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L^p VERSIONS OF HARDY'S UNCERTAINTY PRINCIPLE ON HYPERBOLIC SPACES

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Abstract. Hardy's uncertainty principle states that it is impossible for a function and its Fourier transform to be simultaneously very rapidly decreasing. In this paper we prove L^p versions of this principle for the Jacobi transform and for the Fourier transform on real hyperbolic spaces.

1. Introduction

The uncertainty principles roughly state that a non-zero function f and its Fourier transform $\widehat{f}$ cannot both be sharply localised. This is already evident in the Paley–Wiener theorem; the Fourier transform of a compactly supported smooth function extends to an entire function, hence it cannot have compact support. The Fourier transform of a rapidly decreasing function, i.e., a Schwartz function, is on the other hand again a rapidly decreasing function. Hardy's uncertainty principle (see [8]) tells us, however, that they cannot both be *very* rapidly decreasing. Generalisations to a L^p set-up have been studied and proved by, among others, Beurling ([12]) and Cowling–Price:

Theorem 1.1 (Cowling–Price [4])**.** *Let f be a measurable function on $\mathbb{R}$. If $e^{\alpha|\cdot|^2} f \in L^p(\mathbb{R})$ and $e^{\beta|\cdot|^2}\widehat{f} \in L^q(\mathbb{R})$, with $\min(p,q) < \infty$ and $\alpha\beta \geq \frac{1}{4}$, then $f = 0$.*

The case $p = q = \infty$, $\alpha\beta > \frac{1}{4}$ is Hardy's uncertainty principle.

Analogues of Hardy's uncertainty principle and its L^p versions for the Fourier transform on (semisimple) Lie groups have been the object of interest in several recent papers; see [5], [15] and the references therein. The Riemannian symmetric spaces of the non-compact type have also been studied; see [17] and [18].

The aim of this paper is to prove a version of Theorem 1.1 for the Jacobi transform and for the Fourier transform on the real hyperbolic spaces

$$SO_o(m,n)/SO_o(m-1,n), \qquad m,n \in \mathbb{N}.$$

The proof of the latter case is based on the observation that the Fourier transform of functions of fixed K-type can be expressed in terms of modified Jacobi functions. This approach can be expanded to cover all hyperbolic spaces.

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2. Jacobi functions and the Jacobi transform

Let $a, b, \lambda \in \mathbb{C}$ and $0 < t < \infty$. We consider the differential equation

$$\left(\Delta^{a,b}(t)\right)^{-1} \frac{d}{dt}\left(\Delta^{a,b}(t)\frac{du(t)}{dt}\right) = -(\lambda^2+\rho^2)u(t), \tag{1}$$

where $\rho = a+b+1$ and $\Delta^{a,b}(t) = (2\sinh t)^{2a+1}(2\cosh t)^{2b+1}$. Using the substitution $x = -\sinh^2 t$, we can rewrite (1) as a hypergeometric differential equation with parameters $\frac{1}{2}(\rho+i\lambda)$, $\frac{1}{2}(\rho-i\lambda)$ and $a+1$ (see [6, 2.1.1]). Let ${}_2F_1$ denote the Gauß hypergeometric function. The Jacobi function (of order (a,b)),

$$\varphi_\lambda^{a,b}(t) := {}_2F_1\left(\frac{1}{2}(\rho+i\lambda), \frac{1}{2}(\rho-i\lambda), a+1; -\sinh^2 t\right),$$

is for $a \notin -\mathbb{N}$ the unique solution to (1) satisfying $\varphi_\lambda^{a,b}(0) = 1$ and $\frac{d}{dt}\big|_{t=0}\varphi_\lambda^{a,b} = 0$. The Jacobi functions satisfy the following growth estimates:

Lemma 2.1. *There exists for each* $a, b \in \mathbb{C}$ *a constant* $C > 0$ *such that*

$$\left|\Gamma(a+1)^{-1}\varphi_\lambda^{a,b}(t)\right| \le C(1+|\lambda|)^k(1+t)e^{(|\Im\lambda|-\Re\rho)t},$$

for all $\lambda \in \mathbb{C}$ *and all* $t \ge 0$, *where* $k = 0$ *if* $\Re a > -\frac{1}{2}$ *and* $k = [\frac{1}{2} - \Re a]$ *if* $\Re a \le -\frac{1}{2}$.

Proof. See [13, Lemma 2.3]. □

Here $[\cdot]$ denotes the integer part. We note that $\Gamma(a+1)^{-1}\varphi_\lambda^{a,b}(t)$ is an entire function in the variables a, b and $\lambda \in \mathbb{C}$ (also for $a \in -\mathbb{N}$). The Jacobi transform (of order (a,b)) is defined by

$$\widehat{f}^{a,b}(\lambda) := \int_{\mathbb{R}_+} f(t)\varphi_\lambda^{a,b}(t)\Delta^{a,b}(t)dt,$$

for all even functions f and all complex numbers λ for which the right-hand side is well-defined. The Paley–Wiener theorem for the Jacobi transform, [13, Theorem 3.4], states that the (normalised) application $f \mapsto \Gamma(a+1)^{-1}\widehat{f}^{a,b}$ is a bijection from $C_c^\infty(\mathbb{R})_{\text{even}}$ onto $\mathcal{H}(\mathbb{C})_{\text{even}}$, the space of even entire rapidly decreasing functions of exponential type, for all $a, b \in \mathbb{C}$.

The Jacobi functions of the second kind

$$\phi_\lambda^{a,b}(t) = (2\cosh t)^{i\lambda-\rho}{}_2F_1\left(\frac{1}{2}(\rho-i\lambda), \frac{1}{2}(a-b+1-i\lambda), 1-i\lambda; \cosh^{-2} t\right),$$

defines for $\lambda \notin -i\mathbb{N}$ another solution of (1), characterised by the property that $\phi_\lambda^{a,b}(t) \sim e^{i(\lambda-\rho)t}$ for $t \to \infty$. We remark that $\phi_\lambda^{a,b}$ is singular if and only if $\lambda \in -i\mathbb{N}$, with simple poles. Define the meromorphic Jacobi c-functions as

$$c^{a,b}(\lambda) := 2^{\rho-i\lambda}\frac{\Gamma(a+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\rho))\Gamma(\frac{1}{2}(i\lambda+a-b+1))}. \tag{2}$$

Then

$$\varphi_\lambda^{a,b} = c^{a,b}(\lambda)\phi_\lambda^{a,b} + c^{a,b}(-\lambda)\phi_{-\lambda}^{a,b}, \tag{3}$$

as a meromorphic identity; see [14, (2.15–18)]. The inversion formula for the Jacobi transform can be written as (with $\mu \ge 0$, $\mu > -\Re(a \pm b + 1)$)

$$f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}^{a,b}(\lambda+i\mu)\phi_{\lambda+i\mu}^{a,b}(t)\frac{d\lambda}{c^{a,b}(-\lambda-i\mu)} \qquad (t > 0) \tag{4}$$

for $f \in C_c^\infty(\mathbb{R})_{\text{even}}$; see [14, Theorem 2.2]. Using residual calculus we can rewrite (4) as follows:

Theorem 2.2. *Assume that* $a \notin -\mathbb{N}$. *Let* $D_{a,b}$ *denote the finite set of zeroes for* $c^{a,b}(-\lambda)$ *with* $\Im\lambda \geq 0$. *Let* $\eta = 0$ *if* $D_{a,b} \cap \mathbb{R} = \{\emptyset\}$ *and otherwise choose* $0 < \eta < \frac{1}{2}$ *such that* $c^{a,b}(\pm\lambda) \neq 0$ *for* $\Im\lambda \in [-\eta, \eta]\backslash\{0\}$. *Then*

$$f(t) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\widehat{f}^{a,b}(\lambda + i\eta)\varphi^{a,b}_{\lambda+i\eta}(t)}{c^{a,b}(-\lambda - i\eta)c^{a,b}(\lambda + i\eta)}d\lambda - \sum_{\nu \in D_{a,b}} ik_\nu Res_{\lambda=\nu}\left\{\frac{\widehat{f}^{a,b}(\lambda)\varphi^{a,b}_\lambda(t)}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\}$$

for $f \in C_c^\infty(\mathbb{R})_{even}$, *where* $k_\nu := 1/2$ *if* $\nu \in i\mathbb{N} \cup \mathbb{R}$ *and* $k_\nu := 1$ *otherwise.*

Proof. The set $D_{a,b}$ is determined by the poles of the Γ-functions of (2). It follows that $D_{a,b}$ consists of those elements $\nu \neq 0$, with $\Im\nu \geq 0$, which are of the form $\nu = i(\pm b - a - 1 - 2m)$, $m \in \mathbb{N} \cup \{0\}$.

Let $\nu \in D_{a,b}$, that is, $c^{a,b}(-\nu) = 0$. Assume first that $\nu \notin i\mathbb{N}$; then $c^{a,b}(\nu) \neq 0$ by the condition $a \notin -\mathbb{N}$, and

$$\begin{aligned}\mathrm{Res}_{\lambda=\nu}\left\{\frac{\widehat{f}^{a,b}(\lambda)\phi^{a,b}_\lambda(t)}{c^{a,b}(-\lambda)}\right\} &= \mathrm{Res}_{\lambda=\nu}\left\{\widehat{f}^{a,b}(\lambda)\left(\frac{\phi^{a,b}_\lambda(t)}{c^{a,b}(-\lambda)} + \frac{\phi^{a,b}_{-\lambda}(t)}{c^{a,b}(\lambda)}\right)\right\} \\ &= \mathrm{Res}_{\lambda=\nu}\left\{\frac{\widehat{f}^{a,b}(\lambda)\varphi^{a,b}_\lambda(t)}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\},\end{aligned}$$

by (3), since $\phi^{a,b}_{-\lambda}(t)/c^{a,b}(\lambda)$ is regular at $\lambda = \nu$.

Now assume $\nu \in i\mathbb{N} \cap D_{a,b}$. Then ν is a zero for $c^{a,b}(-\lambda)$ of order 1; a double pole in the denominator of $c^{a,b}(-\lambda)$ at $\nu \in i\mathbb{N}$ would imply $a \in -\mathbb{N}$, which we have excluded. The c-function $c^{a,b}(\lambda)$ is regular and non-zero at $\lambda = \nu$, as the poles arising from the Γ-functions in (2) cancel each other (we have excluded the cases with double poles in the denominator). We also note that $\phi^{a,b}_\lambda(t)$ is regular at $\lambda = \nu$.

Write $\nu = i(\pm b - a - 1 - 2m)$. Fix a and m, and define, for λ in some small neighbourhood of ν, a continuous function $b(\lambda)$ by the condition

$$\lambda = i(\pm b(\lambda) - a - 1 - 2m).$$

It follows that $c^{a,b(\lambda)}(-\lambda) = 0$ and $\varphi^{a,b(\lambda)}_\lambda(t) = c^{a,b(\lambda)}(\lambda)\phi^{a,b(\lambda)}_\lambda(t)$, for $\lambda \neq \nu$, by (3), and $b(\nu) = b$. Since $\lim_{\lambda\to n}\frac{\Gamma((\lambda-n)/2)}{\Gamma(\lambda)} = 2$ for $n \in -\mathbb{N} \cup \{0\}$, it can be seen from (2) that $\lim_{\lambda\to\nu} c^{a,b(\lambda)}(\lambda) = 2\lim_{\lambda\to\nu} c^{a,b}(\lambda)$, and thus, by continuity of the Jacobi functions in all the variables,

$$\mathrm{Res}_{\lambda=\nu}\left\{\frac{\widehat{f}^{a,b}(\lambda)\phi^{a,b}_\lambda(t)}{c^{a,b}(-\lambda)}\right\} = \frac{1}{2}\mathrm{Res}_{\lambda=\nu}\left\{\frac{\widehat{f}^{a,b}(\lambda)\varphi^{a,b}_\lambda(t)}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\},$$

since $2\phi^{a,b}_\nu(t) = 2\lim_{\lambda\to\nu}\phi^{a,b(\lambda)}_\lambda(t) = 2\lim_{\lambda\to\nu}\frac{\varphi^{a,b(\lambda)}_\lambda(t)}{c^{a,b(\lambda)}(\lambda)} = \lim_{\lambda\to\nu}\frac{\varphi^{a,b(\lambda)}_\lambda(t)}{c^{a,b}(\lambda)} = \frac{\varphi^{a,b}_\nu(t)}{c^{a,b}(\nu)}$.

Let $0 \leq \eta < \frac{1}{2}$ and let $\delta > 0$. We can extend the estimates in [13, Lemma 2.1] (for $\Im\lambda \geq 0$) as follows: There exists a positive constant C such that

$$|\phi^{a,b}_\lambda(t)| \leq Ce^{-(\Im\lambda + \Re\rho)t}$$

for all $t \geq \delta$ and all $\lambda \in \mathbb{C}$ with $\Im\lambda \geq -\eta$; see [3] for details. The polynomial estimates on $|c^{a,b}(-\lambda)^{-1}|$ away from the poles given by [13, Lemma 2.2] can also be extended to $\Im\lambda \geq -\eta$.

Now choose η as in the theorem. By the above estimates and since $\widehat{f}^{a,b}$ satisfies the usual Paley–Wiener growth estimates, we can shift the contour toward the real axis, and (4) becomes

$$\begin{aligned} f(t) =& \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}^{a,b}(\lambda+i\eta)\phi^{a,b}_{\lambda+i\eta}(t)\frac{d\lambda}{c^{a,b}(-\lambda-i\eta)} + \text{Residual terms} \\ =& \frac{1}{4\pi}\int_{\mathbb{R}} \widehat{f}^{a,b}(\lambda+i\eta)\phi^{a,b}_{\lambda+i\eta}(t)\frac{d\lambda}{c^{a,b}(-\lambda-i\eta)} \\ +& \frac{1}{4\pi}\int_{\mathbb{R}} \widehat{f}^{a,b}(-\lambda-i\eta)\phi^{a,b}_{-\lambda-i\eta}(t)\frac{d\lambda}{c^{a,b}(\lambda+i\eta)} + \text{Residual terms,} \end{aligned}$$

where we have moved half the integral across the real axis if $D_{a,b}\cap\mathbb{R}\neq\{\emptyset\}$ and made a sign change $\lambda\mapsto-\lambda$ in the integral over the line $\Im\lambda=-\eta$. Since $\widehat{f}^{a,b}$ is even, we get our inversion formula from the identity (3). $\square$

Remark 2.3. Theorem 2.2 and its proof were communicated to us by H. Schlichtkrull. For $a>-1$, $b\in\mathbb{R}$ (which implies $\eta=0$), it is due to [7, Appendix 1] (a minor error has been corrected with the introduction of the constant k_ν).

Our proof of the L^p-version of Hardy's uncertainty principle for the Jacobi transform is inspired by the proof of the semisimple Lie group case; see [15]. The following lemma is crucial:

Lemma 2.4. *Let $1\le q<\infty$. Assume that h is an entire function on $\mathbb{C}$ such that*

$$|h(\lambda)|\le Ce^{\gamma|\Re\lambda|^2}(1+|\Im\lambda|)^N,\ \lambda\in\mathbb{C} \qquad \textit{and} \qquad \int_{\mathbb{R}}|h(\lambda)|^q d\lambda<\infty$$

for positive constants γ, N and C. Then $h=0$.

Proof. See [15, Lemma 2.3]. $\square$

Theorem 2.5. *Let $a,b\in\mathbb{C}$, $a\notin-\mathbb{N}$, and suppose that $1\le p\le\infty$, $1\le q<\infty$. Assume that f is an even measurable function on $\mathbb{R}$ satisfying*

$$e^{\alpha|\cdot|^2}e^{(1-\frac{2}{p})\Re\rho|\cdot|}f\in L^p(\mathbb{R},\,|\Delta^{a,b}(t)|) \qquad \textit{and} \qquad e^{\beta|\cdot|^2}\widehat{f}^{a,b}\in L^q(\mathbb{R}),$$

for positive constants α,β such that $\alpha\beta\ge\frac{1}{4}$. Then $f=0$ almost everywhere.

Proof. Let f be an even measurable function satisfying the above growth conditions. The very rapid decay implies that $f\in L^1(\mathbb{R}_+,|\Delta^{a,b}(t)|dt)\cap L^2(\mathbb{R}_+,|\Delta^{a,b}(t)|dt)$ and that $\widehat{f}^{a,b}(\lambda)$ defines an analytic function in $\lambda\in\mathbb{C}$ for all $a,b\in\mathbb{C}$. We note that it suffices to prove the theorem for $\alpha\beta=\frac{1}{4}$. First let $p<\infty$. Using Lemma 2.1, we get the following estimates on $\widehat{f}^{a,b}(\lambda)$ (for different constants $C, N > 0$):

$$\begin{aligned}
|\widehat{f}^{a,b}(\lambda)| &\le C\int_{\mathbb{R}_+} |f(t)|(1+|\lambda|)^k(1+t)e^{(|\Im\lambda|-\Re\rho)t}|\Delta^{a,b}(t)|dt\\
&\le C(1+|\lambda|)^k\int_{\mathbb{R}_+}|f(t)|e^{\alpha t^2}e^{(1-\frac{2}{p})\Re\rho t}(1+t)e^{|\Im\lambda|t}e^{-\alpha t^2}e^{(\frac{2}{p}-2)\Re\rho t}|\Delta^{a,b}(t)|dt\\
&\le C(1+|\lambda|)^k\left(\int_{\mathbb{R}_+}\left((1+t)e^{|\Im\lambda|t}e^{-\alpha t^2}e^{-\frac{2}{p'}\Re\rho t}\right)^{p'}|\Delta^{a,b}(t)|dt\right)^{\frac{1}{p'}}\\
&\le C(1+|\lambda|)^k\left(\int_{\mathbb{R}_+}(1+t)^{p'}e^{p'|\Im\lambda|t}e^{-p'\alpha t^2}dt\right)^{\frac{1}{p'}}\\
&= C(1+|\lambda|)^k e^{|\Im\lambda|^2/4\alpha}\left(\int_{\mathbb{R}_+}(1+t)^{p'}e^{-p'\alpha(t-|\Im\lambda|/2\alpha)^2}dt\right)^{\frac{1}{p'}}\\
&= C(1+|\lambda|)^k e^{|\Im\lambda|^2/4\alpha}\left(\int_{-|\Im\lambda|/2\alpha}^{\infty}(1+t+|\Im\lambda|/2\alpha)^{p'}e^{-p'\alpha t^2}dt\right)^{\frac{1}{p'}}\\
&\le C(1+|\lambda|)^N e^{|\Im\lambda|^2/4\alpha}
\end{aligned}$$

for $\lambda \in \mathbb{C}$, using translation invariance of dt, the Hölder inequality (with $\frac{1}{p}+\frac{1}{p'}=1$) and the inequality $|\Im\lambda| \le |\lambda|$. For $p=\infty$, we have $|f(t)| \le Ce^{-\alpha|t|^2}e^{-\Re\rho|t|}$ for some positive constant C, and we get

$$\begin{aligned}
|\widehat{f}^{a,b}(\lambda)| &\le C\int_{\mathbb{R}_+} e^{-\alpha t^2}e^{-\Re\rho|t|}(1+|\lambda|)^k(1+t)e^{(|\Im\lambda|-\Re\rho)t}|\Delta^{a,b}(t)|dt\\
&\le C(1+|\lambda|)^k e^{|\Im\lambda|^2/4\alpha}\int_{\mathbb{R}_+}(1+t)e^{-\alpha(t-|\Im\lambda|/2\alpha)^2}dt\\
&\le C(1+|\lambda|)^k e^{|\Im\lambda|^2/4\alpha}\int_{-|\Im\lambda|/2\alpha}^{\infty}(1+t+|\Im\lambda|/2\alpha)e^{-\alpha t^2}dt\\
&\le C(1+|\lambda|)^N e^{|\Im\lambda|^2/4\alpha}
\end{aligned}$$

for $\lambda \in \mathbb{C}$.

Define $g(\lambda) := \widehat{f}^{a,b}(\lambda)e^{\langle\lambda,\lambda\rangle/4\alpha}$. Then

$$|g(\lambda)| \le C(1+|\lambda|)^N e^{|\Re\lambda|^2/4\alpha} \le C(1+|\Im\lambda|)^N e^{|\Re\lambda|^2/4\alpha'},$$

for some $0 < \alpha' < \alpha$. Furthermore we see that

$$\int_{\mathbb{R}}|g(\lambda)|^q d\lambda = \int_{\mathbb{R}}\left(|\widehat{f}^{a,b}(\lambda)|e^{\beta|\lambda|^2}\right)^q d\lambda < \infty,$$

so Lemma 2.4 implies that g, and hence also $\widehat{f}^{a,b}$, is identically zero on $\mathbb{C}$.

Using (the proof of) Theorem 2.2, we see that

$$\begin{aligned}\int_{\mathbb{R}_+} f(t)h(t)\Delta^{a,b}(t)dt &= \frac{1}{2\pi}\int_{\mathbb{R}_+}\int_{\mathbb{R}} f(t)\widehat{h}^{a,b}(\lambda+i\mu)\phi^{a,b}_{\lambda+i\mu}(t)\frac{d\lambda\Delta^{a,b}(t)dt}{c^{a,b}(-\lambda-i\mu)} \\ &= \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\frac{f(t)\varphi^{a,b}_{\lambda+i\eta}(t)\widehat{g}^{a,b}(\lambda+i\eta)}{c^{a,b}(-\lambda-i\eta)c^{a,b}(\lambda+i\eta)}\Delta^{a,b}(t)dtd\lambda \\ &\qquad - \sum_{\nu\in D_{a,b}} ik_\nu \mathrm{Res}_{\lambda=\nu}\left\{\frac{\int_{\mathbb{R}_+} f(t)\varphi^{a,b}_{\lambda}(t)\widehat{h}^{a,b}(\lambda)\Delta^{a,b}(t)dt}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\} \\ &= \frac{1}{4\pi}\int_{\mathbb{R}}\frac{\widehat{f}^{a,b}(\lambda+i\eta)\widehat{h}^{a,b}(\lambda+i\eta)}{c^{a,b}(-\lambda-i\eta)c^{a,b}(\lambda+i\eta)}d\lambda \\ &\qquad - \sum_{\nu\in D_{a,b}} ik_\nu \mathrm{Res}_{\lambda=\nu}\left\{\frac{\widehat{f}^{a,b}(\lambda)\widehat{h}^{a,b}(\lambda)}{c^{a,b}(-\lambda)c^{a,b}(\lambda)}\right\}\end{aligned}$$

is identically zero for any $h \in C_c^\infty(\mathbb{R})_{\text{even}}$, and we conclude that f is zero almost everywhere. ☐

3. THE FOURIER TRANSFORM ON REAL HYPERBOLIC SPACES

Let $m \geq 1$ and $n \geq 2$ be two integers and consider the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{m+n}$ given by

$$\langle x,y\rangle = x_1y_1 + \cdots + x_my_m - x_{m+1}y_{m+1} - \cdots - x_{m+n}y_{m+n}, \quad x,y \in \mathbb{R}^{m+n}.$$

Let $G = SO_o(m,n)$ denote the connected group of $(m+n)\times(m+n)$ matrices preserving $\langle\cdot,\cdot\rangle$ and let $H = SO_o(m-1,n) \subset G$ denote the isotropy subgroup of the point $(1,0,\ldots,0) \in \mathbb{R}^{m+n}$. Let $K = SO(m)\times SO(n) \subset G$ be the (maximal compact) subgroup of elements fixed by the classical Cartan involution on G: $\theta(g) = (g^*)^{-1}$.

The space $\mathbb{X} := G/H$ is a semisimple symmetric space (an involution τ of G fixing H is given by $\tau(g) = JgJ$, where J is the diagonal matrix with entries $(1,-1,\ldots,-1)$). The map $g \mapsto g\cdot(1,0,\ldots,0)$ induces an embedding of $\mathbb{X}$ in $\mathbb{R}^{m+n}$ as the hypersurface (with $x_1 > 0$ if $m = 1$)

$$\mathbb{X} = \{x \in \mathbb{R}^{m+n} | \langle x,x\rangle = 1\}.$$

Let $\mathbb{Y} := \mathbb{S}^{m-1}\times\mathbb{S}^{n-1}$. We introduce spherical coordinates on $\mathbb{X}$ as

$$x(t,y) = (v\cosh t, w\sinh t),\ t\in\mathbb{R}_+,\ y=(v,w)\in\mathbb{Y}.$$

The map is injective, continuous and maps onto a dense subset of $\mathbb{X}$. The (K-invariant) metric distance from $x \in \mathbb{X}$ to the origin is given by $|x| = |x(t,y)| = |t|$.

The unique (up to a constant) G-invariant measure on $\mathbb{X}$ is in spherical coordinates given by

$$\int_{\mathbb{X}} f(x)dx = \int_{\mathbb{R}_+\times\mathbb{Y}} f(x(t,y))J(t)dtdy$$

(see [9, Part II, Example 2.3]), where $J(t) = \cosh^{m-1}t\sinh^{n-1}t$ is the Jacobian, dt the Lebesgue measure on $\mathbb{R}$ and dy an invariant measure on $\mathbb{Y}$, normalised such that $\int_{\mathbb{Y}} 1dy = 1$.

The action of $SO(m)$ on $C^\infty(\mathbb{S}^{m-1})$ decomposes into irreducible representations $\mathcal{H}^r$ of spherical harmonics of degree $|r|$ (see [10, Introduction]) characterised as the eigenfunctions of the Laplace–Beltrami operator Δ_m on $\mathbb{S}^{m-1}$ with eigenvalue $-r(r+m-2)$. Here $r=0$ if $m=1$, $r\in\mathbb{Z}$ for $m=2$ and $r\in\mathbb{N}\cup\{0\}$ for $m>2$.

Let $\mathcal{H}^{r,s}=\mathcal{H}^r\otimes\mathcal{H}^s$ and denote the representation of K on $\mathcal{H}^{r,s}$ by $\delta_{r,s}$. Let $d_{r,s}=\dim\mathcal{H}^{r,s}$ and $\chi_{r,s}$ denote the dimension and the character of $\delta_{r,s}$. A function in $L^2(\mathbb{X})$ is said to be of K-type (r,s) if its translates under the left regular action of K span a vector space which is equivalent to $\mathcal{H}^{r,s}$ as a K-module. We write $L^2(\mathbb{X})^{r,s}$ for the collection of functions of K-type (r,s). The projection $\mathbf{P}^{r,s}$ of $L^2(\mathbb{X})$ onto $L^2(\mathbb{X})^{r,s}$ is given by

$$\mathbf{P}^{r,s}f(x)=d_{r,s}\int_K\chi_{r,s}(k^{-1})f(k\cdot x)dk,\qquad f\in L^2(\mathbb{X}),$$

for $x\in\mathbb{X}$; see [10, Chapter V, §3] and [11, Chapter III, §5]. There are similar definitions and results for functions in $L^2(\mathbb{Y})$ and also for functions in $C^\infty(\mathbb{X})$ and $C^\infty(\mathbb{Y})$.

The algebra of left-G-invariant differential operators on $\mathbb{X}$ is generated by the Laplace–Beltrami operator $\Delta_{\mathbb{X}}$ (see [9, Part II, Example 4.1]), which in spherical coordinates is given by

$$\Delta_{\mathbb{X}}f=\frac{1}{J(t)}\frac{\partial}{\partial t}\left(J(t)\frac{\partial f}{\partial t}\right)-\frac{1}{\cosh^2 t}\Delta_m f+\frac{1}{\sinh^2 t}\Delta_n f,\qquad f\in C^\infty(\mathbb{X});$$

see [16, p. 455]. It reduces to a differential operator $\Delta_{\mathbb{X}}^{r,s}$ in the t-variable when acting on functions of K-type (r,s):

$$\begin{aligned}\Delta_{\mathbb{X}}^{r,s}f&=\Delta_{\mathbb{X}}f\\&=\frac{1}{J(t)}\frac{\partial}{\partial t}\left(J(t)\frac{\partial f}{\partial t}\right)+\frac{r(r+m-2)}{\cosh^2 t}f-\frac{s(s+n-2)}{\sinh^2 t}f,\qquad f\in C^\infty(\mathbb{X})^{r,s}.\end{aligned}$$

Consider the differential equation

$$\Delta_{\mathbb{X}}f=\Delta_{\mathbb{X}}^{r,s}f=(\lambda^2-\rho^2)f,\qquad f\in C^\infty(\mathbb{X})^{r,s},\tag{5}$$

where $\rho=\frac{1}{2}(m+n-2)$. Altering the proof of [11, Chapter I, Proposition 2.7] to fit our setup, we see that we can write any function $f\in C^\infty(\mathbb{X})^{r,s}$ in spherical coordinates as

$$f(x(t,y))=\sum_i f_i(t)\phi_i^{r,s}(y),\tag{6}$$

where $\{\phi_i^{r,s}\}=\{\phi^r\otimes\phi^s\}_i$ is a (finite) basis for $\mathcal{H}^{r,s}$, and f_i is a function of the form $f_i(t)=t^{|s|}f_{i,o}(t)$, with $f_{i,o}$ even. Let $x=-\sinh^2 t$ and $g=(1-x)^{-|r|/2}(-x)^{-|s|/2}f_i$. Then g is a solution to the hypergeometric differential equation with parameters $1/2(\lambda+\rho+|r|+|s|)$, $1/2(-\lambda+\rho+|r|+|s|)$ and $q/2+|s|$. Let $\Phi_\lambda^{r,s}$ denote the regular (for generic λ) solution to this hypergeometric differential equation satisfying the asymptotic condition $\Phi_\lambda^{r,s}(t)\sim e^{(\lambda-\rho)t}$ for $t\to\infty$ (for $\Re\lambda>0$ and when defined). Then

$$\begin{aligned}\Phi_\lambda^{r,s}(t)&=2^{\lambda-\rho-|r|-|s|}\cosh^{|r|}t\sinh^{|s|}t\\&\quad\times\frac{\Gamma(\frac{1}{2}(\lambda+\rho+|r|+|s|))\Gamma(\frac{1}{2}(\lambda-\rho+n-|r|+|s|))}{\Gamma(\lambda)\Gamma(\frac{n}{2}+|s|)}\\&\quad\times{}_2F_1\left(\frac{1}{2}(\lambda+\rho+|r|+|s|),\frac{1}{2}(-\lambda+\rho+|r|+|s|);\frac{n}{2}+|s|;-\sinh^2 t\right),\end{aligned}$$

for $\Re\lambda > 0$; see [1, pp. 72 and 76]. We also note that the function $x(t,y) \mapsto \Phi^{r,s}_\lambda(t)\phi(y)$ extends to a solution of (5) on $\mathbb{X}$ for any $\phi \in \mathcal{H}^{r,s}$.

Let $\epsilon \in \{0,1\}$ and define $C^\infty_\epsilon(\mathbb{Y}) := \{\phi \in C^\infty(\mathbb{Y}) | \phi(-y) = (-1)^\epsilon \phi(y)\}$. The Poisson transform, $F_{\epsilon,\lambda} : C^\infty_\epsilon(\mathbb{Y}) \to C^\infty(\mathbb{X})$, is defined as

$$F_{\epsilon,\lambda}\phi(x) := \int_{\mathbb{Y}} |\langle x,y\rangle|^{(-\lambda-\rho)} \operatorname{sign}^\epsilon \langle x,y\rangle \phi(y)dy, \qquad \phi \in C^\infty_\epsilon(\mathbb{Y}),$$

when $-\Re\lambda \geq \rho$.

Lemma 3.1. *Let $\phi \in C^\infty_\epsilon(\mathbb{Y})$. The (meromorphic extension of the) function $F_{\epsilon,\lambda}\phi$ is an eigenfunction of the Laplace–Beltrami operator $\Delta_{\mathbb{X}}$ with eigenvalue $\lambda^2 - \rho^2$ (when defined), i.e.,*

$$\Delta_{\mathbb{X}} F_{\epsilon,\lambda}\phi = (\lambda^2 - \rho^2) F_{\epsilon,\lambda}\phi\,.$$

The asymptotic behaviour of $F_{\epsilon,\lambda}\phi$ for $t \to \infty$ is given by (when defined)

$$F_{\epsilon,\lambda}\phi(x(t,y)) \sim e^{(\lambda-\rho)t} c(\epsilon,\lambda)\phi(y)$$

for $\Re\lambda > 0$, where $c(\epsilon,\lambda)$ is the so-called c-function for $\mathbb{X}$ given by

$$c(\epsilon,\lambda) = \frac{2^{2\rho-1}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\pi} \frac{\Gamma(\lambda)}{\Gamma(\lambda+\rho)} \begin{cases} \tan\left(\frac{\pi}{2}(\lambda+\rho+\epsilon)\right) & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}. \end{cases}$$

Proof. The function $F_{\epsilon,\lambda}\phi$ extends meromorphically to $\mathbb{C}$ by distribution theory; see [16, Lemma 5(a)]. Differentiating under the integral sign for $\Re(\lambda+\rho)$ very negative and then using meromorphic continuation shows that it is an eigenfunction of the Laplace–Beltrami operator $\Delta_{\mathbb{X}}$ with eigenvalue $\lambda^2 - \rho^2$. The asymptotic behaviour is computed in [16, Appendix A]; see also [16, Lemma 4 and Lemma 5]. ☐

We define the (normalised) Fourier transform $\mathcal{F}f$ of any function $f \in C^\infty_c(\mathbb{X})$ as

$$\mathcal{F}f(\epsilon,\lambda,y) := c(\epsilon,-\lambda)^{-1} \int_{\mathbb{X}} |\langle x,y\rangle|^{(\lambda-\rho)} \operatorname{sign}^\epsilon \langle x,y\rangle f(x)dx,$$

for $\epsilon \in \{0,1\}$, $\Re\lambda \geq \rho$ and $y \in \mathbb{Y}$. Let $f \in C^\infty_c(\mathbb{X})^{r,s}$ for some fixed K-type (r,s). We can (re)write the Fourier transform of f as (with $\epsilon \equiv r+s \mod 2$)

$$\mathcal{F}f(\epsilon,\lambda,y) = \int_{\mathbb{R}_+} \Phi^{r,s}_{-\lambda}(t) f(x(t,y)) J(t)dt,$$

using spherical coordinates, Schur's Lemma and properties of the Poisson transform; see [1, pp. 74–76] for details. We see that $\mathcal{F}f(\epsilon,\lambda,y)$ extends to a meromorphic function in the λ-variable, with zeros and poles completely determined by the above expression for $\Phi^{r,s}_\lambda$. Due to the factor $\Gamma(\lambda)$ in the denominator there are no poles for purely imaginary λ.

Theorem 3.2. *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$. Assume that f is a measurable function on $\mathbb{X}$ satisfying*

$$e^{\alpha|\cdot|^2} e^{(1-\frac{2}{p})\rho|\cdot|} f \in L^p(\mathbb{X}) \qquad \text{and} \qquad e^{\beta|\cdot|^2}\mathcal{F}f \in L^q(\{0,1\} \times i\mathbb{R} \times \mathbb{Y})$$

for positive constants α, β such that $\alpha\beta \geq \frac{1}{4}$. Then $f = 0$ almost everywhere.

Proof. Let f be a measurable function satisfying the above growth conditions. The very rapid decay again implies that $f \in L^1(\mathbb{X}) \cap L^2(\mathbb{X})$ and that the Fourier transform $\mathcal{F}f$ is well-defined.

Define $\tilde{\rho} = \rho + |r| + |s|$, $a = |s| + \frac{n}{2} - 1$ and $b = |r| + \frac{m}{2} - 1$ (i.e., $\tilde{\rho} = a + b + 1$). Then

$$\Phi_\lambda^{r,s}(t) = 2^{\lambda-\tilde{\rho}} \cosh^{|r|} t \sinh^{|s|} t \frac{\Gamma(\frac{1}{2}(\lambda+\tilde{\rho}))\Gamma(\frac{1}{2}(\lambda-\tilde{\rho}+n+2|s|))}{\Gamma(\frac{n}{2}+|s|)\Gamma(\lambda)} \varphi_{-i\lambda}^{a,b}(t).$$

Let $f_{r,s}(t,y) := \mathbf{P}^{r,s} f(x(t,y))/\cosh^{|r|} t \sinh^{|s|} t$. By (6) and continuity of the projection $\mathbf{P}^{r,s}$ we see that $f_{r,s}$ is a measurable function on $\mathbb{R} \times \mathbb{Y}$, even in the t-variable. Also let

$$Q_{r,s}(\lambda) := 2^{\lambda-3\tilde{\rho}} \frac{\Gamma(\frac{1}{2}(\lambda+\tilde{\rho}))\Gamma(\frac{1}{2}(\lambda-\tilde{\rho}+n+2|s|))}{\Gamma(\frac{n}{2}+|s|)\Gamma(\lambda)}.$$

Then

$$\widehat{f}_{r,s}^{a,b}(i\lambda, y) := \int_{\mathbb{R}_+} f_{r,s}(t,y)\varphi_{i\lambda}^{a,b}(t)\Delta^{a,b}(t)dt = Q_{r,s}(\lambda)^{-1}\mathcal{F}\mathbf{P}^{r,s}f(\epsilon,\lambda,y).$$

We note that $\widehat{f}_{r,s}^{a,b}(\lambda, y)$ is well-defined for all $\lambda \in \mathbb{C}$. Using spherical coordinates and the definition of $\mathbf{P}^{r,s}$, we get the following estimates of $f_{r,s}$ and $\widehat{f}_{r,s}^{a,b}$:

$$e^{\alpha|\cdot|^2} e^{(1-\frac{2}{p})\tilde{\rho}|\cdot|} f_{r,s}(\cdot, y) \in L^p(\mathbb{R}, |\Delta^{a,b}(t)|) \quad \text{and} \quad |Q_{r,s}(i\cdot)| e^{\beta|\cdot|^2} \widehat{f}_{r,s}^{a,b}(\cdot, y) \in L^q(\mathbb{R}),$$

for $y \in \mathbb{Y}$. We now note that $|Q_{r,s}(\lambda)|$ is bounded away from zero for $|\Im\lambda| > 0$ (see [6, 1.18]), which implies that

$$e^{\alpha|\cdot|^2} e^{(1-\frac{2}{p})\tilde{\rho}|\cdot|} f_{r,s}(\cdot, y) \in L^p(\mathbb{R}, |\Delta^{a,b}(t)|) \qquad \text{and} \qquad e^{\beta|\cdot|^2} \widehat{f}_{r,s}^{a,b}(\cdot, y) \in L^q(\mathbb{R}),$$

for $y \in \mathbb{Y}$. It follows from Theorem 2.5 that $f_{r,s}^{a,b}$ is zero almost everywhere, and thus also that $\mathbf{P}^{r,s} f$ is zero almost everywhere. We conclude the theorem since $f = \sum_{r,s} \mathbf{P}^{r,s} f$. □

4. REMARKS AND FURTHER RESULTS

Hardy's uncertainty principle (the $p = q = \infty$, $\alpha\beta > \frac{1}{4}$ case) for the Jacobi transform and for the Fourier transform on real hyperbolic spaces was proved by the author in [2].

The space $\mathbb{X} = SO_o(m,n)/SO_o(m-1,n)$ is a semisimple Riemannian symmetric space of the non-compact type when $m = 1$ and of the non-Riemannian type when $m > 1$. Our statement and proof generalise to all rank 1 Riemannian symmetric spaces of the non-compact type, using the fact that the Fourier transform of K-finite functions can be expressed by Jacobi functions.

Versions of Theorem 3.2 for Riemannian symmetric spaces were proved by J. Sengupta (see [17, Theorem 2]) and by E. K. Narayanan and S. K. Ray; see [15, Theorem 3.2] (as a corollary of this theorem, valid for functions on semisimple Lie groups). The former result does not have the $e^{(1-\frac{2}{p})\rho|x|}$ factor in the estimate for the function f and is only valid for $\alpha\beta > \frac{1}{4}$. The latter result is stated using the Harish-Chandra function instead of the factor $e^{-\rho|x|}$, and thus by [15, Proposition 2.1 iii)] differs from our result by a factor $(1 + |t|)$.

Let $a = b = \frac{1}{2}$, i.e., $\rho = 2$. In this case the Jacobi transform can be viewed as the spherical Helgason–Fourier transform on $SL(2,\mathbb{C})$. Let $\alpha\beta = \frac{1}{4}$ and assume that $2 < p < \infty$. Following [15], it can be shown that the $e^{(1-\frac{2}{p})2|t|}$ factor is optimal;

there exists for any number s such that $0 < s < 1 - \frac{2}{p}$ a non-zero even function f such that

$$\int_{\mathbb{R}} \left(|f(t)|e^{\alpha|t|^2}e^{s2|t|}\right)^p |\Delta^{a,b}(t)|dt < \infty \qquad \text{and} \qquad \int_{\mathbb{R}} \left(|\widehat{f}^{a,b}(\lambda)|e^{\beta|\lambda|^2}\right)^q d\lambda < \infty.$$

Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{H}$ and let $x \mapsto \bar{x}$ be the standard (anti)-involution of $\mathbb{F}$. Let m and n be two positive integers and let $[\,,\,]$ be the Hermitian form on $\mathbb{F}^{m+n}$ given by

$$[x, y] = x_1\bar{y}_1 + \cdots + x_m\bar{y}_m - x_{m+1}\bar{y}_{m+1} - \cdots - x_{m+n}\bar{y}_{m+n},$$

for $x, y \in \mathbb{F}^{m+n}$. Let $G = U(m, n; \mathbb{F})$ denote the group of all $(m+n)\times(m+n)$ matrices over $\mathbb{F}$ preserving $[\,,\,]$. Thus $U(m, n; \mathbb{C}) = U(m, n)$ and $U(m, n; \mathbb{H}) = Sp(m, n)$ in standard notation. Let H be the subgroup of G stabilising the line $\mathbb{F}(1, 0, \ldots, 0)$ in $\mathbb{F}^{m+n}$. We can identify H with $U(1, 0; \mathbb{F}) \times U(m-1, n; \mathbb{F})$ and the homogeneous space G/H (which is a reductive symmetric space) with the projective image of the space $\{z \in \mathbb{F}^{m+n} | [z, z] = 1\}$. The statement and proof of the L^p version of Hardy's uncertainty principle for the Fourier transform on G/H follows from above, either embedding G/H into $SO_o(dm, dn)/SO_o(dm-1, dn)$, with $d = \dim_{\mathbb{R}} \mathbb{F}$, or again expressing the Fourier transform of K-finite functions using modified Jacobi functions. See [1, p. 117] for more details.

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