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p-HYPONORMAL OPERATORS ARE SUBSCALAR

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ABSTRACT. We prove that if $R, S \in B(\mathbf{X})$, R, S are injective, then RS is subscalar if and only if SR is subscalar. As corollaries, it is shown that p-hyponormal operators ($0 < p \leq 1$) and log-hyponormal operators are subscalar; also w-hyponormal operators T with $\mathrm{Ker}T \subset \mathrm{Ker}T^*$ and their generalized Aluthge transformations $T(r, 1-r)$ ($0 < r < 1$) are subscalar.

1. Introduction and notations

Let $\mathbf{X}$ denote a Banach space. $T \in B(\mathbf{X})$ is said to be generalized scalar ([1]) if there exists a continuous algebra homomorphism

$$\Phi : \quad \varepsilon(\mathbb{C}) \longrightarrow B(\mathbf{X})$$

with $\Phi(1) = I$ and $\Phi(z) = T$. Here $\varepsilon(\mathbb{C})$ denotes the algebra of all infinitely differentiable functions on $\mathbb{C}$ with the topology defined by the uniform convergence of such functions and their derivatives ([2]). An operator similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces is called subscalar. Subscalar operators are subdecomposable ([1]).

Let $\mathbf{H}$ denote an infinite-dimensional separable Hilbert space, $T \in B(\mathbf{H})$. T is said to be p-hyponormal ($p > 0$) if $(T^*T)^p \geq (TT^*)^p$, i.e., $|T|^{2p} \geq |T^*|^{2p}$. A p-hyponormal operator is called hyponormal if $p = 1$ and semi-hyponormal if $p = \frac{1}{2}$ ([3], [4]). The Löwner-Heinz inequality implies that if T is q-hyponormal, then it is p-hyponormal for any $0 < p \leq q$. An invertible operator T is said to be log-hyponormal ([5]) if $\log(T^*T) \geq \log(TT^*)$.

For $T \in B(\mathbf{H})$, let $T = U|T|$ be the polar decomposition of T, and for $s, t \geq 0$, $T(s,t) = |T|^s U |T|^t$ is called the generalized Aluthge transformation of T. $T(\frac{1}{2}, \frac{1}{2}) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is denoted by $\hat{T}$. It is known that if T is p-hyponormal ($p > 0$), then $\hat{T}$ is semihyponormal and $(\hat{\hat{T}}) = \hat{T}(\frac{1}{2}, \frac{1}{2})$ (denoted by $\tilde{T}$) is hyponormal. If T is log-hyponormal, then $\hat{T}$ is semihyponormal. T is said to be w-hyponormal ([6]) if $|\hat{T}| \geq |T| \geq |\hat{T}^*|$, so $\hat{T}$ is semihyponormal if T is w-hyponormal. It is shown in [6] that p-hyponormal ($p > 0$) and log-hyponormal operators are w-hyponormal.

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M. Putinar showed ([7]) that hyponormal operators are subscalar; Y. Chu showed ([8]) that semihyponormal operators are also subscalar. Are all p-hyponormal ($p > 0$) operators subscalar? Some authors have discussed this problem and obtained certain partial answers. We are going to derive a result: if $R, S \in B(\mathbf{X}), R, S$ injective, then RS is subscalar if and only SR is subscalar. Appling this result to generalized Aluthge transformation of operators, we get an affirmative and stronger answer to the above question: all p-hyponormal operators and log-hyponormal operators are subscalar; w-hyponormal operators T with Ker$T \subset$KerT^* and their generalized Aluthge transformation $T(r, 1-r)$ $(0 < r < 1)$ are also subscalar.

2. Main results

Let $\mathbf{X}$ denote a Banach space. $R, S \in B(\mathbf{X})$, operators RS and SR have many common properties, including spectral properties, subdecomposability, nontriviality of invariant and hyperinvariant subspaces, etc. (see [9], [10]). It is proved in [9] that if R, S are injective, then RS is subdecomposable if and only if SR is subdecomposable, whence one can derive that p-hyponormal operators ($p > 0$) are subdecomposable (see also [11]). We are now going to present a similar result for subscalarity.

Theorem A. *If R, $S \in B(\mathbf{X})$, where R and S are injective, then RS is subscalar if and only if SR is subscalar.*

Some lemmas are to be considered first. $\varepsilon(U, \mathbf{X})$ denotes the Fréchet space of all $\mathbf{X}$-valued C^∞-functions, i.e., infinitely continuously differentiable functions on U ([1]). T is said to have property $(\beta)_\varepsilon$ (denoted by $T \in (\beta)_\varepsilon$) if for each open set U in $\mathbb{C}$ the operator

$$T_z : \quad \varepsilon(U, \mathbf{X}) \longrightarrow \varepsilon(U, \mathbf{X}), \qquad f \longmapsto (T - z)f$$

is a topological monomorphism, i.e., $T_z f_n \to 0$ implies $f_n \to 0$ for $f_n \in \varepsilon(U, \mathbf{X})$.

Lemma 1. *For $T \in B(\mathbf{X})$, the following statements are equivalent:*

(1) T is subscalar;

(2) $T \in (\beta)_\varepsilon$, i.e., for each open subset U of $\mathbb{C}, f_n \in \varepsilon(U, \mathbf{X}), (T - z)f_n \longrightarrow 0$ $(n \to \infty)$ implies $f_n \longrightarrow 0$ $(n \to \infty)$;

(3) $T_z = T - z$ acts one-to-one and with closed range on $\varepsilon(U, \mathbf{X})$ for each open subset U of $\mathbb{C}$.

Proof. (1)$\Longleftrightarrow$ (2). See corollary 4.6 of [1].

(2)$\Longrightarrow$ (3). Suppose $f \in \varepsilon(U, \mathbf{X}), T_z f = 0$. Take $f_n = f$ $(n = 1, 2, \cdots)$; then $T_z f_n \longrightarrow 0$ $(n \to \infty)$, $f = \lim_{n\to\infty} f_n = 0$. T_z is one-to-one on $\varepsilon(U, \mathbf{X})$.

Suppose that $f_n, g \in \varepsilon(U, \mathbf{X})$, $\lim_{n\to\infty} T_z f_n = g$. Then $T_z(f_n - f_m) \longrightarrow 0$ $(n, m \to \infty)$. This implies that $f_n - f_m \longrightarrow 0$ $(n, m \to \infty)$ and $f_n \longrightarrow f \in \varepsilon(U, \mathbf{X})$ $(n \to \infty)$ since the Fréchet space $\varepsilon(U, \mathbf{X})$ is complete. It follows that $g = \lim_{n\to\infty} T_z f_n = T_z f$. Therefore the range of T_z is closed.

(3)$\Longrightarrow$ (2). Suppose that $f_n \in \varepsilon(U, \mathbf{X}), n = 1, 2, \cdots, T_z f_n \longrightarrow 0$ $(n \to \infty)$. Let $E = \{T_z f : f \in \varepsilon(U, \mathbf{X})\}$; $\mathbf{E}$ is a closed subspace of $\varepsilon(U, \mathbf{X})$ and is therefore a Fréchet space, too. $S = T_z : \varepsilon(U, \mathbf{X}) \longrightarrow E$ is one-to-one and subjective. In view that S is a continuous linear mapping from $\varepsilon(U, \mathbf{X}) \longrightarrow \mathbf{E}$, the open mapping theorem implies that S^{-1} is continuous and hence that $f_n = S^{-1} T_z f_n \longrightarrow 0$ $(n \to \infty)$. $\square$

Lemma 2. *Let $U \subset \mathbb{C}, U$ be open, $0 \in U$; $g(z) \in \varepsilon(U, \mathbf{X}), \varphi(z) \in \varepsilon(U \setminus \{0\}, \mathbf{X})$, $Q \in B(\mathbf{X})$ being injective, $g(z) = (Q - z)\varphi(z), z \in U \setminus \{0\}$. $\{z_n\}$ and $\{t_n\}$ are two sequences in $\mathbb{C}$ converging to 0. Let (1) $\Phi(z) = \varphi(z)$ or (2) $\Phi(z) = \frac{\partial^k \varphi(z)}{\partial u_1 \cdots \partial u_k}$ ($u_i = x$ or $y, i = 1, 2, \cdots, k$), φ and its derivatives of order $n < k$ are continuous at $z = 0$. If $\lim_{n\to\infty} \Phi(z_n)$ and $\lim_{n\to\infty} \Phi(t_n)$ both exist, then $\lim_{n\to\infty} \Phi(z_n) = \lim_{n\to\infty} \Phi(t_n)$.*

Proof. (1) $\Phi(z) = \varphi(z)$. By hypothesis,

$$g(z_n) = (Q - z_n)\varphi(z_n), g(0) = \lim_{n\to\infty} g(z_n) = Q \lim_{n\to\infty} \varphi(z_n).$$

Similarly $g(0) = Q \lim_{n\to\infty} \varphi(t_n)$. The injectivity of Q then implies that

$$\lim_{n\to\infty} \Phi(z_n) = \lim_{n\to\infty} \Phi(t_n).$$

(2) $\Phi(z) = \frac{\partial^k \varphi(z)}{\partial u_1 \cdots \partial u_k}$ It is clear by calculation and induction that

$$G(z) = \frac{\partial^k g(z)}{\partial u_1 \cdots \partial u_k} = (Q - z)\Phi(z) - \sum_{i=1}^{m} a_i \psi_i(z), \quad z \in U \setminus \{0\}.$$

Here $\psi_i = \varphi$ or derivatives of φ of order $n < k, a_i$ are constants, $1 \le i \le m$. Since $g \in \varepsilon(U, \mathbf{X})$, hence

$$G(0) = \lim_{n\to\infty} G(z_n) = Q \lim_{n\to\infty} \Phi(z_n) - \sum_{i=1}^{m} a_i \lim_{n\to\infty} \psi_i(z_n),$$

$$G(0) = \lim_{n\to\infty} G(t_n) = Q \lim_{n\to\infty} \Phi(t_n) - \sum_{i=1}^{m} a_i \lim_{n\to\infty} \psi_i(t_n).$$

The continuity of ψ_i at $z = 0$ implies that $\lim_{n\to\infty} \psi_i(z_n) = \lim_{n\to\infty} \psi_i(t_n)$, $1 \le i \le m$. Consequently we have $Q \lim_{n\to\infty} \Phi(z_n) = Q \lim_{n\to\infty} \Phi(t_n)$ and $\lim_{n\to\infty} \Phi(z_n) = \lim_{n\to\infty} \Phi(t_n)$ by the injectivity of Q. □

Lemma 3. *Let $0 \in U \subset \mathbb{C}, U$ be open, $h(z), g(z) \in \varepsilon(U, \mathbf{X})$, $h(0) = 0, Q \in B(\mathbf{X})$ be injective. If $g(z) = (Q - z)\frac{h(z)}{z}, \quad z \in U \setminus \{0\}$, then we can define*

$$\varphi(z) = \begin{cases} h(z)/z\ , & z \in U \setminus \{0\}, \\ y_0\ , & z = 0 \end{cases}$$

($y_0 \in \mathbf{X}$), such that $\varphi(z) \in \varepsilon(U, \mathbf{X})$.

Proof. It follows from the Taylor formula that

$$h(z) = xh'_x(0) + yh'_y(0) + R(z), \quad z \in U. \tag{1}$$

Here $R(z) = o(|z|)$, i.e., $\frac{R(z)}{|z|} \longrightarrow 0 \ \ (z \to 0)$. Hence

$$\varphi(z) = \frac{h(z)}{z} = \frac{xh'_x(0) + yh'_y(0)}{x + iy} + \frac{R(z)}{z}, \quad z \in U \setminus \{0\}.$$

By (1), $\lim_{x\to 0} \varphi(x) = \lim_{x\to 0} \frac{h(x)}{x} = h'_x(0)$ and $\lim_{y\to 0} \varphi(iy) = \lim_{y\to 0} \frac{h(iy)}{iy} = \frac{h'_y(0)}{i}$ both exist. Lemma 2 implies that $h'_x(0) = \frac{h'_y(0)}{i}$, whence

$$\begin{aligned} \varphi(z) &= \frac{h'_x(0)(x + iy)}{x + iy} + \frac{R(z)}{z} \\ &= h'_x(0) + \frac{R(z)}{z}, \quad z \in U \setminus \{0\}. \end{aligned}$$

Define $y_0 = h'_x(0)$; then we have

$$(C_0) \qquad \varphi(z) = \begin{cases} h'_x(0) + \frac{R(z)}{z}, & z \in U \setminus \{0\}, \\ h'_x(0), & z = 0, \end{cases}$$

$R(z) \in \varepsilon(U, \mathbf{X})$, $R(z) = o(|z|)$ and the continuity of $\varphi(z)$ in U follows at once.

Suppose that Φ is a kth order derivative of φ (including $\varphi^{(0)} = \varphi$) and that φ and all derivatives of φ of order $n \leq k$ are continuous at $z = 0$. Φ satisfies the condition

$$(C_k) \qquad \Phi(z) = \begin{cases} a + \frac{G(z)}{z^{k+1}}, & z \in U \setminus \{0\}, \\ a, & z = 0, \end{cases}$$

$G(z) \in \varepsilon(U, \mathbf{X}), G(z) = o(|z|^{k+1}), a \in \mathbf{X}$ being a constant vector.

We are going to show that Φ'_x and Φ'_y exist and are continuous at $z = 0$ and satisfy condition (C_{k+1}) similar to (C_k).

Applying the Taylor formula, we have

$$(2) \qquad G(z) = G(0) + G_1(z) + \cdots + G_i(z) + \cdots + G_{k+2}(z) + R_1(z),$$
$$G_i(z) = \frac{1}{i!}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})^i G(0),\ 1 \leq i \leq k+2, R_1(z) = o(|z|^{k+2}).$$

The hypothesis $G(z) = o(|z|^{k+1})$ implies that

$$G(0) \equiv G_1(z) \equiv \cdots \equiv G_{k+1}(z) \equiv 0$$

and hence that

$$(3) \qquad \frac{\partial^n G}{\partial^m x \partial^{n-m} y}(0) = 0, \quad 1 \leq n \leq k+1, \quad 0 \leq m \leq n.$$

It follows from (2) and (3) that

$$(4) \qquad G(z) = \frac{G_x^{(k+2)}(0)}{(k+2)!}x^{k+2} + a_1 x^{k+1} y + \cdots + a_{k+2} y^{k+2} + R_1(z).$$

Here $a_1, a_2, \cdots, a_{k+2}$ are constant vectors.

Similarly we have (noticing that $G'_x(z) \in \varepsilon(U, \mathbf{X})$, too), by (3),

$$G'_x(z) = \frac{1}{(k+1)!}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})^{k+1} G'_x(0) + R_2(z)$$
$$(5) \qquad = \frac{G_x^{k+2}(0)}{(k+1)!}x^{k+1} + b_1 x^k y + \cdots + b_{k+1} y^{k+1} + R_2(z).$$

Here $b_1, b_2, \cdots, b_{k+1}$ are constant vectors, $R_2(z) = o(|z|^{k+1})$.

From (4) and (5), we obtain

$$\Phi'_x(0) = \lim_{x \to 0} \frac{\Phi(x) - \Phi(0)}{x} = \lim_{x \to 0} \frac{G(x)}{x^{k+2}} = \frac{G_x^{(k+2)}(0)}{(k+2)!} = a_0,$$

say, and

$$\Phi'_x(z) = \frac{zG'(z) - (k+1)G(z)}{z^{k+2}}$$
$$= \frac{a_0 x^{k+2} + \alpha_1 x^{k+1} y + \cdots + \alpha_{k+2} y^{k+2}}{(x+iy)^{k+2}} + \frac{F(z)}{z^{k+2}} \quad (z \in U \setminus \{0\}).$$

Here $\alpha_1, \alpha_2, \cdots, \alpha_{k+2}$ are constant vectors, $F(z) = zR_2(z) - (k+1)R_1(z) = o(|z|^{k+2})$.

Substituting $x = r\cos\theta, y = r\sin\theta$ into the first term of the right-hand side, we obtain

$$\Phi'_x(z) = f(\theta) + \frac{F(z)}{z^{k+2}}, \quad z \in U \setminus \{0\}.$$

$\lim_{r\to 0} \Phi'_x(z) = \lim_{r\to 0} \Phi'_x[r(\cos\theta + i\sin\theta)] = f(\theta)$ exists for every $\theta,\ 0 \le \theta \le 2\pi$, therefore it follows from Lemma 2 that

$$f(\theta) = f(0) = a_0 = \Phi'_x(0).$$

$\Phi'_x(z)$ satisfies the following condition:

$$(C_{k+1}) \qquad \Phi'_x(z) = \begin{cases} a_0 + \frac{F(z)}{z}, & z \in U \setminus \{0\}, \\ a_0, & z = 0, \end{cases}$$

$F(z) \in \varepsilon(U, \mathbf{X}), F(z) = o(|z|^{k+2})$, whence Φ'_x is continuous at $z = 0$.

Similarly we have that Φ'_y exists and is continuous at $z = 0$ and satisfies

$$(C_{k+1}) \qquad \Phi'_y(z) = \begin{cases} b_0 + \frac{E(z)}{z^{k+2}}, & z \in U \setminus \{0\}, \\ b_0, & z = 0, \end{cases}$$

$E(z) \in \varepsilon(U, \mathbf{X}), E(z) = o(|z|^{k+2})$, here b_0 being a constant vector.

We have proved that $\varphi^{(0)} = \varphi$ is continuous at $z = 0$ and satisfies condition (C_0). It follows by mathematical induction that φ has continuous derivatives of any order at $z = 0$, i.e., $\varphi \in \varepsilon(U, \mathbf{X})$. □

Proof of Theorem A. By Lemma 1, it is sufficient to show that if R and S are injective and $RS - z$ is one-to-one and has closed range, then $SR - z$ is one-to-one and has closed range, too.

Let U be any open subset of $\mathbb{C}, F(z) \in \varepsilon(U, \mathbf{X}), (SR - z)F(z) = 0$. It follows that

$$(RS - z)RF(z) = R(SR - z)F(z) = 0 \quad (z \in U).$$

Since $RS - z$ is one-to-one, hence $RF(z) = 0$ and the injectivity of R implies $F(z) = 0$ $(z \in U)$ so that $SR - z$ is one-to-one on $\varepsilon(U, \mathbf{X})$.

Suppose that $g \in \varepsilon(U, \mathbf{X}),\ f_n \in \varepsilon(U, \mathbf{X})$ $(n = 1, 2, \cdots)$ and $(SR - z)f_n(z) \longrightarrow g(z)$ on $\varepsilon(U, \mathbf{X})$. Then $(RS - z)Rf_n(z) = R(SR - z)f_n(z) \longrightarrow Rg(z) \in \varepsilon(U, \mathbf{X})$. Since $RS - z$ has closed range, there exists $f(z) \in \varepsilon(U, \mathbf{X})$ such that $Rg(z) = (RS - z)f(z)$ and hence

$$(6) \qquad RSf(z) = Rg(z) + zf(z).$$

Let $h(z) = Sf(z) - g(z)$. Then $h(z) \in \varepsilon(U, \mathbf{X})$ and

$$Rh(z) = R(Sf(z) - g(z)) = zf(z)$$

$$(SR - z)h(z) = S[zf(z)] - zSf(z) + zg(z) = zg(z)$$

$$(7) \qquad (SR - z)\frac{h(z)}{z} = g(z), \quad z \in U \setminus \{0\}.$$

If $0 \notin U$, then $\frac{h(z)}{z} \in \varepsilon(U, \mathbf{X}), g(z)$ belongs to the range of $SR - z$ on $\varepsilon(U, \mathbf{X})$, that is, $SR - z$ has closed range on $\varepsilon(U, \mathbf{X})$.

If $0 \in U$, then by (6), $Rg(0) = RSf(0), g(0) = Sf(0)$, hence $h(0) = Sf(0) - g(0) = 0$. By Lemma 3, taking $Q = SR$, we can define

$$\varphi(z) = \begin{cases} \frac{h(z)}{z}, & z \in U \setminus \{0\}, \\ y_0, & z = 0, \end{cases}$$

such that $\varphi(z) \in \varepsilon(U, \mathbf{X})$. By (7) and the continuity of φ at $z = 0$, we have

$$(SR - z)\varphi(z) = g(z), \quad z \in U,$$

and hence that $SR - z$ has closed range on $\varepsilon(U, \mathbf{X})$. The conclusion that SR is subscalar now follows. □

Now let us turn to the applications of Theorem A. Let $\mathbf{H}$ denote a separable Hilbert space in the sequel.

Corollary 1. *Let $T \in B(\mathbf{H}), KerT \subset KerT^*, s \geq 0, 0 < r < t$. Then $T(s,t)$ is subscalar if and only if $T(s+r, t-r)$ is subscalar; especially $T = T(0,1)$ is subscalar if and only if $T(r, 1-r)$ $(0 < r < 1)$ is subscalar.*

Proof. Since $\text{Ker}T \subset \text{Ker}T^*$, $\text{Ker}T$ reduces T and $T = \theta \oplus T_1$; here $\theta = T\mid_{\text{Ker}T}, T_1 = T\mid_{(\text{Ker}T)^\perp}, \text{Ker}T_1 = \{0\}$. It is clear that $T(\alpha, \beta) = \theta \oplus T_1(\alpha, \beta)$ $(\alpha, \beta > 0)$. Suppose that $T_1 = U_1|T_1|$ is the polar decomposition of $T_1, R = |T_1|^r, S = |T_1|^s U_1 |T_1|^{t-r}$; then $\text{Ker}R = \text{Ker}S = \{0\}, RS = T_1(s+r, t-r), SR = T_1(s,t)$. By Theorem A, it concludes that $T_1(s,t)$ is subscalar if and only if $T_1(s+r, t-r)$ is subscalar and hence that $T(s,t)$ is subscalar if and only if $T(s+r, t-r)$ is subscalar. □

Corollary 2. *Let $T \in B(\mathbf{H})$ be a p-hyponormal $(0 < p \leq 1)$ operator or a log-hyponormal operator. Then T is subscalar.*

Proof. Suppose T is a p-hyponormal $(0 < p \leq 1)$ operator. Then we have $\text{Ker}\,T \subset \text{Ker}\,T^*$ ([12], Lemma 1), $\tilde{T} = \hat{T}(\frac{1}{2}, \frac{1}{2})$ is hyponormal and therefore subscalar ([7]). By Corollary 1, this implies that $\hat{T} = T(\frac{1}{2}, \frac{1}{2})$ and $T = T(0,1)$ are subscalar.

Suppose T is a log-hyponormal operator. Then T is invertible, $\text{Ker}T = \text{Ker}T^* = \{0\}$. Since $\hat{T} = T(\frac{1}{2}, \frac{1}{2})$ is $\frac{1}{2}$-hyponormal and therefore subscalar, T is also subscalar by Corollary 1. □

Corollary 3. *Let $T \in B(\mathbf{H})$ be w-hyponormal and $KerT \subset KerT^*$. Then T and their generalized Aluthge transformation $T(r, 1-r)$ $(0 < r < 1)$ are subscalar.*

Proof. By definition, $|\hat{T}| \geq |\hat{T}^*|$. This implies that $\hat{T} = T(\frac{1}{2}, \frac{1}{2})$ is $\frac{1}{2}$-hyponormal and subscalar. Since $\text{Ker}T \subset \text{Ker}T^*$, it follows from Corollary 1 that T and $T(r, 1-r)$ $(0 < r < 1)$ are subscalar. □

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