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A NOTE ON DIVERGENCE OF L^p-INTEGRALS OF SUBHARMONIC FUNCTIONS AND ITS APPLICATIONS

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Abstract. A non L^p-integrability condition of non-constant non-negative subharmonic functions on a general complete manifold (M, g) is given in an optimal form. As an application in differential geometry, several topics related to parabolicity of manifolds, the Liouville theorem for harmonic maps and conformal deformation of metrics are shown without any assumption on the Ricci curvature of (M, g).

Introduction

In this paper we always denote by (M, g) a non-compact complete and connected Riemannian manifold of dimension m, and by Δ_g the Laplacian defined by $\Delta_g := \mathrm{Trace}_g \nabla\nabla$. In section 1, for any $p > 1$ and a function u satisfying $u\Delta_g u \geq k$ with a suitable function k on M, we show a divergence property of the L^p-integral of u over every geodesic sphere of radius r as r tends to infinity. The idea of the proof is due to an abstract vanishing theorem for the gradient length of L^p-integrable functions (see Theorem 1.1). This gives a refinement of known results by several authors (cf. [Y], [K], [S], [LY]). As an application, in section 2 we give an alternative proof of Li-Tam's result which gives a sufficient condition for (M, g) to be parabolic in terms of the area of the geodesic sphere. In section 3 using this condition we show Liouville theorems for harmonic maps from a class of parabolic manifolds to either manifolds of non-positive curvature or hyperbolic Kähler manifolds. In section 4 we study the upper estimate and uniqueness of solutions of certain non-linear differential equations, i.e., the Poisson equation and the scalar curvature one.

1. On L^p-integrals of a smooth function u satisfying the inequality $u\Delta_g u \geq k$

The following implies a vanishing of gradient length of L^p-integrable functions.

Theorem 1.1. *Let $K_\pm(r)$, $\mathcal{I}(r)$, and $E(r)$ be non-negative absolutely continuous and non-decreasing functions on $\mathbf{R}_+ := [1, +\infty)$, and let C_1 and C_2 be positive constants respectively such that $K_-(+\infty) := \lim_{r\to+\infty} K_-(r) < +\infty$, $\frac{d}{dr}\mathcal{I}(r) > 0$ and*

$$K_+(r) + C_1 E(r) \leq K_-(r) + C_2\sqrt{\frac{d}{dr}\mathcal{I}(r)\frac{d}{dr}E(r)} \tag{1}$$

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for almost all $r \geq r_0 >> 1$. *Suppose* $K_-(+\infty) \leq K_+(+\infty) := \lim_{r\to+\infty} K_+(r) \leq +\infty$. *Then either* $E(r) \equiv 0$ *or* $1/\frac{d}{dr}\mathcal{I}(r) \in L^1(\mathbf{R}_+)$.

Proof. By reduction to absurdity assume that $E(r) > 0$ for any $r \geq r_0$ and $1/\frac{d}{dr}\mathcal{I}(r) \notin L^1(\mathbf{R}_+)$. By dividing (1) by $\frac{d}{dr}\mathcal{I}(r)E(r) > 0$ and integrating from r_0 up to $r > r_0$, we can see the following:

$$\frac{K_+(r_0)}{E(r)} + C_1 \leq C_2 \Bigg/ \sqrt{E(r_0)\int_{r_0}^{r} \frac{dt}{\frac{d}{dt}\mathcal{I}(t)}} + \frac{K_-(r)}{E(r_0)} \ . \tag{2}$$

If $E(+\infty) := \lim_{r\to+\infty} E(r) = +\infty$, then letting $r \to +\infty$ and later $r_0 \to +\infty$, (2) implies $0 < C_1 \leq 0$, which is a contradiction. Hence $0 < E(+\infty) < +\infty$ and so by the same procedure we see that $K_+(+\infty) < +\infty$ and $K_+(+\infty) - K_-(+\infty) \leq -C_1 E(+\infty) < 0$. □

First Theorem 1.1 induces the following result which implies not only Theorem 2.1 but also Theorem 2.2 in [K] simultaneously, and is a generalization of Theorem 1 b) in [S] for the Laplacian Δ_g in view of the Lemma given in section 5.

Theorem 1.2. *Suppose* u *is a non-constant smooth function satisfying the inequality* $u\Delta_g u \geq 0$ *on* M. *Then for any* $p > 1$, $r > 0$ *and* $x \in M$, *the function* $\mathcal{I}_x^p(u,r)$ *defined by* $\mathcal{I}_x^p(u,r) := \int_{B_x(r)} |u|^p \, dv_g$ *satisfies* $1/\frac{d}{dr}\mathcal{I}_x^p(u,r) \in L^1(\mathbf{R}_+)$, *where* $B_x(r)$ *is the geodesic ball centered at* $x \in M$ *and of radius* r.

Remark 1. Since the distance function r_x from a point $x \in M$ is Lipschitz continuous on M and satisfies $|\nabla r_x| \equiv 1$ within the cut locus of x, letting σ_r be the $(m-1)$-dimensional Hausdorff measure of the geodesic sphere $S_x(r)$ induced by g, it follows that $\frac{d}{dr}\mathcal{I}_x^p(u,r)$ coincides with the integral $\int_{S_x(r)} |u|^p \, \sigma_r$ for almost all $r > 0$ by the co-area formula (cf. [F], 3.2.12, Theorem and 3.2.46).

Proof of Theorem 1.2. Let ψ be a non-negative Lipschitz continuous function on M with $\mathrm{Supp}(\psi) \subset \overline{B_x(r)}$. If $u \not\equiv 0$ satisfies $u\Delta_g u \geq k$ for a locally integrable function k on M, then $|u|^{p-2}|\nabla u|^2$ is locally integrable and the following estimate is valid:

$$\int k\psi^2|u|^{p-2}dv_g + \int \psi^2|u|^{p-2}|\nabla u|^2 dv_g \leq \frac{2[\nabla\psi]}{C(p)}\sqrt{\int_T |u|^p dv_g \int_T \psi^2|u|^{p-2}|\nabla u|^2 dv_g} \ , \tag{3}$$

where $[\nabla\psi] := \sup_M |\nabla\psi|$, $C(p) := \min\{1, p-1\} > 0$ and $T = \mathrm{Supp}(\nabla\psi)$ (cf. (2.3) in the proof of Theorems 2.1 and 2.2 in [K]). For any $r > 0$ and $p > 1$, we set $E^p(u,r) := \int_{B_x(r)} |u|^{p-2}|\nabla u|^2 \, dv_g < +\infty$. For any $\varepsilon > 0$, we take a Lipschitz continuous function ω_ε such that $0 \leq \omega_\varepsilon \leq 1$, $\mathrm{Supp}(\omega_\varepsilon) \subset \overline{B_x(r)}$, $\omega_\varepsilon \equiv 1$ on $B_x(r-\varepsilon)$, and $|\nabla\omega_\varepsilon| \leq 1/\varepsilon$. Putting $\psi = \omega_\varepsilon$, $T = B_x(r) \setminus B_x(r-\varepsilon)$ and $k \equiv 0$, and letting $\varepsilon \to 0$ in (3), we get

$$E^p(u,r) \leq C_* \sqrt{\frac{d}{dr}\mathcal{I}_x^p(u,r)\frac{d}{dr}E^p(u,r)} \tag{4}$$

for almost all $r > 0$ and $C_* := 2/C(p)$. If u is not constant, then $E^p(u,r) > 0$ for any $r \geq r_0 >> 0$. Applying Theorem 1.1 to $K_\pm(r) \equiv 0$, $E(r) = E^p(u,r)$, $\mathcal{I}(r) = \mathcal{I}_x^p(u,r)$ and $C_1 = 1$, we get the conclusion. □

In case $p = 2$ we can relax the non-negativity condition of k as follows.

Theorem 1.3. *Let u be a smooth non-constant solution satisfying the inequality $u\Delta_g u \geq k$ for a locally integrable function k on M. If $k_- := \max\{-k, 0\} \in L^1(M)$ and $\int_M k\, dv_g \geq 0$, then $1\Big/\frac{d}{dr}\mathcal{I}_x^2(u,r) \in L^1(\mathbf{R}_+)$ for any $x \in M$.*

Proof. We set $K_\pm(r) := \int_{B_x(r)} k_\pm\, dv_g$ for $k = k_+ - k_-$ with $k_+ = \max\{k, 0\}$. By using the estimate (3) for $p = 2$ and the same procedure as in the proof of Theorem 1.2, we get

$$K_+(r) + E^2(u,r) \leq K_-(r) + C_*\sqrt{\frac{d}{dr}\mathcal{I}_x^2(u,r)\frac{d}{dr}E^2(u,r)}$$

for almost all $r > 0$. Hence the conclusion follows from Theorem 1.1. □

The following is a generalization of Theorem 2.1 in [LY] and Theorem 1 in [Y].

Theorem 1.4. *Let u be a smooth non-constant solution satisfying the inequality $\Delta_g \log u \geq k$ on $M_+ := \{u > 0\}$. If either $k \equiv 0$ or $k_- \in L^1(M)$ and $\int_M k\, dv_g > 0$, then $1\Big/\frac{d}{dr}\mathcal{I}_x^p(u,r) \in L^1(\mathbf{R}_+)$ for any $p > 0$ and $x \in M$.*

Remark 2. Under the same situation Li and Yau showed $\liminf_{r\to+\infty} \mathcal{I}_x^p(u,r)/r^2 > 0$ in [LY], which follows from $1\Big/\frac{d}{dr}\mathcal{I}_x^p(u,r) \in L^1(\mathbf{R}_+)$ in view of the Lemma in section 5.

Proof of Theorem 1.4. We only have to show the case $p = 1$. For any $c > 0$ and the function ω_ε taken in the proof of Theorem 1.2, a direct calculation shows the following:

$$\operatorname{div}\left(\omega_\varepsilon^2 \nabla \log(u+c)\right) \geq \frac{\omega_\varepsilon^2 uk}{u+c} + \frac{\omega_\varepsilon^2 c|\nabla u|^2}{u(u+c)^2} - 2|\nabla\omega_\varepsilon||\omega_\varepsilon \nabla \log(u+c)|\ .$$

We set $K_\pm(r,c) := \int_{B_x(r)} \frac{uk_\pm}{u+c}\, dv_g$ and $E(r,c) := \int_{B_x(r)} \frac{|\nabla u|^2}{u(u+c)^2}\, dv_g$. Integrating the above inequality we can see $E(r,c) < +\infty$ and letting $\varepsilon \to 0$, we get the following:

$$K_+(r,c) + cE(r,c) \leq K_-(r,c) + 2\sqrt{\frac{d}{dr}\mathcal{I}_x^1(u,r)\frac{d}{dr}E(r,c)}\ .$$

In case $k \equiv 0$ the conclusion follows from Theorem 1.1. To see the latter case assume that u is not constant and $1\Big/\frac{d}{dr}\mathcal{I}_x^1(u,r) \notin L^1(\mathbf{R}_+)$. The proof of Theorem 1.1 implies $K_+(+\infty,c) + cE(+\infty,c) \leq K_-(+\infty,c)$ for any $c > 0$. By letting $c \to 0$, we get $K_+(+\infty) \leq K_-(+\infty)$, which contradicts the assumption. □

Corollary 1.5. *Let u be a smooth non-negative solution of the inequality $\Delta_g \log u \geq k$ on M_+. If $k_- \in L^1(M)$, $1/\max\left\{1, \frac{d}{dr}\mathcal{I}_x^p(u,r)\right\} \notin L^1(\mathbf{R}_+)$ for some $p > 0$ and a point $x \in M$, and (M,g) admits a positive Green function, then u should be identically zero.*

Proof. By hypothesis (M,g) admits a smooth non-constant subharmonic function φ with $0 \leq \varphi \leq 1$ on M. Putting $u_* := u\exp(L\varphi^2)$ for a constant $L > 0$, u_* satisfies $\Delta_g \log u_* \geq k_* := k + 2L|\nabla\varphi|^2$ and $1/\max\left\{1, \frac{d}{dr}\int_{B_x(t)} u_*^p\, dv_g\right\} \notin L^1(\mathbf{R}_+)$. The proof of Theorem 1.4 implies that k, $|\nabla\varphi|^2 \in L^1(M)$ and $\int_M k\, dv_g + L\int_M |\nabla\varphi|^2\, dv_g \leq 0$. However this is a contradiction because L can be chosen

arbitrarily large so that $\int_M k\ dv_g + L\int_M |\nabla\varphi|^2\ dv_g > 0$. Hence u_* should be constant and so $u \equiv 0$ by Theorem 2.1 below. $\square$

2. A note on parabolicity of manifolds

(M, g) is said to be *parabolic* if (M, g) admits no positive Green's function. Varopoulos showed that (M, g) is parabolic if $r/V_x(r) \notin L^1(\mathbf{R}_+)$ for some point $x \in M$, where $V_x(r) := \mathrm{Vol}(B_x(r))$ is the volume of $B_x(r)$ relative to g (cf. [V], Theorem 2, and [Gri], Corollary 2). Later Li and Tam proved the same result under the condition $1\big/\frac{d}{dr}V_x(r) \notin L^1(\mathbf{R}_+)$ (cf. [LT1], Corollary 2.3). Here we note that $\frac{d}{dr}V_x(r)$ is the area of the geodesic sphere $S_x(r)$ induced by g for almost all $r > 0$. In view of the Lemma in section 5, Li and Tam's assertion is sharper than Varopoulos' one. The idea of the proof of Theorem 1.2 allows us to give an alternative and elementary proof of their result by showing the following.

Theorem 2.1. *Suppose (M, g) admits a non-constant continuous subharmonic function bounded from above. Then $1\big/\frac{d}{dr}V_x(r) \in L^1(\mathbf{R}_+)$ for any point $x \in M$.*

Remark 1. Here a continuous function u is said to be subharmonic if D is any relatively compact open subset in M, $\Delta_g v \equiv 0$ on D, and $u \le v$ on $\overline{D} \setminus D$; then $u \le v$ on D. If u is of class C^2, then u satisfies $\Delta_g u \ge 0$. It is known that a positive Green's function produces smooth non-constant bounded subharmonic functions (cf. [CTW], Theorem 1.4).

By admitting Theorem 2.1, we get the following in view of Remark 1.

Corollary 2.2. *If (M, g) admits a point $x \in M$ such that $1\big/\frac{d}{dr}V_x(r) \notin L^1(\mathbf{R}_+)$, then (M, g) is parabolic.*

Remark 2. If (M, g) is rotationally symmetric at a point $x_* \in M$, and if r_* is the distance function from $x_* \in M$, then we can see that (i) $w(x) := \int_1^{r_*(x)} dr\big/\frac{d}{dr}V_{x_*}(r)$ is harmonic on $M \setminus \{x_*\}$ and $\lim_{r_*(x)\to 0} w(x) = -\infty$, (ii) $\int_1^{+\infty} r dr\big/V_{x_*}(r) \ge \int_1^{+\infty} dr\big/\frac{d}{dr}V_{x_*}(r)$ if the radial curvature is non-positive (cf. [GW1] and the Lemma in section 5). In particular $u := \exp w \ge 0$ defines a continuous *bounded* subharmonic function on M if and only if $1\big/\frac{d}{dr}V_{x_*}(r) \in L^1(\mathbf{R}_+)$. If the Ricci curvature of (M, g) is non-negative, then (M, g) is parabolic if and only if $r/V_x(r) \notin L^1(\mathbf{R}_+)$ for some point $x \in M$ (cf. [V], Theorem 2, and [LT2], Theorem 1.9). However the integrability of $r/V_x(r)$ does not always imply the existence of the Green's function on (M, g) generally as observed in [V].

Proof of Theorem 2.1. We may assume that there exists a non-constant continuous subharmonic function v with $\sup_M v \le 0$. By Corollary 1 in [GW2], for a monotone increasing sequence $\{r_j\}$ of positive numbers tending to infinity, there exists a sequence $\{v_j\}$ of smooth subharmonic functions on M such that $\sup_{B_x(r_j)} |v - v_j| \le \frac{1}{j}$ for any $j \in \mathbf{N}$. Putting $u := \exp v$ and $u_j := \exp\left(v_j - \frac{1}{j}\right)$, we get $0 < u_j \le u \le 1$ on $B_x(r_j)$. Here, applying the notation (4) to $u = u_j$ and $p = 2$, we claim that there exists $k_* \in \mathbf{N}$ such that $\lim_{j\to+\infty} E^2(u_j, r_{k_*}) := E_* > 0$. Otherwise $\lim_{j\to+\infty} E^2(u_j, r_k) = 0$ for any $k \ge 1$. Then for any smooth function ω with compact support, by using Schwarz's inequality, we have

$$\int_M u^2 \Delta_g \omega\ dv_g = \lim_{j\to+\infty} \int_M u_j^2 \Delta_g \omega\ dv_g = -\lim_{j\to+\infty} \int_M 2u_j \langle \nabla u_j, \nabla \omega\rangle\ dv_g = 0\ .$$

This implies that u^2 is harmonic and u is also so by the same argument. Since u is smooth by Weyl's lemma, this implies that u is constant on M. This is a contradiction. Since $\frac{d}{dr}\mathcal{I}_x^2(u_j,r) \le \frac{d}{dr}V_x(r)$ for almost all $r>0$ with $r \le r_j$ and $j \in \mathbf{N}$, by applying (4) in the proof of Theorem 1.2 to $p=2$, we can see

$$\int_{r_{k_*}}^{r_j} \frac{dr}{\frac{d}{dr}V_x(r)} \le C_* \int_{r_{k_*}}^{r_j} \frac{\frac{d}{dt}E^2(u_j,t)}{E^2(u_j,t)^2}dt \le \frac{C_*}{E^2(u_j,r_{k_*})} \le \frac{2C_*}{E_*} < +\infty$$

for any $j > k_*$. By letting $j \to +\infty$, the above inequality implies the conclusion. □

3. Liouville theorems of harmonic maps for a class of parabolic manifolds

First we show the following (cf. [LY], Theorem 2.1, and [Y], Corollary).

Theorem 3.1. *Let u be a smooth function satisfying the inequality $\Delta_g u \ge k$ for a locally integrable function k on M. Suppose $\int_{B_x(r)} |\nabla u|^2\, dv_g = o\left(\int_1^r dt/\frac{d}{dt}V_x(t)\right)$ for some point $x \in M$ and $k_- := \max\{-k,0\} \in L^1(M)$. Then $\int_M k_+ dv_g \le \int_M k_-\, dv_g$.*

Proof. Let ω_ε be the function taken in the proof of Theorem 1.2. Since $\operatorname{div}(\omega_\varepsilon \nabla u) = \omega_\varepsilon \Delta_g u + \langle \nabla \omega_\varepsilon, \nabla u\rangle$, by integrating this and letting $\varepsilon \to 0$ we have

$$K_+(r) \le K_-(r) + \sqrt{\frac{d}{dr}V_x(r)\frac{d}{dr}E^2(u,r)}\ .$$

Here we have used the same notation as in the proof of Theorem 1.3. Dividing this inequality by $\frac{d}{dr}V_x(r)$ and integrating from $r_0 >> 0$ up to $r > r_0$, we get

$$K_+(r_0) \le K-(r) + \sqrt{E^2(u,r) \Big/ \int_{r_0}^r \frac{dt}{\frac{d}{dt}V_x(t)}}\ .$$

By letting $r \to +\infty$ and later $r_0 \to +\infty$, we attain the conclusion. □

In the case $k \ge 0$ and $k \not\equiv 0$, we can show the following stronger result than Theorem 3.1.

Theorem 3.2. *Let $f : (M,g) \to (N,h)$ be a smooth map from (M,g) to a Riemannian manifold N provided with a smooth function φ and a continuous function $\chi > 0$ such that $\operatorname{Hess}(\varphi) \ge \chi h$ and $|\nabla\varphi| \le C$ for a constant $C > 0$ on N. Suppose f is harmonic, i.e., the tension field $\tau(f) \in \mathcal{C}(M, f^*TN)$ of f vanishes identically, and the energy density $\mathbf{e}(f) := (1/2)|df|^2$ of f satisfies the following condition $(*)$: $\int_{B_x(r)} \mathbf{e}(f)\, dv_g = o\left(\int_1^r dt/\frac{d}{dt}V_x(t)\right)$ for some point $x \in M$. Then f is a constant map.*

Proof. The pull back $u := f^*\varphi$ of φ satisfies $\Delta_g u \ge f^*\chi\ \mathbf{e}(f)$ by the composition law of maps (cf. [EL], (2.20), Proposition), and moreover $|\nabla u| \le C\sqrt{\mathbf{e}(f)}$ by a direct calculation. If f is not a constant map, then $k := f^*\chi\ \mathbf{e}(f) \ge 0$ and $k \not\equiv 0$. On the other hand Theorem 3.1 implies $k \equiv 0$, which is a contradiction. Hence f should be constant. □

As a corollary we obtain the following Liouville theorem for harmonic maps to a manifold of asymptotically non-positive curvature.

Corollary 3.3. *Let $f : (M, g) \to (N, h)$ be a harmonic map to a complete Riemannian manifold (N, h) with a pole $x \in N$ whose radial curvature R_N satisfies $R_N \leq 1/[4(1 + r_x)^2]$ on N, where r_x is the distance function from $x \in M$. Suppose the energy density $\mathbf{e}(f)$ of f satisfies the condition $(*)$ in Theorem 3.2. Then f is a constant map.*

Proof. Setting $\varphi := \sqrt{r_x^2 + 1}$, we can see that $\mathrm{Hess}(\varphi) \geq h/\max\left\{\varphi^3, 2\varphi\sqrt{r_x + 1}\right\}$ and $|\nabla\varphi| \leq 1$ on N (cf. [GW1], Theorem A, Proposition 2.20, and [T1], Proof of Theorem 3). Hence the conclusion follows from Theorem 3.2. □

Remark. It is known that if (M, g) is parabolic, then any harmonic map of finite energy from (M, g) to any Hadamard manifold (N, h), i.e., N is simply connected and $R_N \leq 0$ on N, is constant (cf. [CTW], Proposition 2.1 and Theorem 3.2). On the other hand there exists a non-degenerate harmonic map from a two-dimensional Euclidean space $\mathbf{R}^2$ with flat metric to a hyperbolic plane of constant curvature -1 (cf. [CT]). The energy of such a map on $B_x(r) \subset \mathbf{R}^2$ with $x = (0, 0) \in \mathbf{R}^2$ diverges no slower than $\log r$ by Theorem 3.2.

However the analyticity of such a map yields the following Liouville theorem of holomorphic maps to a manifold of negative curvature bounded away from zero.

Theorem 3.4. *Let $f : (M, \omega_M) \to (N, \omega_N)$ be a holomorphic map from a complete Kähler manifold (M, ω_M) of dimension $m = \dim_{\mathbb{C}} M$ to a Kähler manifold (N, ω_N). If (M, ω_M) admits a point $x \in M$ such that $1\big/\frac{d}{dr}V_x(r) \notin L^1(\mathbf{R}_+)$, and (N, ω_N) admits a smooth 1-form θ such that $\omega_N = d\theta$ and $C := \sup_N |\theta|_g < +\infty$, then f is a constant map.*

Proof. If f is not constant, then integrating $d\left[\omega_\varepsilon f^*\theta \wedge \omega_M^{m-1}\right]$ and letting $\varepsilon \to 0$ we get the inequality $E(f, r) \leq \sqrt{C\frac{d}{dr}E(f, r)\frac{d}{dr}V_x(r)}$, where $E(f, r) := \int_{B_x(r)} f^*\omega_N \wedge \omega_M^{m-1} > 0$ for any $r > 0$. Theorem 1.1 implies $1\big/\frac{d}{dr}V_x(r) \in L^1(\mathbf{R}_+)$. □

Many kinds of hyperbolic Kähler manifolds admit such a Kähler metric (cf. [Gro]).

4. Upper estimate and uniqueness of solutions of certain non-linear differential equations and its applications

We begin with the following (cf. [CY], Theorem 8 and Corollary, and [T2], Proposition).

Theorem 4.1. *Let u be a smooth function satisfying the inequality $\Delta_g u \geq k\lambda(u)$ where λ (resp. $k \geq 0$) is a continuous function on $\mathbf{R}$ (resp. M) satisfying $\lambda(t) > 0$ for any $t > \alpha$ (resp. $k \not\equiv 0$). Suppose there exist a constant $p > 1$ and a point $x \in M$ such that $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p([u - \alpha]_+, r), 1\right\} \notin L^1(\mathbf{R}_+)$, where $[u - \alpha]_+ := \max\{u - \alpha, 0\}$. Then $\sup_M u \leq \alpha$. In particular, if $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p(u - \alpha, r), 1\right\} \notin L^1(\mathbf{R}_+)$, then we obtain the following assertions*: (i) *If (M, g) is not parabolic, then $\sup_M u = \alpha$, and* (ii) *If $\alpha = \sup\{t \in \mathbf{R}\ ;\ \lambda(t) < 0\}$ and u satisfies the equality $\Delta_g u = k\lambda(u)$, then $u \equiv \alpha$.*

Proof. Assume $\{u > \alpha\} \neq \emptyset$ and take a smooth function $\tau : \mathbf{R} \to \mathbf{R}$ such that $\tau(t) \equiv 0$ if $t \leq 0$, $\tau(t) > 0$, $\tau'(t) > 0$, $\tau''(t) \geq 0$ if $t > 0$, and $\tau'(t) \equiv 1$ if $t \geq \delta$ for a sufficiently small $\delta > 0$. Setting $v := \tau(u - \alpha) \not\equiv 0$, a direct calculation

shows $\Delta_g v \geq \tau'(u-\alpha)k\lambda(u) \geq 0$, and v satisfies $v \leq [u-\alpha]_+$. By Theorem 1.2 v should be constant. Hence we get $u \equiv \alpha^* > \alpha > 0$ on M which implies $k \equiv 0$. This is a contradiction. Hence $\sup_M u \leq \alpha$. To see (i) if $\sup_M u < \alpha$, then $1\big/\frac{d}{dr}V_x(r) \notin L^1(\mathbf{R}_+)$ by hypothesis. Hence Corollary 2.2 implies the conclusion. To see (ii), setting $w = u - \alpha \leq 0$, w satisfies $w\Delta_g w = wk\lambda(u) \geq 0$ on M and so $w \equiv 0$ by Theorem 1.2 and the definition of α. □

The above proof implies the following which is a complement of Theorem 1.2.

Corollary 4.2. *If u is a non-constant smooth subharmonic function on (M,g), then $1\big/\frac{d}{dr}\mathcal{I}_x^p\left([u-\alpha]_+, r\right) \in L^1(\mathbf{R}_+)$ for any $\alpha < \sup_M u \leq +\infty$, $p > 1$ and $x \in M$.*

Since $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p([u]_+, r), 1\right\} \notin L^1(\mathbf{R}_+)$ implies $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p([u-\alpha]_+, r), 1\right\} \notin L^1(\mathbf{R}_+)$ for any $\alpha > 0$, we get the following by changing an orientation.

Corollary 4.3. *Let u be a smooth solution of the equality $\Delta_g u = k\lambda(u)$ on M. Assume that λ (resp. $k \geq 0$) is a continuous function on $\mathbf{R}$ (resp. M) such that $-\infty < \alpha_- := \inf\{t \in \mathbf{R}\ ;\ \lambda(t) \geq 0\} \leq \alpha_+ := \sup\{t \in \mathbf{R}\ ;\ \lambda(t) \leq 0\} < +\infty$ (resp. $k \not\equiv 0$), and $1\big/\max\{\frac{d}{dr}\mathcal{I}_x^p(u,r), 1\} \notin L^1(\mathbf{R}_+)$ for some constant $p > 1$ and a point $x \in M$. Then $\min\{\alpha_-, 0\} \leq \inf_M u \leq \sup_M u \leq \max\{\alpha_+, 0\}$. In particular, $u \equiv 0$ if $\alpha_+ = \alpha_- = 0$.*

The following assertion is a variant of Theorem 1.4 (cf. [LY], Theorem 2.1 and Corollary 2.2, and [Y], Theorems 1 and 5).

Theorem 4.4. *Let u be a smooth non-negative function satisfying the inequality $\Delta_g \log u \geq k\lambda(u)$ on $\{u > 0\} \subset M$ where λ (resp. $k \geq 0$) is a continuous function on $\mathbf{R}$ (resp. M) satisfying $\lambda(t) > 0$ for any $t > \alpha \geq 0$ (resp. $k \not\equiv 0$). Then we obtain the following assertions*: (i) *If there exist a constant $p > 0$ and a point $x \in M$ such that $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p(u,r), 1\right\} \notin L^1(\mathbf{R}_+)$, then $\sup_M u \leq \alpha$. In particular, $u \equiv 0$ if $\alpha = 0$.* (ii) *If $\alpha = \sup\{t \in \mathbf{R}\ ;\ \lambda(t) < 0\}$, $u > 0$ satisfies the equality $\Delta_g \log u = k\lambda(u)$ on M and $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^q(\log(u/\alpha), r), 1\right\} \notin L^1(\mathbf{R}_+)$ for some $q > 1$, then $u \equiv \alpha$.*

Proof. By taking a constant δ with $p > \delta > 0$ and putting $w = u^\delta$, w satisfies $\Delta_g w \geq \delta k w \lambda\left(w^{1/\delta}\right)$ on $\{w > 0\}$. By hypothesis, $1\big/\frac{d}{dr}\mathcal{I}_x^{p_*}([w-\alpha^\delta]_+, r) \notin L^1(\mathbf{R}_+)$ for $p_* := p/\delta > 1$. Hence $\sup_M u \leq \alpha$ by Theorem 4.1. If u satisfies $\Delta_g \log u = k\lambda(u)$, then, putting $v = u/\alpha$, v satisfies $\log v \Delta_g \log v = k\lambda(u)\log v \geq 0$. Hence $v \equiv 1$ by hypothesis. □

In Theorem 4.1 a special choice of λ implies the following (cf. [BRS], Theorems 1.3 and 3.5).

Theorem 4.5. *Let u be a non-negative smooth function satisfying the inequality $\Delta_g u \geq -hu + ku^{\sigma+1}$ on M for a constant $\sigma > 0$ where h and $k \geq 0$ with $k \not\equiv 0$ are continuous functions satisfying $h \leq Ak$ on M for some constant $A \geq 0$. Suppose there exist a constant $p > 1$ and a point $x \in M$ such that*

$$1\Big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p\left(\left[u - A^{1/\sigma}\right]_+, r\right), 1\right\} \notin L^1(\mathbf{R}_+).$$

Then $\sup_M u \leq A^{1/\sigma}$. In particular, if $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p(u - A^{1/\sigma}, r), 1\right\} \notin L^1(\mathbf{R}_+)$, then we obtain the following assertions: (i) *If (M,g) is not parabolic and $u \not\equiv 0$, then $\sup_M u = A^{1/\sigma}$, and* (ii) *If u satisfies $\Delta_g u = ku(u^\sigma - A)$, then either $u \equiv A^{1/\sigma}$ or $u \equiv 0$.*

Theorem 4.5 implies the following gap theorem for solutions of a non-linear equation.

Corollary 4.6. *Let u be a positive smooth solution of the equality $\Delta_g u = ku(u^\sigma - A)$ as in Theorem 4.5(ii) for $A > 0$. If there exist constants $\beta > 0$, $\gamma > 0$ and $C > 0$ such that $\limsup_{r\to+\infty} V_y(r)/r^\beta < +\infty$, i.e., (M, g) has polynomial volume growth and $\left|u - A^{1/\sigma}\right| < C/[1+r_y]^\gamma$ on M, where r_y is the distance function from $y \in M$, then $u \equiv A^{1/\sigma}$.*

For a given conformal diffeomorphism $f : (M, g) \to (N, h)$ of Riemannian manifolds of dimension $m \geq 3$ (resp. $m = 2$), the pull-back metric f^*h can be written in the form $f^*h = u^{4/(m-2)}g$ (resp. $f^*h = ug$) for some positive smooth function u on M. The conformal factor u satisfies the following equality on M:

$$\begin{cases} c_m \Delta_g u - s_g u + K_{f^*h} u^{(m+2)/(m-2)} \equiv 0 & \text{if } m \geq 3, \\ \Delta_g \log u - s_g + K_{f^*h} u \equiv 0 & \text{if } m = 2, \end{cases}$$

where $c_m := 4(m-1)/(m-2)$, and s_g and K_{f^*h} are the scalar curvatures of g and f^*h respectively. f is said to be *isometric* (resp. *preserve the scalar curvature*) if $u \equiv 1$ (resp. $s_g = K_{f^*h}$). Applying Theorem 4.5 to $k = -(1/c_m)s_g$, $A = 1$ and $\sigma = 4/(m-2)$, we can get the following which is known for the case $p = 2$ (cf. [BRS], Theorems 1.5).

Theorem 4.7. *Let $f : (M, g) \to (N, h)$ be a conformal diffeomorphism of manifolds of dimension $m \geq 3$ which preserves the scalar curvature s_g. If s_g satisfies $s_g \leq 0$ with $s_g \not\equiv 0$ on M, and there exist a constant $p > 1$ and a point $x \in M$ such that the conformal factor u of f satisfies $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p(u-1, r), 1\right\} \notin L^1(\mathbf{R}_+)$, then f is isometric.*

As a corresponding result to Corollary 4.6, we get the following.

Corollary 4.8. *Let $f : (M, g) \to (N, h)$ be the same as in Theorem 4.7. If $s_g \leq 0$ with $s_g \not\equiv 0$ on M, (M, g) has polynomial volume growth, and there exist constants $\gamma > 0$ and $C > 0$ such that the conformal factor u of f satisfies $|u - 1| < C/[1+r_y]^\gamma$ on M, then f is isometric.*

As a related topic, the following holds (cf. [BRS], Theorem 4.1 for the case $p = 2$).

Theorem 4.9. *Let u_i be a non-negative smooth solution of the equality $\Delta_g u + hu - ku^{\sigma+1} = 0$ on M for $\sigma > 0$ with $i = 1, 2$. If $h \leq Ak$ for a constant $A \geq 0$, $k \geq 0$ with $k \not\equiv 0$, and there exist a constant $p > 1$ and a point $x \in M$ such that $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p(u_1 - u_2, r), 1\right\} \notin L^1(\mathbf{R}_+)$, then $|u_1 - u_2| \leq A^{1/\sigma}$. In particular, if $h \leq 0$, then $u_1 \equiv u_2$.*

Proof. Putting $w_\pm := \pm(u_1 - u_2)$, the conclusion follows from Theorem 4.5 because $w_\pm$ satisfies the inequality $\Delta_g w_\pm \geq k w_\pm (w_\pm^\sigma - A)$ on $\{w_\pm \geq 0\}$. □

The two-dimensional version of Theorem 4.7 is contained in the following.

Theorem 4.10. *Let $\{u_i\}$ $(i = 1, 2)$ be positive smooth solutions satisfying the equality $\Delta_g \log u = k(u-1)$ where k is a non-negative continuous function on M with $k \not\equiv 0$. If there exist a constant $p > 1$ and a point $x \in M$ such that $w := u_1/u_2$ satisfies either condition* (i) $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p(\log w, r), 1\right\} \notin L^1(\mathbf{R}_+)$ *or* (ii) $1\big/\max\left\{\frac{d}{dr}\mathcal{I}_x^p(w-1, r), \frac{d}{dr}\mathcal{I}_x^p(w^{-1}-1, r), 1\right\} \notin L^1(\mathbf{R}_+)$, *then $u_1 \equiv u_2$.*

Proof. If condition (i) holds, then the conclusion follows from Theorem 4.4(ii) because w satisfies $\Delta_g \log w = ku_2(w-1)$. On the other hand w satisfies the inequality $\Delta_g w \geq ku_2 w(w-1)$ on M. Moreover w^{-1} satisfies the same inequality by replacing $ku_2 \geq 0$ by $ku_1 \geq 0$. Therefore the conclusion follows from Theorem 4.5 if condition (ii) is satisfied. Finally we note that $u \equiv 1$ is also a solution of the equality. □

5. Appendix

Lemma. *Let $v(r) > 0$ be an absolutely continuous function on $[0, +\infty)$ such that $\frac{d}{dr}v(r) > 0$ for almost all $r \in [0, +\infty)$. Then v satisfies the following integral inequality*:

$$\int_2^r \frac{tdt}{v(t)} \leq 4\int_1^r \frac{dt}{\frac{d}{dt}v(t)} \quad \textit{for any } r > 2 .$$

If $v(r)/r$ is non-decreasing (in particular, $v(r)$ is convex and $v(0) = 0$), then $r/v(r) \in L^1(\mathbf{R}_+)$ if and only if $1/\frac{d}{dr}v(r) \in L^1(\mathbf{R}_+)$.

Proof. By integration by parts and Schwarz's inequality we get the following:

$$\int_1^r \frac{t-1}{v(t)}\, dt \leq 2\int_1^r \frac{dt}{\frac{d}{dt}v(t)} \quad \text{for any } r > 1$$

which implies the desired inequality. □

References

[BRS] Brandolini, L., Rigoli, M., Setti, A. G., *Positive solutions of Yamabe type equations on complete manifolds and applications*, J. Functional Analysis **160** (1998), 176-222. MR **2000a:**35051

[CN] Chen, K. S., Ni, W. M., *On the structure of the conformal scalar curvature equation on* $\mathbb{R}^n$, Indiana Univ. Math. J. **41** (1992), 261-278. MR **93g:**35040

[CT] Choi, H. I., Treibergs, A., *Gauss maps of spacelike constant mean curvature hypersurfaces of Minkowski space*, J. Differential Geometry **32** (1990), 775-817. MR **92e:**58051

[CTW] Cheng, S. Y., Tam, L. F., Wan, T. Y-H., *Harmonic maps with finite total energy*, Proc. Amer. Math. Soc. **124** (1996), 275-28 4. MR **96d:**58031

[CY] Cheng, S. Y., Yau, S. T., *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333-354. MR **52:**6608

[EL] Eells, J., Lemaire, L., *Selected topics in harmonic maps*, A.M.S., Providence, 1983. MR **85g:**58030

[F] Federer, H., *Geometric Measure Theory*, Grundlehren series, Vol. 153, Springer, 1969. MR **41:**1976

[Gri] Grigor'yan, A. A., *On the existence of positive fundamental solution on Riemannian manifolds*, Math. USSR. Sb. **56** (1987), 349-358. MR **88g:**58175

[Gro] Gromov, M., *Kähler hyperbolicity and L_2-Hodge theory*, J. Differential Geometry **33** (1991), 263-292. MR **92a:**58133

[GW1] Greene, R. E., Wu, H., *Function theory on manifolds which possess a pole*, Lecture Notes in Mathematics, vol. 699, Berlin-Heiderberg-New York, Springer, 1979. MR **81a:**53002

[GW2] Greene, R. E., Wu, H., *C^∞ approximation of convex, subharmonic, and plurisubharmonic functions*, Ann. Scient. Éc. Norm. Sup. **12** (1979), 47-84. MR **80m:**53055

[K] Karp, L., *Subharmonic functions on real and complex manifolds*, Math. Z. **179** (1982), 535-554. MR **84d:**53042

[LT1] Li, P., Tam, L. F., *Complete surfaces with finite total curvature*, J. Differential Geometry **33** (1991), 139-168. MR **92e:**53051

[LT2] Li, P., Tam, L. F., *Green's functions, harmonic functions, and volume comparison*, J. Differential Geometry **41** (1995), 277-318. MR **96f:**53054

[LY] Li, P.,Yau, S. T., *Curvature and holomorphic mappings of complete Kähler manifolds*, Compositio Math. **73** (1990), 1 25-144. MR **91g:**32010

[S] Sturm, K-T., *Analysis on local Dirichlet spaces, I. Recurrence, conservativeness and L^p-Liouville properties*, J. Reine Angew. Math. **456** (1994), 173-196. MR **95i:**31003

[T1] Takegoshi, K., *Energy estimates and Liouville theorems for harmonic maps*, Ann. Scient. Éc. Norm. Sup. **23** (1990), 563-592. MR **91j:**58047

[T2] Takegoshi, K., *A maximum principle for P-harmonic maps with L^q finite energy*, Proc. A.M.S. **126** (1998), 3 749-3753. MR **99b:**58065

[V] Varopoulos, N., *Potential theory and diffusion on Riemannian manifolds*, Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vols. I, II, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, pp. 821-837. MR **85a:**58103

[Y] Yau, S.T., *Some function theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), 656-670. MR **54:**5502

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