(x^i)\cap B_{2r_j}(x^j)=\emptyset\qquad\text{for}\qquad i\ne j$$

and $k<1$ in $\overline{B_{r_j}(x^j)}$. Since $M(k(x),2\rho(x))$ is bounded in each of the balls $B_{r_j}(x^j)$ (but not in their union), by sufficiently decreasing each r_j, we can force the function

$$\hat{M}:=\begin{cases}M_j:=\sup_{B_{r_j}(x^j)}M(k(x),2\rho(x)) & \text{in } B_{r_j}(x^j),\\ 0 & \text{in } \mathbf{R}^n-\bigcup_{j=1}^\infty B_{2r_j}(x^j),\\ (2-|x-x^j|/r_j)M_j & \text{in } B_{2r_j}(x^j)-B_{r_j}(x^j)\end{cases}$$

to satisfy

$$N\hat{M}<\frac{1}{2}\rho\qquad\text{in}\qquad\mathbf{R}^n. \tag{2.2}$$

(More precisely, we can force $\hat{M}$ to satisfy (2.2) by choosing r_j positive and so small that the Newtonian potential of the characteristic function of $B_{2r_j}(x^j)$ is less than $\rho/(2^{j+1}M_j)$ in $\mathbf{R}^n$. This is possible because ρ is positive and decays at ∞ like $|x|^{2-n}$.) Since $\hat{M}$ is locally Lipschitz continuous in $\mathbf{R}^n$ we have $\bar{v}:=\rho/2+N\hat{M}\in C^2(\mathbf{R}^n)$ and

$$-\Delta\bar{v}=2^{n^*}\rho^{n^*}+\hat{M}\qquad\text{in}\qquad\mathbf{R}^n \tag{2.3}$$

by (2.1). It follows from (2.2) that

$$\frac{1}{2}\rho<\bar{v}<\rho\qquad\text{in}\qquad\mathbf{R}^n. \tag{2.4}$$

Let ε_j be a sequence of positive numbers such that $\sum\limits_{j=1}^\infty\varepsilon_j=1$. For each positive integer j, choose σ_j positive and so small that $u_j(x):=w(|x-x^j|,\sigma_j)$ satisfies

$$u_j<\varepsilon_j\rho\qquad\text{in}\qquad\mathbf{R}^n-B_{r_j}(x^j), \tag{2.5}$$

$$u_j(x^j)>j\varphi(|x^j|), \tag{2.6}$$

$\sum\limits_{j=1}^\infty u_j\in C^\infty(\mathbf{R}^n)$, and

$$-\Delta\left(\sum_{j=1}^\infty u_j\right)=\sum_{j=1}^\infty u_j^{n^*}\qquad\text{in}\qquad\mathbf{R}^n. \tag{2.7}$$

Define $\underline{H}, H : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ by

$$\underline{H}(x,v) = k(x)\left(v + \sum_{j=1}^{\infty} u_j(x)\right)^{n^*} - \sum_{j=1}^{\infty} u_j(x)^{n^*},$$
$$H(x,v) = \max\{0, \underline{H}(x,v)\}.$$

For $x \in B_{r_j}(x^j)$ and $0 \le v \le \rho(x)$ it follows from (2.5) that

$$\begin{aligned}\underline{H}(x,v) &< k(x)\left(v + u_j(x) + \sum_{i\neq j} u_i(x)\right)^{n^*} - u_j(x)^{n^*}\\ &< k(x)\left(2\rho(x) + u_j(x)\right)^{n^*} - u_j(x)^{n^*}\\ &= f\left(u_j(x), k(x), 2\rho(x)\right)\\ &\le M\left(k(x), 2\rho(x)\right) \le \hat{M}(x) \le -(\Delta\bar{v})(x)\end{aligned}$$

by (2.3). Also, for $x \in \mathbf{R}^n - \bigcup_{j=1}^{\infty} B_{r_j}(x^j)$ and $0 \le v \le \rho(x)$ we have

$$\underline{H}(x,v) < k(x)(2\rho(x))^{n^*} \le -(\Delta\bar{v})(x).$$

Hence

$$0 \le H(x,v) < -(\Delta\bar{v})(x) \qquad \text{for} \qquad x \in \mathbf{R}^n \text{ and } 0 \le v \le \rho(x)$$

because (2.3) implies that $-\Delta\bar{v} > 0$ in $\mathbf{R}^n$.

Thus by (2.4), for each positive integer i we can use $\underline{v} :\equiv 0$ and $\bar{v}$ as sub and super solutions of the problem

$$\begin{aligned} -\Delta v = H(x,v) &\quad \text{in } B_i(0),\\ v = 0 &\quad \text{on } \partial B_i(0)\end{aligned}$$

to conclude that this problem has a C^2 solution v_i satisfying $0 \le v_i \le \rho$. It follows from standard elliptic theory that some subsequence of v_i converges to a C^2 solution u_0 of

$$\left.\begin{aligned} -\Delta u_0 = H(x,u_0)\\ 0 \le u_0 \le \rho \end{aligned}\right\} \quad \text{in} \quad \mathbf{R}^n. \tag{2.8}$$

Clearly $\underline{H} \le H \le \overline{H}$ in $\mathbf{R}^n \times [0,\infty)$ where

$$\overline{H}(x,v) = \left(v + \sum_{j=1}^{\infty} u_j(x)\right)^{n^*} - \sum_{j=1}^{\infty} u_j(x)^{n^*}$$

and thus

$$\underline{H}(x,u_0(x)) \le H(x,u_0(x)) \le \overline{H}(x,u_0(x)) \qquad \text{in} \qquad \mathbf{R}^n$$

which together with (2.7) and (2.8) implies that $u := \sum_{j=0}^{\infty} u_j$ is a C^2 positive solution of (1.1). Hence Theorem 1 follows from (2.6).

Proof of Theorem 2. By scaling u we see that it suffices to prove Theorem 2 under the assumption that $b = 1$.

Choose a sequence $\{r_j\}_{j=1}^{\infty} \subset (0,1)$ such that $B_{2r_j}(x^j) \subset \mathbf{B}^n - \{0\}$,

$$B_{2r_i}(x^i) \cap B_{2r_j}(x^j) = \emptyset \qquad \text{for} \qquad i \neq j,$$

and $k(x)/K_j < 1$ in $\overline{B_{r_j}(x^j)}$, where $K_j = \inf_{B_{r_j}(x^j)} K$. Since $M(k(x)/K_j, 2\rho(x))$ is bounded in each of the balls $B_{r_j}(x^j)$ (but not in their union), by sufficiently decreasing each r_j, we can force the function

$$\hat{M} := \begin{cases} M_j := \sup_{B_{r_j}(x^j)} M(k(x)/K_j, 2\rho(x)) & \text{in } B_{r_j}(x^j), \\ 0 & \text{in } \mathbf{R}^n - \bigcup_{j=1}^{\infty} B_{2r_j}(x^j), \\ (2 - |x - x^j|/r_j)M_j & \text{in } B_{2r_j}(x^j) - B_{r_j}(x^j) \end{cases}$$

to satisfy

$$N\hat{M} < \frac{1}{2}\rho \qquad \text{in} \qquad \mathbf{R}^n. \tag{2.9}$$

(See the parenthetical remark after equation (2.2).) Since $\hat{M}$ is locally Lipschitz continuous in $\mathbf{R}^n - \{0\}$ we have $\bar{v} := \rho/2 + N\hat{M} \in C^2(\mathbf{R}^n - \{0\})$ and

$$-\Delta \bar{v} = 2^{n^*}\rho^{n^*} + \hat{M} \qquad \text{in} \qquad \mathbf{R}^n - \{0\} \tag{2.10}$$

by (2.1). It follows from (2.9) that

$$\frac{1}{2}\rho < \bar{v} < \rho \qquad \text{in} \qquad \mathbf{R}^n. \tag{2.11}$$

Let

$$\underline{v}(x) \equiv \beta := \inf_{\mathbf{B}^n} \bar{v} \qquad \text{in} \qquad \mathbf{B}^n.$$

Clearly $\beta > 0$ and later $\underline{v}$ and $\bar{v}$ will be used as sub and super solutions.

Let ε_j be a sequence of positive numbers such that $\sum_{j=1}^{\infty} \varepsilon_j = a$. For each positive integer j, choose σ_j positive and so small that $u_j(x) := w(|x - x^j|, \sigma_j)/K_j^{1/(n^*-1)}$ satisfies

$$u_j < \beta\varepsilon_j \qquad \text{in} \qquad \mathbf{R}^n - B_{r_j}(x^j), \tag{2.12}$$

$$u_j(x^j) > j\varphi(|x^j|), \tag{2.13}$$

$\sum_{j=1}^{\infty} u_j \in C^{\infty}(\mathbf{R}^n - \{0\})$, and

$$-\Delta\left(\sum_{j=1}^{\infty} u_j\right) = \sum_{j=1}^{\infty} K_j u_j^{n^*} \qquad \text{in} \qquad \mathbf{R}^n - \{0\}. \tag{2.14}$$

Define $\underline{H}, H : (\mathbf{B}^n - \{0\}) \times [\beta, \infty) \to \mathbf{R}$ by

$$\underline{H}(x,v) = k(x)\left(v + \sum_{j=1}^{\infty} u_j(x)\right)^{n^*} - \sum_{j=1}^{\infty} K_j u_j(x)^{n^*},$$

$$H(x,v) = \max\{0, \underline{H}(x,v)\}.$$

For $x \in B_{r_j}(x^j)$ and $\beta \le v \le \rho(x)$ it follows from (2.12) that

$$\begin{aligned}\underline{H}(x,v) &< k(x)\left(v + u_j(x) + \sum_{i\neq j} u_i(x)\right)^{n^*} - K_j u_j(x)^{n^*} \\ &\le k(x)\left(2\rho(x) + u_j(x)\right)^{n^*} - K_j u_j(x)^{n^*} \\ &= K_j f\left(u_j(x), k(x)/K_j, 2\rho(x)\right) \\ &\le K_j M\left(k(x)/K_j, 2\rho(x)\right) \le \hat{M}(x) \le -(\Delta\bar{v})(x)\end{aligned}$$

by (2.10). Also, for $x \in (\mathbf{B}^n - \{0\}) - \bigcup_{j=1}^{\infty} B_{r_j}(x^j)$ and $\beta \le v \le \rho(x)$, we have

$$\underline{H}(x,v) < k(x)(2\rho(x))^{n^*} \le -(\Delta\bar{v})(x).$$

Hence

$$0 \le H(x,v) < -(\Delta\bar{v})(x) \qquad \text{for} \qquad x \in \mathbf{B}^n - \{0\} \text{ and } \beta \le v \le \rho(x)$$

because (2.10) implies that $-\Delta\bar{v} > 0$ in $\mathbf{R}^n - \{0\}$.

Thus by (2.11), for each integer $i > 2$ we can use $\underline{v}$ and $\bar{v}$ as sub and super solutions of the problem

$$\begin{aligned} -\Delta v = H(x,v) \qquad & \text{in } \tfrac{1}{i} < |x| < 1 - \tfrac{1}{i}, \\ v = \beta \qquad & \text{for } |x| = \tfrac{1}{i} \text{ or } |x| = 1 - \tfrac{1}{i} \end{aligned}$$

to conclude that this problem has a C^1 solution v_i satisfying $\beta \le v_i \le \rho$. It follows from standard elliptic theory that some subsequence of v_i converges to a C^1 solution u_0 of

$$\left.\begin{aligned} -\Delta u_0 = H(x,u_0) \\ \beta \le u_0 \le \rho \end{aligned}\right\} \quad \text{in} \quad \mathbf{B}^n - \{0\}. \tag{2.15}$$

Defining $\overline{H} : (\mathbf{B}^n - \{0\}) \times [\beta, \infty) \to \mathbf{R}$ by

$$\overline{H}(x,v) = K(x)\left(v + \sum_{j=1}^{\infty} u_j(x)\right)^{n^*} - \sum_{j=1}^{\infty} K_j u_j(x)^{n^*}$$

we have for $v \ge \beta$ and $x \in (\mathbf{B}^n - \{0\}) - \bigcup_{j=1}^{\infty} B_{r_j}(x^j)$ that

$$\begin{aligned}\overline{H}(x,v) &\ge a\beta^{n^*} - \sum_{j=1}^{\infty} (\varepsilon_j \beta)^{n^*} \\ &= \beta^{n^*}\left(a - \sum_{j=1}^{\infty} \varepsilon_j^{n^*}\right) > 0\end{aligned}$$

and we have for $v \geq \beta$ and $x \in B_{r_j}(x^j)$ that

$$\begin{aligned}\overline{H}(x,v) &\geq K_j(\beta + u_j(x))^{n^*} - K_j u_j(x)^{n^*} - \sum_{i \neq j}^{\infty} (\beta\varepsilon_i)^{n^*} \\ &\geq K_j \beta^{n^*} - \beta^{n^*} \sum_{i \neq j}^{\infty} \varepsilon_i^{n^*} \\ &\geq \beta^{n^*} \left(a - \sum_{j=1}^{\infty} \varepsilon_j^{n^*} \right) > 0.\end{aligned}$$

Therefore $\overline{H}$ is a positive function and hence $\underline{H} \leq H \leq \overline{H}$ in $(\mathbf{B}^n - \{0\}) \times [\beta, \infty)$. Thus

$$\underline{H}(x, u_0(x)) \leq H(x, u_0(x)) \leq \overline{H}(x, u_0(x)) \qquad \text{in} \qquad \mathbf{B}^n - \{0\}$$

which together with (2.14) and (2.15) implies that $u := \sum_{j=0}^{\infty} u_j$ is a C^1 positive solution of (1.2). Hence Theorem 2, except for its last sentence, follows from (2.13).

The last sentence of Theorem 2 follows from the following two observations:

(i) If $k \in C^\alpha(\mathbf{B}^n - \{0\})$, then a C^1 solution u_0 of (2.15) is necessarily a C^2 solution.

(ii) If $K \in C^\alpha(\mathbf{B}^n - \{0\})$, then there exists a function $\hat{k} \in C^\alpha(\mathbf{B}^n - \{0\})$ satisfying $k \leq \hat{k} \leq K$ in $\mathbf{B}^n - \{0\}$ and $\hat{k}(x^j) < K(x^j)$ for $j = 1, 2, \ldots$.

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