A TAUBERIAN THEOREM FOR VILENKIN SERIES

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Abstract. There are a number of papers in the literature which contain Cesàro analogues of results already known for martingale sums of Vilenkin-Fourier series. We show that for Vilenkin systems of bounded type, these are not merely analogues but actually generalizations. Indeed, we prove that convergence of the Cesàro means of a Vilenkin series $S$ implies convergence of martingale partial sums of $S$ itself.

§1. Introduction

Let $\mathbb{N} := \{0, 1, 2, \cdots\}$, and $\mathcal{P} := \{p_0, p_1, \cdots\}$ be any sequence of integers which satisfies $p_n \geq 2$. For each $n \in \mathbb{N}$ set $P_n := p_0p_1 \cdots p_{n-1}$, where the empty product is by definition 1. The multiplicative Vilenkin group associated with $\mathcal{P}$ is the set $G := \{(x_0, x_1, \cdots) : x_k \in \mathbb{N} \text{ and } 0 \leq x_k < p_k\}$ together with the operation

$$x + y := (x_0 \oplus y_0, x_1 \oplus y_1, \cdots),$$

where $x = (x_0, x_1, \cdots)$, $y = (y_0, y_1, \cdots)$ and, for each $k$, $x_k \oplus y_k$ represents the sum of $x_k$ and $y_k$ modulo $p_k$. The dual group of $G$ is the system $(w_n, n \in \mathbb{N})$ defined for $x = (x_0, x_1, \cdots)$ by

$$w_n(x) := \prod_{k=0}^{\infty} \exp\left(\frac{2\pi i n_k x_k}{p_k}\right),$$

where the coefficients $n_k$ are integers which satisfy $0 \leq n_k < p_k$ and $n = \sum_{k=0}^{\infty} n_k P_k$ (see Vilenkin [4] for details). When $p_k := 2$ for all $k$, the group $G$ is called the dyadic group and the characters $w_n$ are called the Walsh system. When $p_k = O(1)$, the system $\{w_n\}$ is called a (multiplicative) Vilenkin system of bounded type.

It is well known that $G$ is a compact group for each collection of radices $\mathcal{P}$, and that the corresponding Vilenkin system $\{w_n\}$ is a complete orthonormal system on $G$. Moreover, the group $G$ can be identified with the interval $[0, 1)$ by taking an $x = (x_0, x_1, \cdots) \in G$ to the number

$$x := \sum_{k=0}^{\infty} x_k P_k^{-1}. $$

Under this identification, Haar measure on $G$ is taken to Lebesgue measure on $[0, 1)$.
A Vilenkin series is a series of the form $S := \sum_{k=0}^{\infty} a_k w_k$, where $a_k$ is some sequence of complex numbers. For each $x \in G$ and $n \in \mathbb{N}$, the partial sums of a Vilenkin series $S$ are defined by

$$S_n(x) := \sum_{k=0}^{n-1} a_k w_k(x).$$

The partial sums $S_n$ form a martingale in $L^2(G)$ which allows one to use martingale convergence theorems on Vilenkin series.

§2. Preliminaries

For each nonnegative integer $n$, define intervals on $G$ by

$$I_0(0) := G,$$  

and

$$I_n(j) := \left\{ x = (x_0, x_1, \cdots) \in G : \sum_{k=0}^{n-1} x_k P_n^{-1} = \frac{j}{P_n} \right\}$$

for $j = 0, 1, \cdots, P_n - 1$, $n = 1, 2, \cdots$. Recall that $\{I_n(0)\}_{n=0}^{\infty}$ is a nested sequence of subgroups of $G$ which forms a neighborhood base at the origin, and for each $n$, $\{I_n(j)\}_{j=0}^{P_n-1}$ is a collection of pairwise disjoint compact sets in $G$ whose union is $G$. In particular, given $x \in G$ and $n \in \mathbb{N}$, there is a unique $0 \leq j < P_n$ such that $x \in I_n(j)$. We shall denote this interval by $I_n(x)$.

Denote the Haar measure of a subset $E$ of $G$ by $m(E)$ and the Lebesgue measure of a subset $E$ of $[0, 1)$ by $|E|$. Notice that under the identification of $G$ with $[0, 1)$, the interval $I_n(j)$ corresponds to the interval $[jP_n^{-1}, (j+1)P_n^{-1})$. In particular,

$$m(I_n(j)) = P_n^{-1}$$

for $0 \leq j < P_n$ and $n \in \mathbb{N}$.

Let $x \in G$. A sequence of measurable sets $E_j$ in $G$ is said to shrink nicely to $x$ if there exist integers $r_j \to 0$, as $j \to \infty$, and an absolute constant $\alpha > 0$ such that $E_j \subset I_{r_j}(x)$ and

$$m(E_j) \geq \alpha \cdot m(I_{r_j}(x)),$$

for $j = 1, 2, \ldots$. By using the identification of $G$ with the unit interval, and of Haar measure with Lebesgue measure, it is easy to check that if $E$ is a measurable subset of $G$ and $h$ represents the characteristic function of $E$, i.e., $h(x) = 1$ for $x \in E$ and $h(x) = 0$ for $x \notin E$, then

$$\frac{1}{m(E_j)} \int_{E_j} h \, dm \to 1$$

almost everywhere $[m]$ on $E$ for any sequence of sets $E_j$ which shrinks nicely to $x$ (see Rudin [2], p. 140). In particular, if $E_j$ shrinks nicely to $x$, then

$$\lim_{j \to \infty} \frac{m(E \cap E_j)}{m(E_j)} = 1$$

for almost every $x \in E$. 

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§3. The main result

For each \( x \in G \) and each integer \( n > 0 \), the Cesàro means of \( S_n \) are defined by

\[
\sigma_n(x) := \frac{1}{n} \sum_{k=1}^{n} S_k(x).
\]

It is well known (and easy to see) that if \( S_k \) converges to some limit \( f \), as \( k \to \infty \), then \( \sigma_n \to f \), as \( n \to \infty \). We will obtain the following partial converse to this result. (The Walsh version of this result was obtained by Shaginyan [3] in 1974.)

**Theorem 1.** Suppose \( E \) is a measurable subset of \( G \) and \( S \) is a Vilenkin series whose Cesàro means satisfy

\[
\lim_{n \to \infty} \sigma_{p_n}(x) = f(x), \quad x \in E,
\]

for some finite-valued, measurable function \( f \). If the Vilenkin system is of bounded type, then

\[
\lim_{n \to \infty} S_{p_n}(x) = f(x)
\]

for almost every \( x \in E \).

**Proof.** We may suppose that \( E \) is of positive Haar measure, i.e., \( m(E) > 0 \). Fix \( 0 < \epsilon < m(E)/2 \). By hypothesis, \( f_n(x) := \sigma_{p_{n+1}}(x) \) converges to \( f(x) \) for \( x \in E \), as \( n \to \infty \). Thus by Egoroff’s Theorem, we can choose a measurable subset \( E_0 \) of \( E \) such that \( f_n \to f \) uniformly on \( E_0 \) and the measure of \( E_0 \) satisfies \( m(E_0) > m(E) - \epsilon \).

Let \( x \in E_0 \) be a point which satisfies (3) for every sequence \( E_j \) which shrinks nicely to \( x \). For each fixed positive integer \( j \), let \( W_j \) represent the collection of indices \( \ell \) which satisfy \( I_{j+1}(\ell) \subset I_j(x) \). If \( \ell \in W_j \) and

\[
B_j := I_{j+1}(\ell) \neq A_j := I_{j+1}(x),
\]

let \( E_j \) represent the union of \( A_j \) and \( B_j \). By construction, \( E_j \subset I_j(x) \) and by (2),

\[
m(E_j) = 2m(I_{j+1}(x)) = \frac{2}{p_j M_p} \geq \frac{2}{M M_p} = \frac{2}{M} m(I_j(x)),
\]

where \( M := \sup \{ p_i : i \in \mathbb{N} \} \). Thus \( A_j \) and \( E_j \) both shrink nicely to \( x \), as \( j \to \infty \).

It follows from (3) that \( m(E_0 \cap A_j)/m(A_j) \to 1 \) and \( m(E_0 \cap E_j)/m(E_j) \to 1 \) as \( j \to \infty \).

Let \( n \in \mathbb{N} \) be so large that \( m(E_0 \cap A_n)/m(A_n) < 5/4 \) and \( m(E_0 \cap E_n)/m(E_n) > 3/4 \). Since \( B_n = E_n \setminus A_n \) and \( m(E_n) = 2m(A_n) = 2m(B_n) \), such a choice for \( n \) implies

\[
\frac{m(E_0 \cap B_n)}{m(B_n)} = \frac{m(E_0 \cap E_n) - m(E_0 \cap A_n)}{m(E_n)} = \frac{2m(E_0 \cap E_n)}{m(E_n)} - \frac{m(E_0 \cap A_n)}{m(E_n)} > \frac{6}{4} - \frac{5}{4} > 0.
\]

In particular, \( E_0 \cap B_n \) is uncountable.

We have proved that if \( n \) is sufficiently large, for each \( \ell \in W_n \) which satisfies \( I_{n+1}(\ell) \neq I_{n+1}(x) \), there exists an \( x_n(\ell) \in E_0 \cap I_{n+1}(\ell) \). The restriction \( I_{n+1}(\ell) \neq I_{n+1}(x) \) is not needed. Indeed, if \( \ell \) satisfies \( I_{n+1}(\ell) = I_{n+1}(x) \), then \( x_n(\ell) := x \) also belongs to \( E_0 \cap I_{n+1}(\ell) \). Thus, for \( n \) sufficiently large, it is possible to choose points \( x_n(\ell) \in E_0 \cap I_{n+1}(\ell) \) for all \( \ell \in W_n \).
We claim that for these points,
\[
\sum_{\ell \in W_n} \sigma_{P_{n+1}}(x_n(\ell)) = \sigma_{P_n}(x) + (p_n - 1)SP_n(x).
\]

To verify (5), let \(y \in G\) and notice by definition that
\[
\sigma_{P_{n+1}}(y) = \frac{1}{P_{n+1}} \sum_{k=1}^{P_{n+1}} S_k(y)
\]
\[
= \frac{1}{P_n P_n n} \sum_{k=1}^{P_n} S_k(y) + \frac{1}{P_n P_n n} \sum_{k=P_n+1}^{P_{n+1}} S_k(y)
\]
\[
= \frac{1}{P_n} \sigma_{P_n}(y) + \frac{1}{P_n P_n n} \sum_{k=P_n+1}^{P_{n+1}} S_k(y) = I_1(y) + I_2(y).
\]

Look at a typical term \(T(y) := a_\ell w_k(y)\) in one of these sums where \(y = x_n(\ell)\). If \(T\) comes from \(I_1\), then \(k\) is less than \(P_n\). Since the Vilenkin function \(w_k\) is constant on \(I_n(x)\) and \(x_n(\ell) \in I_n(x)\), it is clear that \(T(x_n(\ell)) = T(x)\) for all \(\ell \in W_n\). Since the number of intervals of the form \(I_{n+1}(\ell)\) which are subsets of \(I_n(x)\) is \(P_n\), it follows that
\[
\sum_{\ell \in W_n} I_1(x_n(\ell)) = \sum_{\ell \in W_n} \frac{1}{P_n} \sigma_{P_n}(x) = \sigma_{P_n}(x).
\]

On the other hand, suppose that \(T\) comes from \(I_2\) and \(P_n < k \leq P_{n+1}\). Notice that \(\{I_{n+1}(\ell) : \ell \in W_n\}\) contains every subinterval of \(I_n(x)\), hence \(\{w_k(x_n(\ell)), \ell \in W_n\}\) ranges over every \(p_n\)th root of unity. Since the sum of \(p_n\)th roots of unity is zero, we have
\[
\sum_{\ell \in W_n} w_k(x_n(\ell)) = 0.
\]

Consequently, adding up the values of \(I_2\) at \(y = x_n(\ell)\), as \(\ell\) runs over \(W_n\), cancels higher order terms and multiplies lower order terms. Indeed, since the number of intervals of the form \(I_{n+1}(\ell)\) which are subsets of \(I_n(x)\) is \(P_n\) and \(P_{n+1} - P_n = (p_n - 1)P_n\), we have
\[
\sum_{\ell \in W_n} \sum_{k=P_n+1}^{P_{n+1}} S_k(x_n(\ell)) = \sum_{\ell \in W_n} \sum_{k=P_n+1}^{P_{n+1}} S_{P_n}(x) = p_n(p_n - 1)P_n SP_n(x).
\]

It follows from (6) that
\[
\sum_{\ell \in W_n} \sigma_{P_{n+1}}(x_n(\ell)) = \sigma_{P_n}(x) + (p_n - 1)SP_n(x),
\]
i.e., (5) holds.

Solve (5) for \(SP_n\) to write
\[
SP_n(x) = \frac{1}{P_n - 1} \left( \sum_{\ell \in W_n} \sigma_{P_{n+1}}(x_n(\ell)) - \sigma_{P_n}(x) \right).
\]

By construction, \(x_n(\ell)\) belongs to \(I_n(x)\), hence \(x_n(\ell) \to x\), as \(n \to \infty\), for each \(\ell \in W_n\). Since the characters \(w_k\) are continuous on the group, \(w_k(x_n(\ell)) \to w_k(x)\), as \(n \to \infty\), for each \(k \in \mathbb{N}\). But \(f\) is the uniform limit of \(f_n\) on \(E_0\) and each \(f_n\) is a Vilenkin polynomial. Since \(x_n(\ell) \in E_0\), it follows that \(f(x_n(\ell)) \to f(x)\) as \(n \to \infty\).
Let $g$ (respectively, $g_n$) represent the real part of $f$ (respectively, of $f_n$). By construction and the observation just made, we can choose $n$ so large that
\begin{equation}
\Re \left[ \sigma_{P_n+1}(x_n(\ell)) \right] = g_n(x_n(\ell)) \geq g(x_n(\ell)) - \frac{\epsilon}{2} \geq g(x) - \epsilon.
\end{equation}
Substituting this estimate back into (7), we obtain
\begin{equation}
\Re (S\{P_n\}(x)) \geq \frac{1}{p_n-1} \left( p_n g(x) - p_n \epsilon - \Re \left[ \sigma_{P_n}(x) \right] \right)
\end{equation}
and
\begin{equation}
\geq \frac{1}{p_n-1} (p_n g(x) - \Re \left[ \sigma_{P_n}(x) \right]) - M\epsilon.
\end{equation}
Take the limit infimum of this inequality as $n \to \infty$, apply hypothesis (4), and then let $\epsilon \to 0$. It follows that
\begin{equation}
\liminf_{n \to \infty} \Re \left[ S\{P_n\}(x) \right] \geq g(x) = \Re \left[ f(x) \right]
\end{equation}
for almost every $x \in E_0$. Since $|E_0| > |E| - \epsilon$, we can replace $E_0$ by $E$. Hence, \textbf{lim inf}$_{n \to \infty} \Re \left[ S\{P_n\} \right] \geq \Re \left[ f \right]$ almost everywhere on $E$. By repeating the entire argument with the inequalities in (9) reversed, we see that \textbf{lim sup}$_{n \to \infty} \Re \left[ S\{P_n\} \right] \leq \Re \left[ f \right]$ almost everywhere on $E$. A similar argument works for the imaginary parts. We conclude that $S\{P_n\} \to f$ almost everywhere on $E$ as $n \to \infty$.

**References**


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