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AN EXAMPLE IN HOLOMORPHIC FIXED POINT THEORY

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ABSTRACT. If B is the open unit ball in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm $\|\cdot\|$, where $1 < p < \infty$ and $p \neq 2$, then a holomorphic self-mapping f of B has a fixed point if and only if $\sup_{n\in\mathbb{N}} \|f^n(x)\| < 1$ for some $x \in B$.

1. INTRODUCTION

There is an old problem in the fixed point theory of holomorphic mappings to find an example of a complex infinite-dimensional Banach space $(X, \|\cdot\|)$ such that its open unit ball B is holomorphically nonequivalent to the Cartesian product of a finite number of Hilbert balls $\prod_{j=1}^m B_{H_j}$ and each holomorphic self-mapping f of B has a fixed point if and only if $\sup_{n\in\mathbb{N}} \|f^n(x)\| < 1$ for some $x \in B$. In this paper we show that the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm $\|\cdot\|$, where $1 < p < \infty$ and $p \neq 2$, has the claimed properties. Finally, let us recall that the Hilbert ball B_H has the same properties [7], [8], [12]. However, if J is an infinite set of indices,

$$l^\infty(H) = \left\{ x = \{x_j\}_{j\in J} \in \prod_{j\in J} H : \sup_{j\in J} \|x_j\| < \infty \right\},$$

and B_H^∞ the open unit ball in $l^\infty(H)$ with the supremum norm, then B_H^∞ is not holomorphically equivalent to the Cartesian product of a finite number of Hilbert balls $\prod_{j=1}^m B_{H_j}$ and there exists a holomorphic self-mapping f of B_H^∞ without a fixed point and with $\sup_{n\in\mathbb{N}} \|f^n(x)\| < 1$ for each $x \in B_H^\infty$ [14], [15].

2. PRELIMINARIES

All Banach spaces will be complex. Throughout this paper B denotes an open unit ball in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm $\|\cdot\|$ (i.e.,

$$\|(w, z)\| = [\|w\|_2^p + \|z\|_2^p]^{\frac{1}{p}},$$

$w, z \in l^2$ and $\|\cdot\|_2$ is the standard norm in l^2), where $1 < p < \infty$ and $p \neq 2$. By k_B we denote the Kobayashi distance on B [10], [14]. We now recall several useful

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properties of the Kobayashi distance k_B, which are common for all bounded and convex domains in reflexive Banach spaces.

The Kobayashi distance k_B is locally equivalent to the norm [9].

If $x, y, w, z \in B$ and $s \in [0,1]$, then

$$k_B\left(sx + (1-s)\,y, sw + (1-s)\,z\right) \le \max\left\{k_B\left(x,w\right), k_B\left(y,z\right)\right\}.$$

Hence each open (closed) k_B-ball in the metric space (B, k_B) is convex [16].

A subset C of B is said to lie strictly inside B if $dist_{\|\cdot\|}\left(C, \partial B\right) > 0$.

The basic fact about subsets, which lie strictly inside B, is the following: a subset C of B is k_B-bounded if and only if C lies strictly inside B (Proposition 23 in [9]).

We can say more about linear convexity of balls in (B, k_B). Since the open unit ball B in $l^2 \times l^2$ furnished with the l^p-norm $\|\cdot\|$ is strictly convex, then each k_B-ball is also strictly convex in a linear sense [1], [21] (see also [20]).

A mapping $f : B \to B$ is k_B-nonexpansive if

$$k_B\left(f(x), f(y)\right) \le k_B\left(x, y\right)$$

for all $x, y \in B$. Each holomorphic self-mapping $f : B \to B$ is k_B-nonexpansive [2], [5], [7].

If $f : B \to B$ is k_B-nonexpansive, then for each $0 < t < 1$ and $a \in B$ the mapping $f_{t,a} = (1-t)a + tf$ is a contraction and therefore for each $x \in B$ the sequence $\{f^n_{t,a}(x)\}$ tends to a unique fixed point $y_{t,a}$ in B. Additionally, we have $\lim_{t \to 1^-} \|y_{t,a} - f(y_{t,a})\| = 0$ [3].

For k_B-nonexpansive $f : B \to B$ we call a sequence $\{x_n\} \subset B$ an approximating sequence if $\lim_n k_B(x_n, f(x_n)) = 0$ [7].

The open unit ball B_H in a Hilbert space is called the Hilbert ball [2], [7], [18].

Finally, we recall the result due to W. Kaup and H. Upmeier [11].

The complex Banach spaces X_1 and X_2 are isometrically isomorphic if and only if their open unit balls B_1 and B_2 (respectively) are holomorphically equivalent, i.e., there exists a biholomorphic mapping from the open unit ball B_1 onto the unit ball B_2. In the other case we say that the balls B_1 and B_2 are holomorphically nonequivalent.

3. Local uniform convexity of k_B-balls

First we observe that the open unit ball B in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$, is holomorphically nonequivalent to the Cartesian product of a finite number of Hilbert balls $\prod_{j=1}^m B_{H_j}$ [11] (see also [19]). The next important property of B will be stated in the following theorem.

Theorem 3.1. *If B is the open unit ball in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$, then the metric space (B, k_B) is locally linearly uniformly convex, i.e., for each $z \in B$, $R > 0$ and $0 < \epsilon < 2$ we have*

$$\left.\begin{array}{l} k_B\left(z,x\right) \le R \\ k_B\left(z,y\right) \le R \\ k_B\left(x,y\right) \ge \epsilon R \end{array}\right\} \Rightarrow k_B\left(z, \frac{1}{2}x + \frac{1}{2}y\right) \le \left(1 - \delta_0\left(z, R, \epsilon\right)\right) R$$

and

$$\delta\left(R_1, R_2, R_3, \epsilon_1, \epsilon_2\right) = \inf\left\{\delta_0\left(z, R, \epsilon\right) : \epsilon_1 \le \epsilon \le \epsilon_2, \quad \|z\| \le R_1 \quad R_2 \le R \le R_3\right\} > 0$$

for all $0 < R_1$, $0 < R_2 \le R_3$ and $0 < \epsilon_1 \le \epsilon_2 < 2$.

Proof. Choose three points $x = (x^1, x^2)$, $y = (y^1, y^2)$ and $z = (z^1, z^2)$ in B. Let $\{e_1, e_2, ...\}$ be the standard basis in the Hilbert space l^2. Then, it is easy to observe that there exists a linear isometry $T : l^2 \times l^2 \to l^2 \times l^2$ such that

$$Tx, Ty, Tz \in lin\{(e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_2), (0, e_3)\} \cap B.$$

Put

$$B_1 = lin\{(e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_2), (0, e_3)\} \cap B.$$

The set B_1 is the open unit ball in a Cartesian product

$$X_1 = \mathbb{C}^3 \times \mathbb{C}^3$$

furnished with the l^p-norm and B_1 is strictly convex. Since

$$k_B(u, w) = k_{B_1}(u, w)$$

for all $u, w \in B_1$, we get

$$k_B(x, z) = k_{B_1}(Tx, Tz), \qquad k_B(y, z) = k_{B_1}(Ty, Tz)$$

and

$$k_B(x, y) = k_{B_1}(Tx, Ty).$$

Therefore we may restrict our further considerations to the six-dimensional Banach space X_1. Then each k_{B_1}-ball is strictly convex in a linear sense and it is obvious that, by a compactness argument, the metric space (B_1, k_{B_1}) is locally linearly uniformly convex. This implies the same property of (B, k_B). □

Remark 3.1. The construction (given in Theorem 3.1) of the domain which has the locally linearly uniformly convex Kobayashi distance can be generalized but the way of this generalization will be a subject of another paper.

4. Fixed points of holomorphic mappings

In this section we will use the asymptotic center method [4], [6], [7]. Let $\{x_t\}_{t\in T}$ be a k_B-bounded net in the open unit ball B in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$, and let C be a nonempty, k_B-closed and convex subset of B. Consider the functional $r(\cdot, \{x_t\}_{t\in T}) : B \to [0, \infty)$ defined by $r(x, \{x_t\}_{t\in T}) = \limsup_{t\in T} k_B(x, x_t)$. Recall that a point z in C is said to be an asymptotic center of the net $\{x_t\}_{t\in T}$ with respect to C if $r(z, \{x_t\}_{t\in T}) = \inf\{r(x, \{x_t\}_{t\in T}) : x \in C\}$. The infimum of $r(\cdot, \{x_t\}_{t\in T})$ over C is called the asymptotic radius of $\{x_t\}_{t\in T}$ with respect to C and denoted by $r(C, \{x_t\}_{t\in T})$. Let us observe that the function $r(\cdot, \{x_t\}_{t\in T})$ is quasi-convex, i.e.,

$$r((1 - s)x + sy, \{x_t\}_{t\in T}) \leq \max(r(x, \{x_t\}_{t\in T}), r(y, \{x_t\}_{t\in T}))$$

for all x and y in B and $0 \leq s \leq 1$.

Proposition 4.1. *Let B be the open unit ball in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$. Every k_B-bounded net $\{x_t\}_{t\in T}$ in B has a unique asymptotic center with respect to any nonempty, k_B-closed and convex subset C of B.*

Proof. Let $\{x_t\}_{t\in T}$ be a k_B-bounded net in B. Hence the net $\{x_t\}_{t\in T}$ lies strictly inside B and therefore

$$0 < \frac{\sup_{t\in T}\|x_t\|+1}{2} = R < 1.$$

Next, the sets

$$C_n = \{x \in C : r(x, \{x_t\}_{t\in T}) \leq r(C, \{x_t\}_{t\in T}) + \frac{1}{n}\}$$

are nonempty, convex and weakly compact since the function $r(\cdot, \{x_t\}_{t\in T})$ is continuous and quasi-convex. Hence $r(\cdot, \{x_t\}_{t\in T})$ attains its minimum in C. Now, all we have to show is that

$$r(\frac{1}{2}x + \frac{1}{2}y, \{x_t\}_{t\in T}) < \max(r(x, \{x_t\}_{t\in T}), r(y, \{x_t\}_{t\in T}))$$

for every $x \neq y$. To this end, let $r_0 = \max(r(x, \{x_t\}_{t\in T}), r(y, \{x_t\}_{t\in T}))$. Then for each $0 < \epsilon < 1$ there exists $t_\epsilon \in T$ such that

$$k_B(x, x_t) \leq r_0 + \epsilon \qquad \text{and} \qquad k_B(y, x_t) \leq r_0 + \epsilon$$

for all $t \geq t_\epsilon$. Since, by Theorem 3.1, the open unit ball B in the Cartesian product $l^2 \times l^2$ (furnished with the l^p-norm) has the Kobayashi distance k_B which is locally uniformly convex, we get

$$k_B(\frac{1}{2}x + \frac{1}{2}y, x_t) \leq (1 - \delta(R, r_0, r_0+1, \frac{k_B(x,y)}{r_0+1}, \frac{k_B(x,y)}{r_0+1}))(r_0 + \epsilon)$$

for all $t \geq t_\epsilon$ and finally

$$r(\frac{1}{2}x + \frac{1}{2}y, \{x_t\}_{t\in T}) \leq (1 - \delta(R, r_0, r_0+1, \frac{k_B(x,y)}{r_0+1}, \frac{k_B(x,y)}{r_0+1}))r_0 < r_0.$$

This completes the proof. □

Theorem 4.1. *Let B be the open unit ball in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$, and let $f : B \to B$ be a k_B-nonexpansive mapping. Then the following statements are equivalent:*

- (*i*) *f has a fixed point;*
- (*ii*) *there exists a point x in B such that the sequence of iterates $\{f^n(x)\}$ is k_B-bounded;*
- (*iii*) *the sequence of iterates $\{f^n(x)\}$ is k_B-bounded for all x in B;*
- (*iv*) *there exists a k_B-bounded approximating sequence $\{x_n\}$ for f;*
- (*v*) *there exists a closed and f-invariant k_B-ball;*
- (*vi*) *there exists a nonempty, closed, convex, k_B-bounded and f-invariant subset C of the ball B.*

Proof. To prove this theorem it is sufficient to apply the asymptotic center method and the following facts:

1. each nonempty, closed, convex, k_B-bounded and f-invariant subset C of the ball B contains a k_B-bounded approximating sequence for f;
2. if $\{x_n\}$ is a k_B-bounded approximating sequence for f, then

$$r(f(y), \{x_n\}) \leq r(y, \{x_n\})$$

for each $y \in B$;

(3) if $x \in B$ has the k_B-bounded sequence of iterates $\{f^n(x)\}$, then
$$r(f(y), \{f^n(x)\}) \leq r(y, \{f^n(x)\})$$
for each $y \in B$;

(4) by Proposition 4.1 every k_B-bounded sequence $\{x_n\}$ in B has a unique asymptotic center with respect to any nonempty, k_B-closed and convex subset C of B. □

Corollary 4.1. *Theorem* 4.1 *is valid for holomorphic self-mappings of* B.

Proof. Each holomorphic self-mapping of B is k_B-nonexpansive. □

Remark 4.1. Note that in the case of the open unit ball B_H of a Hilbert space H the analogous theorem and corollary are valid [7], [8], [12], but not in the case of B_H^∞ [14], [15].

Next we need the following definition.

Definition 4.1. Let B be the open unit ball in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$. A mapping $f : B \to B$ is said to be asymptotically regular if
$$\lim_n k_B(f^{n+1}(x), f^n(x)) = 0$$
for each $x \in B$.

Remark 4.2. In the case of the Hilbert ball B_H, when we consider averaged k_{B_H}-nonexpansive self-mappings, the asymptotically regular mappings appear in a natural way [14], [17].

As a direct consequence of Theorem 4.1 we get

Proposition 4.2. *Let* B *be the open unit ball in the Cartesian product* $l^2 \times l^2$ *furnished with the* l^p*-norm, where* $1 < p < \infty$ *and* $p \neq 2$ *and let a mapping* $f : B \to B$ *be* k_B*-nonexpansive and asymptotically regular. If* f *is fixed-point-free, then* $lim_n \|f^n(x)\| = 1$ *for each* $x \in B$.

Proof. Assume that there exists a subsequence $\{f^{n_j}(x)\}$ of $\{f^n(x)\}$ such that
$$\sup_j \|f^{n_j}(x)\| < 1.$$
Then $\{f^{n_j}(x)\}$ is a k_B-bounded approximating sequence for f and, by Theorem 4.1, the mapping f has a fixed point, contrary to our assumption. Therefore $\lim_n \|f^n(x)\| = 1$ for each $x \in B$ as claimed. □

Now we will investigate semigroups and their orbits.

Definition 4.2. Let B be the open unit ball in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$. A family $S = \{F_t\}$, where either $t \in \mathbb{R}^+ = \{s \in \mathbb{R} : s \geq 0\}$ or $t \in \mathbb{N} \cup \{0\}$, of self-mappings of B is called a (one-parameter) semigroup if $F_{s+t} = F_s \circ F_t$, $s, t \in \mathbb{R}^+$ ($s, t \in \mathbb{N} \cup \{0\}$), and $F_0 = I_B$, where I_B is the identity operator on B. A semigroup $S = \{F_t\}$, $t \in \mathbb{R}^+$, is said to be (strongly) continuous if the function $F_{(\cdot)}(x) : \mathbb{R}^+ \to X$ is continuous in t for each $x \in B$. If $t \in \mathbb{N} \cup \{0\}$ we say that the semigroup S is discrete.

Definition 4.3. Let B be the open unit ball in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$. Let $S = \{F_t\}$, $t \in \mathbb{R}^+ = \{s \in \mathbb{R} : s \geq 0\}$ (or $t \in \mathbb{N} \cup \{0\}$) be a continuous (or discrete) semigroup acting on B. A point $x \in B$ is said to be a stationary point of S if $F_t(x) = x$ for all $t \in \mathbb{R}^+$ (or $t \in \mathbb{N}$).

Theorem 4.2. *Let B be the open unit ball in the Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$. Let $S = \{F_t\}$ be a continuous or discrete semigroup of k_B-nonexpansive self-mappings of B. Then the following statements are equivalent:*

(*i*) *there is $t_0 > 0$ such that F_{t_0} has a fixed point in B;*
(*ii*) *each F_t has a fixed point;*
(*iii*) *there is a stationary point of the semigroup S.*

Proof. It is clear that $(iii) \Rightarrow (ii) \Rightarrow (i)$. Therefore it is enough to show that $(i) \Rightarrow (iii)$. Indeed, let $x \in B$ be a fixed point of F_{t_0} for some t_0. Then x is a periodic point of S, i.e.,

$$F_{t+t_0}(x) = F_t(F_{t_0}(x)) = F_t(x)$$

for all t. But the set $\{F_t(x) : 0 \leq t \leq t_0\}$ is a compact subset of B and therefore it is k_B-bounded. Applying the asymptotic center method to the k_B-bounded net $\{F_t(x)\}$ and Proposition 4.1 it is easy to show that there exists a common fixed point of S. □

Remark 4.3. In the case of the Hilbert ball B_H the analogous theorem can be found in [13] (see also [14]).

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