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A SIMPLE PROOF OF A THEOREM OF BOLLOBÁS AND LEADER

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ABSTRACT. By using Scherk's lemma we give a simple combinatorial proof of a theorem due to Bollobás and Leader. For any sequence of elements of an abelian group of order k, calling the sum of k terms of the sequence a k-sum, if 0 is not a k-sum, then there are at least $r-k+1$ k-sums.

In [3] Erdös, Ginzburg and Ziv proved an elegant result in Combinatorial Number Theory: let $a_1, \cdots, a_{2k-1}$ be a sequence of elements of an abelian group G of order k. Then some k-sum is 0 (in G), where (here and below) a *t-sum* is a sum of the form $a_{i_1} + \cdots + a_{i_t}$ $(i_1 < \cdots < i_t)$.

There are many proofs and refinements of this result in the literature (see for example [1] and [5, Chap. 2]). Recently, Bollobás and Leader [2] established the following interesting result, which clearly implies the Erdös-Ginzburg-Ziv theorem by taking $r = k-1$.

Theorem. *Let G be an abelian group of order k, and let $r \geq 1$. Let $A = \{a_1, \cdots, a_{k+r}\}$ be a sequence of elements of G. Then if 0 is not a k-sum, the number of k-sums of A is at least $r+1$.*

The proof of the theorem given in [2, pp.30–32] is difficult and complicated. In this note we shall present a simple combinatorial proof of this result. Our argument is based upon the following result of Scherk on addition of subsets of an abelian group (see [6] for a short proof; cf. also [4, Theorem 15′ of Chap. 1]).

Lemma. *Let B and C be two subsets of an abelian group of order k. Suppose $0 \in B \cap C$ and suppose that if $b + c = 0$ with $b \in B$ and $c \in C$, then $b = c = 0$. Then $|B + C| \geq \min(k, |B| + |C| - 1)$, where (here and below) $B + C$ consists of all the elements $b + c$ with $b \in B$ and $c \in C$.*

Now we prove the theorem. Translating (which does not affect k-sums), we may assume that 0 is the most repeated value in A. Let L be the subsequence of all 0's in A, and let l be the cardinality of L. Then $l \leq k-1$. Clearly, we can take a subsequence S of $A \backslash L$ summing to 0 with maximal cardinality s and with $s \leq k-1$ (S may be empty). Then

$$(*) \qquad l + s \leq k - 1,$$

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for, otherwise, S with $k-s$ 0's of L added would be a subsequence of A with length k summing to 0. Hence the cardinality of $A\backslash L \cup S$ is at least $r+1$. Taking a subsequence T of $A\backslash L \cup S$ with length r, and denoting by h the greatest number of times that any element occurs in T, then $h \le l$ by the definition of l. We partition the elements of the sequence T into h *sets* (not sequences) $X_1, \cdots, X_h$, and let $X_i' = X_i \cup \{0\}$ $(i = 1, \cdots, h)$. We note that $0 \notin T$ and that T has no j-sum being 0 for any $1 < j \le h$. For, otherwise, adding such a j-terms sequence to S would give us a subsequence S' of $A\backslash L$ summing to 0, but s', the cardinality of S', satisfies that $s < s' \le h + s \le l + s \le k-1$ (by $(*)$), contradicting the choice of S. Then, by repeatedly applying Scherk's lemma, we have

$$|X_1' + \cdots + X_h'| \ge |X_1| + \cdots + |X_h| + 1 = r + 1$$

(recalling that the cardinality of T is r). In other words, T with h zeros from L appended has at least $r+1$ h-sums. By adding the remaining $k+r-(r+h)$ elements of A to each of these h-sums, we obtain at least $r+1$ k-sums of A. The proof of the theorem is complete.

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