PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 9, Pages 2663–2671
S 0002-9939(03)06990-9
Article electronically published on April 1, 2003

ORTHOCOMPLETE EFFECT ALGEBRAS

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(Communicated by Lance W. Small)

Abstract. We prove that for every orthocomplete effect algebra E the center of E forms a complete Boolean algebra. As a consequence, every orthocomplete atomic effect algebra is a direct product of irreducible ones.

1. Introduction

Effect algebras were introduced by Foulis and Bennett in their paper [5] for the study of logical foundations of quantum mechanics. Independently, Chovanec and Kôpka introduced essentially equivalent structures called *D-posets* (see [14]). Another equivalent structure was introduced by Giuntini and Greuling in [6]. For more information about effect algebras see [4].

The class of effect algebras is a common generalization of several classes of well-established algebraic structures, in particular orthomodular lattices and MV-algebras.

In the present paper we prove that in an orthocomplete effect algebra E, the sums of all orthogonal families of central elements are central elements and that joins and meets of all families of central elements exist in E and that they are central. For finite families, these results were proved in [8]. For countable families, see [11]. As a consequence, an orthocomplete atomic effect algebra is a direct product of irreducible effect algebras. In addition, we prove that an effect algebra is κ-orthocomplete iff every chain of cardinality κ has a supremum.

2. Effect algebras

An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations $0, 1$ satisfying the following conditions:

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.
(E4) If $a \oplus 1$ exists, then $a = 0$.

Received by the editors April 3, 2002.
2000 *Mathematics Subject Classification.* Primary 06F05; Secondary 03G25, 81P10.
Key words and phrases. Effect algebra, central element, orthocomplete.
This research was supported by grant G-1/7625/20 of MŠ SR, Slovakia and grant VEGA 2/7193/20.

In an effect algebra E, we write $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$. It is easy to check that $\leq$ is a partial order on E. In this partial order, 0 is the least and 1 is the greatest element of E. Moreover, it is possible to introduce a new partial operation $\ominus$; $b \ominus a$ is defined iff $a \leq b$ and then $a \oplus (b \ominus a) = b$. It can be proved that $a \oplus b$ is defined iff $a \leq b'$ iff $b \leq a'$. Therefore, it is usual to denote the domain of $\oplus$ by $\perp$. We say that elements a and b in an effect algebra E are *orthogonal* if $a \perp b$. In what follows, when we write $a \oplus b$ we mean that $a \oplus b$ is defined (i.e., $a \perp b$). Owing to associativity (E2), we may omit parentheses in $a_1 \oplus a_2 \oplus a_3$ and $a_1 \oplus a_2 \oplus \cdots \oplus a_n$, the latter term being defined by induction. We will say that the elements $a_1, \dots, a_n$ are **orthogonal** if the element $a_1 \oplus \cdots \oplus a_n$ exists in L. More generally, we say that $\{a_\alpha\}_\alpha$ is an *orthogonal family* if every finite subfamily is orthogonal.

An effect algebra need not be lattice-ordered. However, as proved in [19] and [8], the following relationship between $\wedge$, $\vee$ and $\oplus$ holds: if $a \vee b$ exists and $a \perp b$, then $a \wedge b$ exists and

$$a \oplus b = (a \wedge b) \oplus (a \vee b). \tag{1}$$

Moreover, it is easy to check (see [1]) that, for every subset B of an effect algebra such that $\vee B$ exists and for every $x \geq B$,

$$x \ominus (\vee B) = \wedge \{x \ominus b : b \in B\}. \tag{2}$$

Example 2.1. Let $(L; \wedge, \vee, {}', 0, 1)$ be an orthomodular lattice. Write $a \oplus b = a \vee b$ iff $a \leq b'$, otherwise let $a \oplus b$ be undefined. Then $(L, \oplus, 0, 1)$ is an effect algebra. Effect algebras, which are associated with orthomodular lattices in this way, can be characterized as lattice-ordered effect algebras satisfying the implication

$$a \perp b \implies a \wedge b = 0.$$

Example 2.2. An *MV-algebra* (cf. [2], [15]) is a commutative semigroup $(M; \oplus, \neg, 0)$, satisfying identities $x \oplus 0 = x$, $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$ and

$$x \oplus \neg(x \oplus \neg y) = y \oplus \neg(y \oplus \neg x).$$

There is a natural partial order in an MV-algebra, given by $y \leq x$ iff $x = x \oplus \neg(x \oplus \neg y)$. Every MV-algebra $(M; \oplus, \neg, 0)$ can be considered as an effect algebra $(M; \oplus, 0, \neg 0)$ when we restrict the operation $\oplus$ to the domain $\perp = \{(x, y) : x \leq \neg y\}$. Effect algebras, which are associated with MV-algebras, can be characterized as lattice-ordered effect algebras satisfying the implication

$$a \wedge b = 0 \implies a \perp b.$$

(Cf. [17].)

Example 2.3. Let H be a Hilbert space, and let $S(H)$ denote the partially ordered group of all bounded self-adjoint linear operators on H. Put $E(H) = \{A \in S(H) : 0 \leq A \leq 1\}$; the elements of $E(H)$ are called effects. For $a, b \in E(H)$, write $a \oplus b = a + b$ iff $a + b \in E(H)$, otherwise let $a \oplus b$ be undefined. Then $(E(H); \oplus, 0, 1)$ is an effect algebra. We remark that for $\dim(H) \geq 2$, $E(H)$ is not lattice-ordered.

Let E_1, E_2 be effect algebras. A map $\phi : E_1 \mapsto E_2$ is called a morphism iff it satisfies the following condition:

(H1) $\phi(1) = 1$ and if $a \perp b$, then $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

A morphism $\phi : E_1 \mapsto E_2$ of effect algebras is called *full* iff the following condition is satisfied:

(H2) If $\phi(a) \perp \phi(b)$ and $\phi(a) \oplus \phi(b) \in \phi(E)$, then there exist $a_1, b_1 \in E_1$ such that $a_1 \perp b_1$, $\phi(a) = \phi(a_1)$ and $\phi(b) = \phi(b_1)$.

A bijective, full morphism is called an *isomorphism*. A morphism ϕ is an isomorphism if it is bijective and ϕ^{-1} is also a morphism.

Let E_1 be an effect algebra. A subset $E_2 \subseteq E_1$ is a *subeffect algebra* of E_1 iff $0, 1 \in E_2$, E_2 is closed under the $'$ operation, and $a, b \in E_2$ with $a \perp b \implies a \oplus b \in E_2$.

Another possibility to create a substructure of an effect algebra E is to restrict $\oplus$ to an interval

$$[0, a] = \{x \in E : 0 \leq x \leq a\}$$

as follows. For $x, y \in [0, a]$, $x \oplus y$ is defined iff $x \oplus y$ exists in E and $x \oplus y \in [0, a]$. We can then consider $[0, a]$ as an effect algebra, letting a act as the unit element. In what follows, we denote such effect algebras by $[0, a]_E$.

Let E be an effect algebra. A subset I of E is called *an ideal* of E iff the following condition is satisfied:

$$x, y \in I \text{ and } x \perp y \Leftrightarrow x \oplus y \in I.$$

3. Orthocomplete effect algebras and central elements

In this section, we will prove that the center of an orthocomplete effect algebra is a complete Boolean algebra. This is a generalization of [8] and [11].

Let E be an effect algebra. Suppose that there is an isomorphism $\phi : E \mapsto E_1 \times E_2$. For every such ϕ, the elements $\phi^{-1}(1, 0)$ and $\phi^{-1}(0, 1)$ are called *central elements* of E. We write $C(E)$ for the set of all central elements of an effect algebra E. We say that an effect algebra E is *irreducible* iff $C(E) = \{0, 1\}$.

Recall that an element $a \in E$ is *sharp* if $a \wedge a' = 0$, and a is *principal* if $b, c \leq a$, $b \perp c$ implies $b \oplus c \leq a$. It is easy to see that a principal element is sharp; the opposite implication need not be true, in general. Central elements can be intrinsically characterized by the following properties: (i) c and c' are principal and (ii) every element $x \in E$ admits a decomposition $x = x_1 \oplus x_2$ with $x_1 \leq c$, $x_2 \leq c'$. It can be proved that this decomposition of x is unique. In fact, $x_1 = x \wedge c$, $x_2 = x \wedge c'$. Moreover, for every central element a, the map $x \mapsto a \wedge x$ is a full morphism, which maps E onto $[0, a]_E$ (cf. [12]).

It was proved in [8] that the set of all central elements forms a sub-effect algebra of E, which is a Boolean algebra. Moreover, the joins and meets of elements of $C(E)$ exist in E and coincide with their joins and meets in $C(E)$. If $a, b \in C(E)$ are orthogonal, we have $a \vee b = a \oplus b$ and $a \wedge b = 0$.

Lemma 3.1. *If $x, y \in E$ and $a \in C(E)$, then*

$$a \wedge (x \oplus y) = a \wedge x \oplus a \wedge y. \tag{3}$$

Moreover, if $a, b \in C(E)$, and $x \in E$, then

$$x \wedge (a \oplus b) = x \wedge a \oplus x \wedge b. \tag{4}$$

Proof. Let $x \perp y$, x, y in E, and $a \in C(E)$. Then $x \oplus y = x \wedge a \oplus x \wedge a' \oplus y \wedge a \oplus y \wedge a' = (x \wedge a \oplus y \wedge a) \oplus (x \wedge a' \oplus y \wedge a')$, where the first summand is under a, the second under a'. Uniqueness of the decomposition of $x \oplus y$ then yields $(x \oplus y) \wedge a = x \wedge a \oplus y \wedge a$.

If $a, b \in C(E)$, $a \perp b$ and $x \in E$, then $(a \oplus b) \wedge x = (a \oplus b) \wedge (x \wedge a \oplus x \wedge a') = (a \vee b) \wedge (x \wedge a) \oplus (a \vee b) \wedge (x \wedge a') = a \wedge x \oplus b \wedge x$. ☐

For all $a \in C(E)$, the interval $[0,a]$ is an ideal of E. These ideals are called *central ideals*. By [3], a central ideal in an effect algebra E can be characterized as an ideal I satisfying the following conditions:

- $I = [0,a]$ for some $a \in E$.
- I is a *Riesz ideal*, i.e., if $i \in I$ and $i \leq a \oplus b$, then there exist $i_1, i_2 \in I$, such that $i_1 \leq a$, $i_2 \leq b$, $i \leq i_1 \oplus i_2$.

Let E be an effect algebra, and $\{a_\alpha\}_\alpha$ be an orthogonal family. We define $\oplus_\alpha a_\alpha := \vee_F \oplus (a_\alpha : \alpha \in F)$, where the supremum goes over all finite subfamilies F of α's, if the supremum on the right-hand side exists.

We will say that an effect algebra E is *m-orthocomplete* for an infinite cardinal m if every orthogonal family of at most m elements has an $\oplus$-sum in L. We say that an effect algebra E is *orthocomplete* if it is m-orthocomplete for every cardinal m.

The following theorem is a generalization of [11, Lemma 3.3]. For analogues in orthomodular lattices see [10], in orthoalgebras [9], [18].

Theorem 3.2. *Let E be an effect algebra and let m be a cardinal. The following are equivalent:*

(1) *E is m-orthocomplete.*
(2) *Every chain of at most m elements has a supremum.*

Proof. The implication that (1) implies (2) was proved in [13]. We have to prove that (2) implies (1). Assume that every chain of at most m elements in E has a supremum. Let X be an orthogonal subset of E and let $\mathrm{card}(X) \leq m$. We may assume that X is infinite, and let γ be the first ordinal with $\mathrm{card}(\gamma) = \mathrm{card}(X)$. We will prove that the $\oplus X$ exists and it is equal to $\vee\Sigma$, where

$$\Sigma := (\oplus(x_\alpha : \alpha < \beta) : \beta < \gamma),$$

$(x_\alpha : \alpha < \gamma)$ being an indexing of X with γ. We proceed by induction by $\mathrm{card}(X)$. If X is finite, there is nothing to prove. Let X be infinite, $\mathrm{card}(X) \leq m$, and $\mathrm{card}(X) = \mathrm{card}(\gamma)$. The induction hypothesis is that for all orthogonal sets Y, $\mathrm{card}(Y) = \beta < \gamma$, $\oplus Y$ exists, and

$$\oplus Y = \vee(\oplus(x_\sigma : \sigma < \nu) : \nu < \beta).$$

Let $X = (x_\alpha : \alpha < \gamma)$ be an indexing as desired. By induction hypothesis, the chain

$$\Sigma := (\oplus(x_\alpha : \alpha < \beta) : \beta < \gamma)$$

exists in E. Since $\mathrm{card}(\Sigma) = \mathrm{card}(X) \leq m$, the supremum $s := \vee\Sigma$ exists in E. Let $x_{\alpha_1}, \dots, x_{\alpha_n}$ be an arbitrary finite sequence with $\alpha_1, \dots, \alpha_n < \gamma$. Without loss of generality, we may assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Then $(x_{\alpha_1}, \dots, x_{\alpha_n}) \subseteq (x_\alpha : \alpha < \alpha_n + 1)$ and hence

$$s \geq \oplus(x_\alpha : \alpha < \alpha_n + 1) \geq \oplus(x_{\alpha_1}, \dots, x_{\alpha_n}).$$

This proves that s is an upper bound of all $\oplus(x_\alpha : \alpha \in F)$, F being a finite subset of the index set $(\alpha : \alpha < \gamma)$. To see that s is the desired supremum, let p be an upper bound of all $\oplus(x_\alpha : \alpha \in F)$, where F is a finite subsets of $(\alpha : \alpha < \gamma)$. Then for all $\beta < \gamma$, p is an upper bound of $\oplus(x_\alpha : \alpha < \beta)$. From this it follows that p is an upper bound of Σ, hence $p \geq s$. □

Consequently, in every orthocomplete effect algebra, every chain has a supremum.

Lemma 3.3. *Let E be an orthocomplete effect algebra. Let $(a_\alpha : \alpha \in \Sigma) \subseteq E$ be an orthogonal family of central elements. Let $(x_\alpha : \alpha \in \Sigma)$ be a family of elements satisfying $x_\alpha \leq a_\alpha$, for all $\alpha \in \Sigma$. Then $\vee(x_\alpha : \alpha \in \Sigma)$ exists and equals $\oplus(x_\alpha : \alpha \in \Sigma)$.*

Proof. Obviously, $(x_\alpha : \alpha \in \Sigma)$ is an orthogonal family, so that $\oplus(x_\alpha : \alpha \in \Sigma)$ exists in E by orthocompleteness. Let M be a finite subset of Σ, and let y be any upper bound of $(x_\alpha : \alpha \in M)$. Then clearly $\forall \alpha \in M$, $x_\alpha \leq y \wedge a_\alpha$. Therefore

$$\begin{aligned} \oplus(x_\alpha : \alpha \in M) &\leq \oplus(y \wedge a_\alpha : \alpha \in M) \\ &= y \wedge (\oplus(a_\alpha : \alpha \in M)) \leq y. \end{aligned}$$

Thus, $\oplus(x_\alpha : \alpha \in M)$ is under every upper bound of $(x_\alpha : \alpha \in M)$, and we see that for every finite nonempty $M \subset \Sigma$, we have $\oplus(x_\alpha : \alpha \in M) = \vee(x_\alpha : \alpha \in M)$. This implies that

$$\begin{aligned} \oplus(x_\alpha : \alpha \in \Sigma) &= \vee_F \oplus (x_\alpha : \alpha \in F) \\ &= \vee_F \vee (x_\alpha : \alpha \in F) \\ &= \vee(x_\alpha : \alpha \in \Sigma), \end{aligned}$$

where $\vee_F$ runs over all finite subsets F of Σ. □

Theorem 3.4. *Let E be an orthocomplete effect algebra. Let $(a_\alpha : \alpha \in \Sigma)$ be an orthogonal family of central elements. Denote $a = \oplus(a_\alpha : \alpha \in \Sigma)$. Then a is central and*

$$[0, a]_E \simeq \prod_{\alpha \in \Sigma} [0, a_\alpha]_E.$$

Proof. Define a mapping $\phi : [0, a]_E \to \prod_{\alpha \in \Sigma} [0, a_\alpha]_E$ by $\phi(x) = (x \wedge a_\alpha)_{\alpha \in \Sigma}$. We shall prove that ϕ is an isomorphism.

To prove that ϕ is onto, let $(x_\alpha)_{\alpha \in \Sigma} \in \prod_{\alpha \in \Sigma}[0, a_\alpha]_E$. Observe that $(x_\alpha : \alpha \in \Sigma)$ is an orthogonal family and put $x = \oplus(x_\alpha : \alpha \in \Sigma)$. We will prove that $\phi(x) = (x_\alpha)_{\alpha \in \Sigma}$.

We have

$$\phi(x) = (x \wedge a_\alpha)_{\alpha \in \Sigma} = ((\oplus(x_\alpha : \alpha \in \Sigma)) \wedge a_\alpha)_{\alpha \in \Sigma}.$$

Fix $\beta \in \Sigma$. By associativity of $\oplus$,

$$\begin{aligned} x \wedge a_\beta &= (\oplus(x_\alpha : \alpha \in \Sigma)) \wedge a_\beta \\ &= (x_\beta \oplus (\oplus(x_\alpha : \alpha \in \Sigma \setminus \{\beta\}))) \wedge a_\beta. \end{aligned}$$

Since the family $(x_\alpha : \alpha \in \Sigma \setminus \{\beta\})$ satisfies conditions of Lemma 3.3, we have

$$\begin{aligned} (x_\beta \quad &\oplus \quad (\oplus(x_\alpha : \alpha \in \Sigma \setminus \{\beta\}))) \wedge a_\beta \\ &= \quad (x_\beta \oplus (\vee(x_\alpha : \alpha \in \Sigma \setminus \{\beta\}))) \wedge a_\beta \end{aligned}$$

and since a_β is a central element,

$$\begin{aligned} (x_\beta \ &\oplus (\vee(x_\alpha : \alpha \in \Sigma \setminus \{\beta\}))) \wedge a_\beta = x_\beta \wedge a_\beta \oplus (\vee(x_\alpha : \alpha \in \Sigma \setminus \{\beta\}) \wedge a_\beta) \\ &= x_\beta \oplus (\vee(x_\alpha : \alpha \in \Sigma \setminus \{\beta\}) \wedge a_\beta). \end{aligned}$$

Since, for all $\alpha \in \Sigma \setminus \{\beta\}$, $x_\alpha \wedge a_\beta = 0$, we have

$$(\vee(x_\alpha : \alpha \in \Sigma \setminus \{\beta\}) \wedge a_\beta) = 0.$$

Thus, for all $\alpha \in \Sigma$, $x \wedge a_\alpha = x_\alpha$.

To prove that ϕ is one-to-one, it suffices to prove that, for all $x \in [0, a]$,

$$x = \oplus(x \wedge a_\alpha : \alpha \in \Sigma).$$

As

$$\begin{aligned} \oplus \quad & (x \wedge a_\alpha : \alpha \in \Sigma) = \vee_F(\oplus(x \wedge a_\alpha : \alpha \in F)) \\ = \quad & \vee_F(x \wedge (\oplus a_\alpha : \alpha \in F)), \end{aligned}$$

where we used (4) in the last equality, we see that $\oplus(x \wedge a_\alpha : \alpha \in \Sigma) \leq x$. Moreover, using (2),

$$\begin{aligned} x \ominus (\oplus(x \wedge a_\alpha : \alpha \in \Sigma)) &= x \ominus (\vee_F(\oplus x \wedge a_\alpha : \alpha \in F)) \\ &= x \ominus (\vee_F(x \wedge (\oplus(a_\alpha : \alpha \in F)))) \\ &= \wedge_F(x \ominus (x \wedge \oplus(a_\alpha : \alpha \in F))). \end{aligned}$$

Since $\forall F$, $\oplus(a_\alpha : \alpha \in F)$ is central, we have

$$\begin{aligned} (x \ominus (x \wedge \oplus(a_\alpha : \alpha \in F))) &= x \wedge (\oplus(a_\alpha : \alpha \in F))' \\ &= x \wedge (\vee(a_\alpha : \alpha \in F))' \\ &= x \wedge (\wedge(a'_\alpha : \alpha \in F)). \end{aligned}$$

Therefore

$$\begin{aligned} \wedge_F(x \ominus (x \wedge \oplus(a_\alpha : \alpha \in F))) &= \wedge_F x \wedge (\wedge(a'_\alpha : \alpha \in F)) \\ &\leq \wedge_F a \wedge (\wedge a'_\alpha : \alpha \in F) \\ &= \wedge_F(a \ominus (\vee(a_\alpha : \alpha \in F))') \\ &= a \ominus (\vee_F \vee (a_\alpha : \alpha \in F)) \\ &= 0. \end{aligned}$$

Hence, for all $x \in [0, a]_E$, $x = \oplus(x \wedge a_\alpha : \alpha \in \Sigma)$, and this implies that ϕ is one-to-one.

Let us prove that $[0, a]$ is an ideal. Obviously, $x \oplus y \in [0, a]$ implies $x \perp y$ and $x, y \in [0, a]$. To prove the opposite implication, assume that $x, y \in [0, a]$ and $x \perp y$. By the preceding paragraph, $x = \oplus(x \wedge a_\alpha : \alpha \in \Sigma)$, $y = \oplus(y \wedge a_\alpha : \alpha \in \Sigma)$. Then

$$\begin{aligned} x \oplus y &= (\oplus(x \wedge a_\alpha : \alpha \in \Sigma)) \oplus (\oplus(y \wedge a_\alpha : \alpha \in \Sigma)) \\ &= (\oplus(x \wedge a_\alpha \oplus y \wedge a_\alpha) : \alpha \in \Sigma). \end{aligned}$$

Since $\forall \alpha$, a_α is central,

$$x \wedge a_\alpha \oplus y \wedge a_\alpha = (x \oplus y) \wedge a_\alpha \leq a_\alpha.$$

Using Lemma 3.3,

$$\begin{aligned} x \oplus y &= \vee(x \wedge a_\alpha \oplus y \wedge a_\alpha : \alpha \in \Sigma) \\ &= \vee((x \oplus y) \wedge a_\alpha : \alpha \in \Sigma) \leq a. \end{aligned}$$

Therefore, $[0, a]$ is an ideal. To prove that a is central, we need to prove that $[0, a]$ is a central ideal, i.e. that $[0, a]$ is a Riesz ideal.

Assume that $z \leq x \oplus y$, where $z \in [0,a]$, $x, y \in E$. Then $z = \oplus(z \wedge a_\alpha : \alpha \in \Sigma)$. For $\forall \alpha \in \Sigma$, $z \wedge a_\alpha \leq x \wedge a_\alpha \oplus y \wedge a_\alpha$. Put $z_1 = \oplus(x \wedge a_\alpha : \alpha \in \Sigma)$, $z_2 = \oplus(y \wedge a_\alpha : \alpha \in \Sigma)$. Obviously, $z \leq z_1 \oplus z_2$, $z_1, z_2 \in [0,a]$ and $z_1 \leq x$, $z_2 \leq y$. This proves that $[0,a]$ is a central ideal, i.e. a is central. □

Theorem 3.5. *Let E be an orthocomplete effect algebra. Then the centre $C(E)$ of E is a complete Boolean algebra. Moreover, all suprema and infima in $C(E)$ coincide with those in E.*

Proof. Let $(a_\alpha : \alpha \in \Sigma)$ be a family of elements in $C(E)$ indexed by a set Σ. We will prove that $\vee(a_\alpha : \alpha \in \Sigma)$ exists in E and belongs to $C(E)$. The latter statement is true for any finite set, so we may assume that Σ is infinite. Let σ be the least ordinal corresponding to $\operatorname{card}(\Sigma)$. We may assume that σ is a limit ordinal, and replace the set Σ by the set $(\alpha : \alpha < \sigma)$, so that we are dealing with an ordinal-indexed family. Further we proceed by a transfinite induction. Assume that $y_\alpha = \vee(x_\rho : \rho < \alpha)$ exists and belongs to $C(E)$ for every $\alpha < \sigma$. This family $(y_\alpha : \alpha < \sigma)$ is nondecreasing, and $(y_{\alpha+1} \ominus y_\alpha : \alpha + 1 < \sigma)$ is an orthogonal family. Indeed, choose a finite subset $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ with $\alpha_n + 1 < \sigma$. We then have $y_{\alpha_1} \leq y_{\alpha_1+1} \leq y_{\alpha_2} \leq y_{\alpha_2+1} \leq \cdots \leq y_{\alpha_n} \leq y_{\alpha_n+1}$. Then

$$\begin{aligned}&(y_{\alpha_1+1} \ominus y_{\alpha_1}) \oplus (y_{\alpha_2} \ominus y_{\alpha_1+1}) \oplus \cdots \oplus (y_{\alpha_n+1} \ominus y_{\alpha_n})\\ &\qquad = y_{\alpha_n+1} \ominus y_{\alpha_1} \geq (y_{\alpha+1} \ominus y_{\alpha_1}) \oplus \cdots \oplus (y_{\alpha_n+1} \ominus y_{\alpha_n}).\end{aligned}$$

Hence

$$z = \oplus(y_{\alpha+1} \ominus y_\alpha : \alpha + 1 < \sigma)$$

exists and belongs to $C(E)$ by Theorem 3.4. We will prove that z is the desired join $\vee(x_\rho : \rho < \sigma)$.

First, we note that if z is an upper bound of the set $(x_\rho : \rho < \sigma)$, then it is the least one. For if $w \geq x_\rho$ for all $\rho < \sigma$, then for all $\alpha + 1 < \sigma$,

$$w \geq \vee(x_\rho : \rho < \alpha + 1) = y_{\alpha+1} \geq y_{\alpha+1} \ominus y_\alpha.$$

By Lemma 3.3, $z = \vee(y_{\alpha+1} \ominus y_\alpha : \alpha + 1 < \rho)$, which yields $w \geq z$. Hence it is enough to show that $z \geq x_\beta$ for every $\beta < \sigma$.

If $\beta < \sigma$, σ being a limit ordinal, we have $\beta + 2 < \sigma$, whence

$$\begin{aligned} x_\beta &\leq \vee(x_\rho : \rho \leq \beta + 1) = y_{\beta+1}\\ &= \vee(y_\alpha : \alpha < \beta + 2) = \oplus(y_{\rho+1} \ominus y_\rho : \rho + 1 < \beta + 2) \leq z.\end{aligned}$$

This proves the theorem. □

4. Orthocomplete and atomic effect algebras

Recall that an element $a \neq 0$ in an effect algebra E is called an *atom* if $x \leq a$ implies $x = a$ or $x = 0$. An effect algebra E is *atomic* if every element in E majorizes an atom.

Theorem 4.1. *Let E be an orthocomplete effect algebra and let $x \in E$ be an atom. Then $\wedge(z \in C(E) : z \geq x)$ is an atom in $C(E)$.*

Proof. Put $c(x) = \wedge(z \in C(E) : z \geq x)$. By Theorem 3.5, the element $c(x)$ exists and belongs to $C(E)$. To prove that $c(x)$ is an atom of $C(E)$, assume that $d \in C(E)$, $d \leq c(x)$. Then $c(x) = d \oplus (d' \wedge c(x))$, and

$$x = x \wedge c(x) = x \wedge d \oplus x \wedge (d' \wedge c(x)).$$

If $x \neq d$, then $x \wedge d = 0$, because x is an atom, and hence $x \leq d' \wedge c(x)$. Since $d' \wedge c(x)$ belongs to $C(E)$, we obtain, by the definition of $c(x)$, that $c(x) \leq d' \wedge c(x) \leq d'$. Then we obtain $d \leq c(x)$, $d \leq c(x)'$ hence $d = 0$. This concludes the proof. □

Lemma 4.2. *Let c be a central element of an effect algebra E. The centre of the effect algebra $[0, c]_E$ consists of elements $z \wedge c$, $z \in C(E)$.*

Proof. Since c is central in E, we may write $E \simeq [0, c]_E \times [0, c']_E$. If d is central in $[0, c]_E$, then $E \simeq [0, d]_E \times [0, d' \wedge c]_E \times [0, c']_E$, so that $d \in C(E)$. If $z \in C(E)$, then $z \wedge c, z' \wedge c$ are orthogonal elements of $C(E)$ with $z \wedge c \oplus z' \wedge c = c$, and therefore $[0, c]_E = [0, c \wedge z]_E \times [0, c \wedge z']_E$, hence $z \wedge c$ is central in $[0, c]_E$. □

Theorem 4.3. *Every orthocomplete atomic effect algebra is a direct product of irreducible effect algebras.*

Proof. Since E is atomic, under every element c in the centre $C(E)$ of E there is an atom x of E. Theorem 4.1 implies that the element $c(x) = \wedge(z \in C : x \leq z)$ is an atom of $C(E)$. Clearly, $c(x) \leq c$. It follows that $C(E)$ is an atomic Boolean algebra, and by Theorem 3.5, $C(E)$ is complete. Let $(c_\alpha : \alpha \in \Sigma)$ denote the set of all atoms of $C(E)$. Then $(c_\alpha : \alpha \in \Sigma)$ is an orthogonal set, and by Theorem 3.4, we have

$$E \simeq \prod_{\alpha \in \Sigma} [0, c_\alpha]_E.$$

By Lemma 4.2, the centre of $[0, c_\alpha]_E$ consists of $\{0, c_\alpha\}$, hence $[0, c_\alpha]_E$ is irreducible. □

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