A NOTE ON DIVERGENCE OF $L^p$-INTEGRALS OF SUBHARMONIC FUNCTIONS AND ITS APPLICATIONS

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Abstract. A non $L^p$-integrability condition of non-constant non-negative subharmonic functions on a general complete manifold $(M,g)$ is given in an optimal form. As an application in differential geometry, several topics related to parabolicity of manifolds, the Liouville theorem for harmonic maps and conformal deformation of metrics are shown without any assumption on the Ricci curvature of $(M,g)$.

Introduction

In this paper we always denote by $(M, g)$ a non-compact complete and connected Riemannian manifold of dimension $m$, and by $\Delta_g$ the Laplacian defined by $\Delta_g := \text{Trace}_g \nabla \nabla$. In section 1, for any $p > 1$ and a function $u$ satisfying $u \Delta_g u \geq k$ with a suitable function $k$ on $M$, we show a divergence property of the $L^p$-integral of $u$ over every geodesic sphere of radius $r$ as $r$ tends to infinity. The idea of the proof is due to an abstract vanishing theorem for the gradient length of $L^p$-integrable functions (see Theorem 1.1). This gives a refinement of known results by several authors (cf. [Y], [K], [S], [LY]). As an application, in section 2 we give an alternative proof of Li-Tam’s result which gives a sufficient condition for $(M, g)$ to be parabolic in terms of the area of the geodesic sphere. In section 3 using this condition we show Liouville theorems for harmonic maps from a class of parabolic manifolds to either manifolds of non-positive curvature or hyperbolic Kähler manifolds. In section 4 we study the upper estimate and uniqueness of solutions of certain non-linear differential equations, i.e., the Poisson equation and the scalar curvature one.

1. On $L^p$-integrals of a smooth function $u$ satisfying the inequality $u \Delta_g u \geq k$

The following implies a vanishing of gradient length of $L^p$-integrable functions.

**Theorem 1.1.** Let $K_+(r)$, $I(r)$, and $E(r)$ be non-negative absolutely continuous and non-decreasing functions on $\mathbb{R}_+ := [1, +\infty)$, and let $C_1$ and $C_2$ be positive constants respectively such that $K_+(+\infty) := \lim_{r \to +\infty} K_-(r) < +\infty$, $\frac{d}{dr} I(r) > 0$ and

$$K_+(r) + C_1 E(r) \leq K_-(r) + C_2 \sqrt{\frac{d}{dr} I(r) \frac{d}{dr} E(r)}$$

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for almost all \( r \geq r_0 \gg 1 \). Suppose \( K_-(+\infty) \leq K_+(+\infty) := \lim_{r \to +\infty} K_+(r) \leq +\infty \). Then either \( E(r) \equiv 0 \) or \( 1/\frac{dr}{dt} \in L^1(\mathbb{R}_+) \).

**Proof.** By reduction to absurdity assume that \( E(r) > 0 \) for any \( r \geq r_0 \) and \( 1/\frac{dr}{dt} \not\in L^1(\mathbb{R}_+) \). By dividing (1) by \( \frac{dr}{dt} E(r) > 0 \) and integrating from \( r_0 \) up to \( r > r_0 \), we can see the following:

\[
(2) \quad \frac{K_+(r_0)}{E(r)} + C_1 \leq \frac{E(r_0)}{\sqrt{I(r_0)}} \int_{r_0}^{r} \frac{dt}{I(t)} + \frac{K_-(r)}{E(r_0)}.
\]

If \( E(+\infty) := \lim_{r \to +\infty} E(r) = +\infty \), then letting \( r \to +\infty \) and later \( r_0 \to +\infty \), (2) implies \( 0 < C_1 \leq 0 \), which is a contradiction. Hence \( E(+\infty) < +\infty \) and so by the same procedure we see that \( K_+(+\infty) < +\infty \) and \( K_+(+\infty) - K_-(+\infty) \leq -C_1 E(+\infty) < 0 \). \( \square \)

First Theorem 1.1 induces the following result which implies not only Theorem 2.1 but also Theorem 2.2 in [K] simultaneously, and is a generalization of Theorem 1b in [S] for the Laplacian \( \Delta_g \) in view of the Lemma given in section 5.

**Theorem 1.2.** Suppose \( u \) is a non-constant smooth function satisfying the inequality \( u \Delta u \geq 0 \) on \( M \). Then for any \( p > 1 \), \( r > 0 \) and \( x \in M \), the function \( I^p_r(u, r) \) defined by \( I^p_r(u, r) := \int_{B_r(x)} |u|^p \, dv_g \) satisfies \( 1/\frac{dr}{dt} I^p_r(u, r) \in L^1(\mathbb{R}_+) \), where \( B_r(x) \) is the geodesic ball centered at \( x \in M \) and of radius \( r \).

**Remark 1.** Since the distance function \( r_x \) from a point \( x \in M \) is Lipschitz continuous on \( M \) and satisfies \( |\nabla r_x| \equiv 1 \) within the cut locus of \( x \), letting \( \sigma_r \) be the \((m-1)\)-dimensional Hausdorff measure of the geodesic sphere \( S_x(r) \) induced by \( g \), it follows that \( \frac{dr}{dt} I^p_r(u, r) \) coincides with the integral \( \int_{S_x(r)} |u|^p \, \sigma_r \) for almost all \( r > 0 \) by the co-area formula (cf. [E], 3.2.12, Theorem 3 and 3.2.46).

**Proof of Theorem 1.2.** Let \( \psi \) be a non-negative Lipschitz continuous function on \( M \) with \( \text{Supp}(\psi) \subset B_x(r) \). If \( u \neq 0 \) satisfies \( u \Delta u \geq k \) for a locally integrable function \( k \) on \( M \), then \( |u|^{p-2} |\nabla u|^2 \) is locally integrable and the following estimate is valid:

\[
(3) \quad \int k \psi^2 |u|^{p-2} \, dv_g + \int \psi^2 |u|^{p-2} |\nabla u|^2 \, dv_g \leq \frac{2|\nabla \psi|}{C(p)} \sqrt{\int \frac{1}{T} \int |u|^p \, dv_g \int \frac{1}{T} \psi^2 |u|^{p-2} |\nabla u|^2 \, dv_g} ,
\]

where \( |\nabla \psi| := \sup_M |\nabla \psi| \), \( C(p) := \min\{1, p-1\} > 0 \) and \( T = \text{Supp}(\nabla \psi) \) (cf. (2.3) in the proof of Theorems 2.1 and 2.2 in [K]). For any \( r > 0 \) and \( p > 1 \), we set \( E^p(u, r) := \int_{B_x(r)} |u|^{p-2} |\nabla u|^2 \, dv_g < +\infty \). For any \( \varepsilon > 0 \), we take a Lipschitz continuous function \( \omega_\varepsilon \) such that \( 0 \leq \omega_\varepsilon \leq 1 \), \( \text{Supp}(\omega_\varepsilon) \subset B_x(r-\varepsilon) \), \( \omega_\varepsilon \equiv 1 \) on \( B_x(r-\varepsilon) \), and \( |\nabla \omega_\varepsilon| < 1/\varepsilon \). Putting \( \psi = \omega_\varepsilon \), \( T = B_x(r) \cap B_x(r-\varepsilon) \) and \( k \equiv 0 \), and letting \( \varepsilon \to 0 \) in (3), we get

\[
(4) \quad E^p(u, r) \leq C^* \sqrt{\frac{d}{dr} I^p_r(u, r) \frac{d}{dr} E^p(u, r)}
\]

for almost all \( r > 0 \) and \( C^* := 2/C(p) \). If \( u \) is not constant, then \( E^p(u, r) > 0 \) for any \( r \geq r_0 \gg 0 \). Applying Theorem 1.1 to \( K_\pm(r) \equiv 0 \), \( E(r) = E^p(u, r) \), \( I(r) = I^p_r(u, r) \) and \( C_1 = 1 \), we get the conclusion. \( \square \)
In case $p = 2$ we can relax the non-negativity condition of $k$ as follows.

**Theorem 1.3.** Let $u$ be a smooth non-constant solution satisfying the inequality $u\Delta_g u \geq k$ for a locally integrable function $k$ on $M$. If $k_- := \max\{-k, 0\} \in L^1(M)$ and $\int_M k \, dv_g \geq 0$, then $1/\frac{d}{dr}T^2_x(u, r) \in L^1(\mathbb{R}^+)$ for any $x \in M$.

**Proof.** We set $K_\pm(r) := \int_{B_x(r)} k_\pm \, dv_g$ for $k = k_+ - k_-$ with $k_+ = \max\{k, 0\}$. By using the estimate (3) for $p = 2$ and the same procedure as in the proof of Theorem 1.2, we get

$$K_+(r) + E^2(u, r) \leq K_-(r) + C_0 \sqrt{\frac{d}{dr}T^2_x(u, r) \frac{d}{dr}E^2(u, r)}$$

for almost all $r > 0$. Hence the conclusion follows from Theorem 1.1. \qed

The following is a generalization of Theorem 2.1 in [LY] and Theorem 1 in [Y].

**Theorem 1.4.** Let $u$ be a smooth non-constant solution satisfying the inequality $\Delta_g \log u \geq k$ on $M_+ := \{u > 0\}$. If either $k \equiv 0$ or $k_+ \in L^1(M)$ and $\int_M k \, dv_g > 0$, then $1/\frac{d}{dr}T^p_x(u, r) \in L^1(\mathbb{R}^+)$ for any $p > 0$ and $x \in M$.

**Remark 2.** Under the same situation Li and Yau showed $\lim \inf \frac{\int_{B_x(r)} k \, dv_g}{r^p} > 0$ in [LY], which follows from $1/\frac{d}{dr}T^p_x(u, r) \in L^1(\mathbb{R}^+)$ in view of the Lemma in section 5.

**Proof of Theorem 1.4.** We only have to show the case $p = 1$. For any $c > 0$ and the function $\omega_\varepsilon$ taken in the proof of Theorem 1.2, a direct calculation shows the following:

$$\text{div} \left( \omega_\varepsilon^2 \nabla \log (u + c) \right) \geq \omega_\varepsilon^2 \frac{u_k}{u + c} + \frac{\omega_\varepsilon^2 |\nabla u|^2}{u(u + c)^2} - 2|\nabla \omega_\varepsilon| |\omega_\varepsilon \nabla \log (u + c)|.$$

We set $K_\pm(r, c) := \int_{B_x(r)} \frac{u_k}{u + c} \, dv_g$ and $E(r, c) := \int_{B_x(r)} \frac{1}{u^2(u + c)} \, dv_g$. Integrating the above inequality we can see $E(r, c) < +\infty$ and letting $\varepsilon \to 0$, we get the following:

$$K_+(r, c) + cE(r, c) \leq K_-(r, c) + 2\sqrt{\frac{d}{dr}T^1_x(u, r) \frac{d}{dr}E(r, c)}.$$

In case $k \equiv 0$ the conclusion follows from Theorem 1.1. To see the latter case assume that $u$ is not constant and $1/\frac{d}{dr}T^1_x(u, r) \notin L^1(\mathbb{R}^+)$. The proof of Theorem 1.1 implies $K_+(+\infty, c) + cE(+\infty, c) \leq K_-(+\infty, c)$ for any $c > 0$. By letting $c \to 0$, we get $K_+(+\infty) \leq K_-(+\infty)$, which contradicts the assumption. \qed

**Corollary 1.5.** Let $u$ be a smooth non-negative solution of the inequality $\Delta_g \log u \geq k$ on $M_+$. If $k_- \in L^1(M)$, $1/\max\{1, \frac{d}{dr}T^p_x(u, r)\} \notin L^1(\mathbb{R}^+)$ for some $p > 0$ and a point $x \in M$, and $(M, g)$ admits a positive Green function, then $u$ should be identically zero.

**Proof.** By hypothesis $(M, g)$ admits a smooth non-constant subharmonic function $\varphi$ with $0 \leq \varphi \leq 1$ on $M$. Putting $u_* := u \exp(L\varphi^2)$ for a constant $L > 0$, $u_*$ satisfies $\Delta_g \log u_* \geq k_* := k + 2L|\nabla \varphi|^2$ and $1/\max\{1, \frac{d}{dr}T^p_x(u_*, r)\} \notin L^1(\mathbb{R}^+)$. The proof of Theorem 1.4 implies that $k_*|\nabla \varphi|^2 \in L^1(M)$ and $\int_M k_\varphi + L\int_M |\nabla \varphi|^2 \, dv_g \leq 0$. However this is a contradiction because $L$ can be chosen
Remark 1. Here a continuous function $u$ is said to be subharmonic if $D$ is any relatively compact open subset of $M$, $\Delta g u \equiv 0$ on $D$, and $u \leq v$ on $D \setminus D$; then $u \leq v$ on $D$. If $u$ is of class $C^2$, then $u$ satisfies $\Delta g u \geq 0$. It is known that a positive Green’s function produces smooth non-constant bounded subharmonic functions (cf. [CTW], Theorem 1.4).

By admitting Theorem 2.1, we get the following in view of Remark 1.

Corollary 2.2. If $(M, g)$ admits a point $x \in M$ such that $1/\frac{d}{dr} V_x(r) \notin L^1(\mathbb{R}^+)$, then $(M, g)$ is parabolic.

Remark 2. If $(M, g)$ is rotationally symmetric at a point $x_0 \in M$, and $r_0$ is the distance function from $x_0 \in M$, then we can see that (i) $w(x) := \int_1^{r_0(x)} dr/\frac{d}{dr} V_x(r)$ is harmonic on $M \setminus \{x_0\}$ and $\lim_{r \to 0} w(x) = -\infty$, (ii) $\int_1^{\infty} dr/\frac{d}{dr} V_x(r) \geq \int_1^{\infty} dr/\frac{d}{dr} V_x(r)$ if the radial curvature is non-positive (cf. [GW1] and the Lemma in section 5). In particular $u := \exp w \geq 0$ defines a continuous bounded subharmonic function on $M$ if and only if $1/\frac{d}{dr} V_x(r) \in L^1(\mathbb{R}^+)$. If the Ricci curvature of $(M, g)$ is non-negative, then $(M, g)$ is parabolic if and only if $r/\frac{d}{dr} V_x(r) \notin L^1(\mathbb{R}^+)$ for some point $x \in M$ (cf. [V], Theorem 2, and [LT2], Theorem 1.9). However the integrability of $r/\frac{d}{dr} V_x(r)$ does not always imply the existence of the Green’s function on $(M, g)$ generally as observed in [V].

Proof of Theorem 2.1. We may assume that there exists a non-constant continuous subharmonic function $v$ with $\sup_M v \leq 0$. By Corollary 1 in [GW2], for a monotone increasing sequence $\{r_j\} \subset \mathbb{N}$ of positive numbers tending to infinity, there exists a sequence $\{v_j\}$ of smooth subharmonic functions on $M$ such that $\sup_{B_x(r_j)} |v - v_j| \leq \frac{1}{j}$ for any $j \in \mathbb{N}$. Putting $u := \exp v$ and $u_j := \exp \left( v_j - \frac{j}{2} \right)$, we get $0 \leq u_j \leq u \leq 1$ for all $j \in \mathbb{N}$ (cf. [GW1], Lemma 2.3.1). In particular $u := \exp w \geq 0$ defines a continuous bounded subharmonic function on $M$ if and only if $1/\frac{d}{dr} V_x(r) \in L^1(\mathbb{R}^+)$. If the Ricci curvature of $(M, g)$ is non-negative, then $(M, g)$ is parabolic if and only if $r/\frac{d}{dr} V_x(r) \notin L^1(\mathbb{R}^+)$ for some point $x \in M$ (cf. [V], Theorem 2, and [LT2], Theorem 1.9). However the integrability of $r/\frac{d}{dr} V_x(r)$ does not always imply the existence of the Green’s function on $(M, g)$ generally as observed in [V].
This implies that \( u^2 \) is harmonic and \( u \) is also so by the same argument. Since \( u \) is smooth by Weyl’s lemma, this implies that \( u \) is constant on \( M \). This is a contradiction. Since \( \frac{d}{dr} T^2_x(u_j, r) \leq \frac{d}{dr} V_x(r) \) for almost all \( r > 0 \) with \( r \leq r_j \), and \( j \in \{ \) for some point \( j \in \mathbb{N} \), by applying (4) in the proof of Theorem 1.2 to \( p = 2 \), we can see

\[
\int_{r_k^*}^{r_j} \frac{dr}{\sqrt{V_x(r)}} \leq C_* \int_{r_k^*}^{r_j} \frac{dE^2(u_j, t)}{E^2(u_j, t)^2} dt \leq \frac{C_*}{E^2(u_j, r_k^*)} \leq \frac{2C_*}{E_*} < +\infty
\]

for any \( j > k_\ast \). By letting \( j \rightarrow +\infty \), the above inequality implies the conclusion. \( \square \)

3. LIOUVILLE THEOREMS OF HARMONIC MAPS

FOR A CLASS OF PARABOLIC MANIFOLDS

First we show the following (cf. [LY], Theorem 2.1, and [Y], Corollary).

**Theorem 3.1.** Let \( u \) be a smooth function satisfying the inequality \( \Delta_g u \geq k \) for a locally integrable function \( k \) on \( M \). Suppose \( \int_{B_r(x)} |\nabla u|^2 \, dv_g = o \left( \int_1^r \frac{dt}{\sqrt{V_x(t)}} \right) \) for some point \( x \in M \) and \( k_- := \max\{ -k, 0 \} \in L^1(M) \). Then \( \int_M k_+ dv_g \leq \int_M k_- dv_g \).

**Proof.** Let \( \omega_x \) be the function taken in the proof of Theorem 1.2. Since \( \text{div} \left( \omega_x \nabla u \right) = \omega_x \Delta_g u + \langle \nabla \omega_x, \nabla u \rangle \), by integrating this and letting \( \varepsilon \rightarrow 0 \) we have

\[
K_+(r) \leq K_-(r) + \sqrt{\frac{d}{dr} V_x(r) \frac{d}{dr} E^2(u, r)}.
\]

Here we have used the same notation as in the proof of Theorem 1.3. Dividing this inequality by \( \frac{d}{dr} V_x(r) \) and integrating from \( r_0 \gg 0 \) up to \( r > r_0 \), we get

\[
K_+(r_0) \leq K_-(r) + \sqrt{E^2(u, r)} \int_{r_0}^{r} \frac{dt}{\sqrt{V_x(t)}}.
\]

By letting \( r \rightarrow +\infty \) and later \( r_0 \rightarrow +\infty \), we attain the conclusion. \( \square \)

In the case \( k \geq 0 \) and \( k \neq 0 \), we can show the following stronger result than Theorem 3.1.

**Theorem 3.2.** Let \( f : (M,g) \rightarrow (N,h) \) be a smooth map from \( (M,g) \) to a Riemannian manifold \( N \) provided with a smooth function \( \varphi \) and a continuous function \( \chi > 0 \) such that \( \text{Hess}(\varphi) \geq \chi h \) and \(|\nabla \varphi| \leq C \) for a constant \( C > 0 \) on \( N \). Suppose \( f \) is harmonic, i.e., the tension field \( \tau(f) \in C(M, f^* TN) \) of \( f \) vanishes identically, and the energy density \( e(f) := \frac{1}{2} |df|^2 \) of \( f \) satisfies the following condition (*): \( \int_{B_r(x)} e(f) \, dv_g = o \left( \int_1^r \frac{dt}{\sqrt{V_x(t)}} \right) \) for some point \( x \in M \). Then \( f \) is a constant map.

**Proof.** The pull back \( u := f^* \varphi \) of \( \varphi \) satisfies \( \Delta_g u \geq f^* \chi \, e(f) \) by the composition law of maps (cf. [EI], (2.20), Proposition), and moreover \(|\nabla u| \leq C \sqrt{e(f)} \) by a direct calculation. If \( f \) is not a constant map, then \( k := f^* \chi \, e(f) \geq 0 \) and \( k \neq 0 \). On the other hand Theorem 3.1 implies \( k \equiv 0 \), which is a contradiction. Hence \( f \) should be constant. \( \square \)

As a corollary we obtain the following Liouville theorem for harmonic maps to a manifold of asymptotically non-positive curvature.
Corollary 3.3. Let \( f : (M, g) \to (N, h) \) be a harmonic map to a complete Riemannian manifold \((N, h)\) with a pole \( x \in N \) whose radial curvature \( R_N \) satisfies \( R_N \leq 1/(4(1 + r^2)) \) on \( N \), where \( r \) is the distance function from \( x \in M \). Suppose the energy density \( \mathbf{e}(f) \) of \( f \) satisfies the condition (*) in Theorem 3.2. Then \( f \) is a constant map.

Proof. Setting \( \varphi := \sqrt{r^2 + 1} \), we can see that \( \text{Hess}(\varphi) \geq h/\max \{ \varphi^3, 2\varphi \sqrt{r^2 + 1} \} \) and \( |\nabla \varphi| \leq 1 \) on \( N \) (cf. [GW1], Theorem A, Proposition 2.20, and [TI], Proof of Theorem 3). Hence the conclusion follows from Theorem 3.2. \( \square \)

Remark. It is known that if \((M, g)\) is parabolic, then any harmonic map of finite energy from \((M, g)\) to any Hadamard manifold \((N, h)\), i.e., \( N \) is simply connected and \( R_N \leq 0 \) on \( N \), is constant (cf. [CTW], Proposition 2.1 and Theorem 3.2). On the other hand there exists a non-degenerate harmonic map from a two-dimensional Euclidean space \( \mathbb{R}^2 \) with flat metric to a hyperbolic plane of constant curvature \( -1 \) (cf. [CT]). The energy of such a map on \( B_x(r) \subset \mathbb{R}^2 \) with \( x = (0, 0) \in \mathbb{R}^2 \) diverges no slower than \( \log r \) by Theorem 3.2.

However the analyticity of such a map yields the following Liouville theorem of holomorphic maps to a manifold of negative curvature bounded away from zero.

Theorem 3.4. Let \( f : (M, \omega_M) \to (N, \omega_N) \) be a holomorphic map from a complete Kähler manifold \((M, \omega_M)\) of dimension \( m = \text{dim} \, M \) to a Kähler manifold \((N, \omega_N)\). If \((M, \omega_M)\) admits a point \( x \in M \) such that \( 1/\sqrt{\text{tr}} \omega_M = L^1(\mathbb{R}_+) \), and \((N, \omega_N)\) admits a smooth 1-form \( \theta \) such that \( \omega_N = d\theta \) and \( C := \sup_N |\theta| < +\infty \), then \( f \) is a constant map.

Proof. If \( f \) is not constant, then integrating \( d [\omega_M f^* \theta \wedge \omega_M^{m-1}] \) and letting \( \varepsilon \to 0 \) we get the inequality \( E(f, r) \leq \sqrt{C/\varepsilon \int B_x(r) f^* \omega_N \wedge \omega_M^{m-1}} > 0 \) for any \( r > 0 \). Theorem 1.1 implies \( 1/\sqrt{\text{tr}} \omega_M \in L^1(\mathbb{R}_+) \). \( \square \)

Many kinds of hyperbolic Kähler manifolds admit such a Kähler metric (cf. [Gr]).

4. UPPER ESTIMATE AND UNIQUENESS OF SOLUTIONS
OF CERTAIN NON-LINEAR DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

We begin with the following (cf. [CY], Theorem 8 and Corollary, and [T2], Proposition).

Theorem 4.1. Let \( u \) be a smooth function satisfying the inequality \( \Delta_t u \geq k \lambda(u) \) where \( \lambda \) (resp. \( k \geq 0 \)) is a continuous function on \( \mathbb{R} \) (resp. \( M \)) satisfying \( \lambda(t) > 0 \) for any \( t > \alpha \) (resp. \( k \neq 0 \)). Suppose there exist a constant \( p > 1 \) and a point \( x \in M \) such that \( 1/\max \{ \frac{\text{tr}}{\text{tr}}(u - \alpha)_+, r, 1 \} \notin L^1(\mathbb{R}_+) \), where \( [u - \alpha]_+ := \max\{u - \alpha, 0\} \). Then \( \sup_M u \leq \alpha \). In particular, if \( 1/\max \{ \frac{\text{tr}}{\text{tr}}(u - \alpha), r, 1 \} \notin L^1(\mathbb{R}_+) \), then we obtain the following assertions: (i) If \((M, g)\) is not parabolic, then \( \sup_M u = \alpha \), and (ii) If \( \alpha = \sup\{t \in \mathbb{R} : \lambda(t) < 0\} \) and \( u \) satisfies the equality \( \Delta_t u = k \lambda(u) \), then \( u \equiv \alpha \).

Proof. Assume \( \{u > \alpha\} \neq \emptyset \) and take a smooth function \( \tau : \mathbb{R} \to \mathbb{R} \) such that \( \tau(t) = 0 \) if \( t \leq 0 \), \( \tau(t) > 0 \), \( \tau'(t) > 0 \), \( \tau''(t) \geq 0 \) if \( t > 0 \), and \( \tau'(t) \equiv 1 \) if \( t \geq \delta \) for a sufficiently small \( \delta > 0 \). Setting \( v := \tau(u - \alpha) \neq 0 \), a direct calculation
shows $\Delta_g v \geq \tau'(u - \alpha)k\lambda(u) \geq 0$, and $v$ satisfies $v \leq [u - \alpha]_+$. By Theorem 1.2 $v$ should be constant. Hence we get $u \equiv \alpha^* > \alpha > 0$ on $M$ which implies $k \equiv 0$. This is a contradiction. Hence $\sup_M u \leq \alpha$. To see (i) if $\sup_M u < \alpha$, then $1/\partial_r I_p^r \notin L^1(R_+)$ by hypothesis. Hence Corollary 2.2 implies the conclusion. To see (ii), setting $w = u - \alpha \leq 0$, $v$ satisfies $w \Delta_g w = wk\lambda(u) \geq 0$ on $M$ and so $w \equiv 0$ by Theorem 1.2 and the definition of $\alpha$. □

The above proof implies the following which is a complement of Theorem 1.2.

**Corollary 4.2.** If $u$ is a non-constant smooth subharmonic function on $(M, g)$, then $1/\partial_r I_p^r ([u - \alpha]_+, r) \in L^1(R_+)$ for any $\alpha < \sup_M u \leq +\infty$, $p > 1$ and $x \in M$.

Since $1/\max \{\partial_r I_p^r ([u - \alpha]_+, r), 1\} \notin L^1(R_+)$ implies $1/\max \{\partial_r I_p^r ([u - \alpha]_+, r), 1\} \notin L^1(R_+)$ for any $\alpha > 0$, we get the following by changing an orientation.

**Corollary 4.3.** Let $u$ be a smooth solution of the equality $\Delta_g u = k\lambda(u)$ on $M$. Assume that $\lambda$ (resp. $k \geq 0$) is a continuous function on $R$ (resp. $M$) such that $-\infty < \alpha_- := \inf\{t \in R : \lambda(t) \geq 0\} \leq \alpha_+ := \sup\{t \in R : \lambda(t) \leq 0\} < +\infty$ (resp. $k \neq 0$), and $1/\max \{\partial_r I_p^r (u_+, r), 1\} \notin L^1(R_+)$ for some constant $p > 1$ and a point $x \in M$. Then $\min\{\alpha_-, 0\} \leq \inf_M u \leq \sup_M u \leq \max\{\alpha_+, 0\}$. In particular, $u \equiv 0$ if $\alpha_+ = \alpha_- = 0$.

The following assertion is a variant of Theorem 1.4 (cf. [LY], Theorem 2.1 and Corollary 2.2, and [Y], Theorems 1 and 5).

**Theorem 4.4.** Let $u$ be a smooth non-negative function satisfying the inequality $\Delta_g \log u \geq k\lambda(u)$ on $\{u > 0\} \subset M$ where $\lambda$ (resp. $k \geq 0$) is a continuous function on $R$ (resp. $M$) satisfying $\lambda(t) > 0$ for any $t > 0$ (resp. $k \neq 0$). Then we obtain the following assertions: (i) If there exist a constant $p > 0$ and a point $x \in M$ such that $1/\max \{\partial_r I_p^r (u_+, r), 1\} \notin L^1(R_+)$, then $\sup_M u \leq \alpha$. In particular, $u \equiv 0$ if $\alpha = 0$. (ii) If $\alpha = \sup\{t \in R : \lambda(t) < 0\}$, $u \geq 0$ satisfies the equality $\Delta_g \log u = k\lambda(u)$ on $M$ and $1/\max \{\partial_r I_p^r (u_+, r), 1\} \notin L^1(R_+)$ for some $q > 1$, then $u \equiv 0$.

Proof. By taking a constant $\delta$ with $p > \delta > 0$ and putting $w = u^\delta$, $v$ satisfies $\Delta_g v \geq k\delta w\lambda (w^{1/\delta})$ on $\{u > 0\}$. By hypothesis, $1/\partial_r I_p^r ([u - \alpha]_+, r) \notin L^1(R_+)$ for $p_r := p/\delta > 1$. Hence $\sup_M u \leq \alpha$ by Theorem 4.1. If $u$ satisfies $\Delta_g \log u = k\lambda(u)$, then, putting $v = u/\alpha$, $v$ satisfies $\Delta_g \log v = k\lambda(u)\log v \geq 0$. Hence $v \equiv 1$ by hypothesis. □

In Theorem 4.1 a special choice of $\lambda$ implies the following (cf. [BRS], Theorems 1.3 and 3.5).

**Theorem 4.5.** Let $u$ be a non-negative smooth function satisfying the inequality $\Delta_g u \geq -hu + ku^{\sigma + 1}$ on $M$ for a constant $\sigma > 0$ where $h$ and $k \geq 0$ with $k \neq 0$ are continuous functions satisfying $h \leq Ak$ on $M$ for some constant $A \geq 0$. Suppose there exist a constant $p > 1$ and a point $x \in M$ such that

$$1/\max \{\partial_r I_p^r ([u - A^{1/\sigma}]_+, r), 1\} \notin L^1(R_+).$$

Then $\sup_M u \leq A^{1/\sigma}$. In particular, if $1/\max \{\partial_r I_p^r (u - A^{1/\sigma}, r), 1\} \notin L^1(R_+)$, then we obtain the following assertions: (i) If $(M, g)$ is not parabolic and $u \equiv 0$, then $\sup_M u = A^{1/\sigma}$, and (ii) If $u$ satisfies $\Delta_g u = ku(u^\sigma - A)$, then either $u \equiv A^{1/\sigma}$ or $u \equiv 0$. □
Theorem 4.5 implies the following gap theorem for solutions of a non-linear equation.

**Corollary 4.6.** Let \( u \) be a positive smooth solution of the equality \( \Delta_g u = ku(u^\sigma - A) \) as in Theorem 4.5(ii) for \( A > 0 \). If there exist constants \( \beta > 0 \), \( \gamma > 0 \) and \( C > 0 \) such that \( \limsup_{r \to +\infty} V_g(r)/r^\beta < +\infty \), i.e., \((M, g)\) has polynomial volume growth and \( |u - A^{1/\sigma}| < C/[1 + r_y]^{\gamma} \) on \( M \), where \( r_y \) is the distance function from \( y \in M \), then \( u \equiv A^{1/\sigma} \).

For a given conformal diffeomorphism \( f : (M, g) \to (N, h) \) of Riemannian manifolds of dimension \( m \geq 3 \) (resp. \( m = 2 \)), the pull-back metric \( f^* h \) can be written in the form \( f^* h = u^{4/(m-2)} g \) (resp. \( f^* h = u g \)) for some positive smooth function \( u \) on \( M \). The conformal factor \( u \) satisfies the following equality on \( M \):

\[
\begin{aligned}
    &c_m \Delta_g u - s_g u + K_{f^* h} u^{(m+2)/(m-2)} = 0 \quad \text{if } m \geq 3, \\
    &\Delta_g \log u - s_g + K_{f^* h} u = 0 \quad \text{if } m = 2,
\end{aligned}
\]

where \( c_m := (m-1)/(m-2) \), and \( s_g \) and \( K_{f^* h} \) are the scalar curvatures of \( g \) and \( f^* h \) respectively. \( f \) is said to be isometric (resp. preserve the scalar curvature) if \( u \equiv 1 \) (resp. \( s_g = K_{f^* h} \)). Applying Theorem 4.5 to \( k = -(1/c_m)s_g \), \( A = 1 \) and \( \sigma = 4/(m-2) \), we can get the following which is known for the case \( p = 2 \) (cf. [BRS], Theorems 1.5).

**Theorem 4.7.** Let \( f : (M, g) \to (N, h) \) be a conformal diffeomorphism of manifolds of dimension \( m \geq 3 \) which preserves the scalar curvature \( s_g \). If \( s_g \) satisfies \( s_g \leq 0 \) with \( s_g \not\equiv 0 \) on \( M \), and there exist a constant \( p > 1 \) and a point \( x \in M \) such that the conformal factor \( u \) of \( f \) satisfies \( 1/\max \{ \frac{d}{dr} T^p_x(u - 1), 1 \} \not\in L^1(\mathbb{R}^+) \), then \( f \) is isometric.

As a corresponding result to Corollary 4.6, we get the following.

**Corollary 4.8.** Let \( f : (M, g) \to (N, h) \) be the same as in Theorem 4.7. If \( s_g \leq 0 \) with \( s_g \not\equiv 0 \) on \( M \), \( (M, g) \) has polynomial volume growth, and there exist constants \( \gamma > 0 \) and \( C > 0 \) such that the conformal factor \( u \) of \( f \) satisfies \( |u - 1| < C/[1 + r_y]^{\gamma} \) on \( M \), then \( f \) is isometric.

As a related topic, the following holds (cf. [BRS], Theorem 4.1 for the case \( p = 2 \)).

**Theorem 4.9.** Let \( u_i \) be a non-negative smooth solution of the equality \( \Delta_g u + hu_ku^{\sigma+1} = 0 \) on \( M \) for \( \sigma > 0 \) with \( i = 1, 2 \). If \( h \leq Ak \) for a constant \( A \geq 0 \), \( k \geq 0 \) with \( k \not\equiv 0 \), and there exist a constant \( p > 1 \) and a point \( x \in M \) such that \( 1/\max \{ d/dr T^p_x(u_1 - u_2, r), 1 \} \not\in L^1(\mathbb{R}^+) \), then \( |u_1 - u_2| \leq A^{1/\sigma} \). In particular, if \( h \leq 0 \), then \( u_1 \equiv u_2 \).

**Proof.** Putting \( w_\pm := \pm(u_1 - u_2) \), the conclusion follows from Theorem 4.5 because \( w_\pm \) satisfies the inequality \( \Delta_g w_\pm \geq ku_\pm(u_\pm^\sigma - A) \) on \( \{w_\pm \geq 0\} \).

The two-dimensional version of Theorem 4.7 is contained in the following.

**Theorem 4.10.** Let \( \{u_i\} \ (i = 1, 2) \) be positive smooth solutions satisfying the equality \( \Delta_g \log u = k(u - 1) \) where \( k \) is a non-negative continuous function on \( M \) with \( k \not\equiv 0 \). If there exist a constant \( p > 1 \) and a point \( x \in M \) such that \( w := u_1/u_2 \) satisfies either condition (i) \( 1/\max \{ d/dr T^p_x(\log w, r), 1 \} \not\in L^1(\mathbb{R}^+) \) or (ii) \( 1/\max \{ d/dr T^p_x(w - 1, r), d/dr T^p_x(w^{-1} - 1, r), 1 \} \not\in L^1(\mathbb{R}^+) \), then \( u_1 = u_2 \).
Proof. If condition (i) holds, then the conclusion follows from Theorem 4.4(ii) because \( w \) satisfies \( \Delta w \geq ku_2(w-1) \) on \( M \). On the other hand \( w \) satisfies the inequality \( \Delta w \geq ku_1w(w-1) \) on \( M \). Moreover \( w^{-1} \) satisfies the same inequality by replacing \( ku_2 \geq 0 \) by \( ku_1 \geq 0 \). Therefore the conclusion follows from Theorem 4.5 if condition (ii) is satisfied. Finally we note that \( u \equiv 1 \) is also a solution of the equality. \( \Box \)

5. Appendix

Lemma. Let \( v(r) > 0 \) be an absolutely continuous function on \( [0, +\infty) \) such that \( \frac{d}{dr}v(r) > 0 \) for almost all \( r \in [0, +\infty) \). Then \( v \) satisfies the following integral inequality:

\[
\int_2^r \frac{tdt}{v(t)} \leq 4\int_1^r \frac{dt}{v(t)} \quad \text{for any} \quad r > 2.
\]

If \( v(r)/r \) is non-decreasing (in particular, \( v(r) \) is convex and \( v(0) = 0 \)), then \( r/v(r) \in L^1(\mathbb{R}_+) \) if and only if \( 1/\frac{d}{dr}v(r) \in L^1(\mathbb{R}_+) \).

Proof. By integration by parts and Schwarz’s inequality we get the following:

\[
\int_1^r \frac{t-1}{v(t)} \, dt \leq 2\int_1^r \frac{dt}{v(t)} \quad \text{for any} \quad r > 1
\]

which implies the desired inequality. \( \Box \)

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