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A NOTE ON THE IMBEDDING THEOREM OF BROWDER AND TON

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Abstract. The imbedding theorem of Browder and Ton states that for any real separable Banach space X there exist a real separable Hilbert space H and a compact linear injection $\psi : H \to X$ such that $\psi(H)$ is dense in X. We shall give a short and elementary new proof to this result. We also briefly discuss the corresponding result without the completeness assumption.

1. Introduction

The imbedding theorem of Browder and Ton [7] can be viewed as an abstract version of classical imbedding theorems familiar in the context of function spaces. Indeed, let

$$W^{m,p}(\Omega) = \{u | D^\alpha u \in L_p(\Omega) \quad \text{for all } |\alpha| \le m\},$$

where Ω is a bounded open set in $\mathbb{R}^n$ satisfying the uniform cone condition. If $1 \le p < \infty$, $k - m \ge 1$ and

$$\frac{1}{p} > \frac{1}{2} - \frac{k-m}{n},$$

then by the Sobolev imbedding theorem (see [1], for instance) the natural injection $i : W^{k,2}(\Omega) \to W^{m,p}(\Omega)$ is a compact linear map having a dense range in $W^{m,p}(\Omega)$.

In 1968 F. Browder and B.A. Ton proved the abstract version of the imbedding theorem. It states, on a purely abstract level, that for any real separable Banach space X there exist a real separable Hilbert space H and a compact linear injection $\psi : H \to X$ such that $\psi(H)$ is dense in X.

In their original paper, Browder and Ton used the imbedding theorem to obtain the so-called 'elliptic super-regularization' for operators from X into the dual space X^*. Their approach is a generalization of the method of elliptic regularization used by Lions, Nirenberg and others (see the references given in [7]). The idea is to replace a given nonlinear elliptic equation by a mildly nonlinear elliptic equation of higher order, in which the nonlinear term is considered as a perturbation. A similar idea is later used for instance in [3], [2], [4], [5], [6], [8], [10] and [11].

The original proof of the imbedding theorem in [7] is quite lengthy. A shorter version based on the same reasoning can be found in [9]. We give a short and elementary new proof. Let X be a real separable Banach space and $S = \{v_1, v_2, \dots\}$ an infinite set of linearly independent vectors such that $\|v_k\|_X = 1$ for all $k \in \mathbb{Z}_+$

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and sp S is dense in X. Taking into account a suitably restricted set of infinite linear combinations of vectors of S we find a linear space V such that $\operatorname{sp} S \subset V \subset X$ and V can be naturally identified with a compact injective image of a closed subspace of l^2. Actually, we shall give a variant of the imbedding theorem without the completeness assumption. The imbedding theorem of Browder and Ton is then obtained as a corollary.

2. The result

Let X be a real normed space and $\tilde{X}$ the essentially unique completion of X. The norm in $\tilde{X}$ is denoted by $\|\cdot\|_{\tilde{X}}$ and $\|x\|_{\tilde{X}} = \|x\|_X$ whenever $x \in X$.

Theorem 2.1. *Let X be a normed space and $S \subset X$ a countable subset. Then there exist a separable Hilbert space H and a compact linear injection $\psi : H \to \tilde{X}$ such that* $\operatorname{sp} S \subset \psi(H) \cap X$.

Proof. Without loss of generality we can assume that $S = \{v_1, v_2, \dots\}$ is an infinite set of linearly independent vectors such that $\|v_k\|_X = 1$ for all $k \in \mathbb{Z}_+$. Let $(a_k)_{k=1}^{\infty}$ be a real sequence such that $(a_k)_{k=1}^{\infty} \in l^2$. Then the series

$$\sum_{k=1}^{\infty} \frac{a_k}{k} v_k$$

converges in $\tilde{X}$. Indeed, denoting $s_n = \sum_{k=1}^{n} \frac{a_k}{k} v_k$ we have

$$\|s_{n+p} - s_n\|_X \leq \sum_{k=n+1}^{n+p} \frac{|a_k|}{k} \leq \sqrt{\sum_{k=n+1}^{n+p} \frac{1}{k^2}} \sqrt{\sum_{k=n+1}^{n+p} |a_k|^2} \tag{2.1}$$

for all $n \in \mathbb{Z}_+$ and $p = 1, 2, 3, \dots$. Hence (s_n) is a Cauchy sequence in X and it converges in $\tilde{X}$. Note that the representation $u = \sum_{k=1}^{\infty} \frac{a_k}{k} v_k$, $(a_k)_{k=1}^{\infty} \in l^2$, is not necessarily unique. Define the map $i : l^2 \to \tilde{X}$ by setting

$$i(\vec{a}) = \sum_{k=1}^{\infty} \frac{a_k}{k} v_k$$

for all $\vec{a} = (a_k)_{k=1}^{\infty} \in l^2$. Then it is easy to see that i is linear and by the estimate (2.1) (with the usual convention that the sum over an empty set is zero)

$$\|i(\vec{a})\|_{\tilde{X}} \leq c_0 \|\vec{a}\|_{l^2},$$

where $c_0 = \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}}$. Hence the mapping $i : l^2 \to \tilde{X}$ is continuous. Moreover, the map i is compact since it is a uniform limit of operators having finite dimensional range. Indeed, denoting $i_n(\vec{a}) = \sum_{k=1}^{n} \frac{a_k}{k} v_k$ we get by (2.1)

$$\|i(\vec{a}) - i_n(\vec{a})\|_{\tilde{X}} \leq \sqrt{\sum_{k=n+1}^{\infty} \frac{1}{k^2}} \sqrt{\sum_{k=n+1}^{\infty} |a_k|^2}.$$

Thus

$$\|i - i_n\| = \sup_{\|\vec{a}\|=1} \|i(\vec{a}) - i_n(\vec{a})\|_{\tilde{X}} \leq \sqrt{\sum_{k=n+1}^{\infty} \frac{1}{k^2}},$$

proving the assertion. Denote $W_0 = \mathrm{Ker}(i)$, which is a closed linear subspace of l^2. Define a real separable Hilbert space H by setting

$$H = W_0^{\perp} = \{\vec{a} \in l^2 | \vec{a} \perp W_0\}.$$

Then $H \cong l^2/W_0 = l^2/\mathrm{Ker}(i)$ and consequently the map $\psi = i|_H : H \to \tilde{X}$ is a linear compact injection. Moreover, denoting by $P : l^2 \to H$ the orthonormal projection, we have $i(\vec{e}_j) = \psi(P\vec{e}_j) = v_j/j \in X$ for all $j \in \mathbb{Z}_+$, where $\vec{e}_j = (\delta_{j,k})_{k=1}^{\infty}$. Clearly the subset $S_H := \psi^{-1}(S)$ of H is countable and $\psi(\mathrm{sp}\, S_H) = \mathrm{sp}\, S$. Hence $\mathrm{sp}\, S \subset \psi(H) \cap X$, completing the proof. □

Corollary 2.2. *Let X be a real separable space. Then there exist a separable Hilbert space H and a compact linear injection $\psi : H \to \tilde{X}$ such that $\psi(H) \cap X$ is dense in X.*

Proof. Let $S = \{v_1, v_2, \dots\}$ be an infinite set of linearly independent vectors such that $\|v_k\| = 1$ for all $k \in \mathbb{Z}_+$ and $\mathrm{sp}\, S$ is dense in X. Clearly $\mathrm{sp}\, S$ is also dense in $\tilde{X}$. Thus by Theorem 2.1 there exist a real separable Hilbert space H and a linear compact injection $\psi : H \to \tilde{X}$ such that $\mathrm{sp}\, S \subset \psi(H) \cap X$, completing the proof. □

Corollary 2.3 (Imbedding Theorem of Browder and Ton)**.** *Let X be a real separable Banach space. Then there exist a separable Hilbert space H and a compact linear injection $\psi : H \to X$ such that $\psi(H)$ is dense in X.*

Proof. Now $X = \tilde{X}$ and the the conclusion follows from Corollary 2.2 □

We close this note with a few remarks, which may clarify the reasoning. Let X be a real separable Banach space and let $S = \{v_1, v_2, \dots\}$ be an infinite set of linearly independent vectors such that $\|v_k\| = 1$ for all $k \in \mathbb{Z}_+$ and $\mathrm{sp}\, S$ is dense in X. In view of the proofs above it is relevant to define a linear subspace of X by setting

$$V = \{u \in X | u = \sum_{k=1}^{\infty} \frac{a_k}{k} v_k, (a_k)_{k=1}^{\infty} \in l^2\}.$$

Clearly $\mathrm{sp}\, S \subset V \subset X$ and hence V is dense in X. Identifying any pair of sequences $(a_k)_{k=1}^{\infty} \in l^2$ and $(b_k)_{k=1}^{\infty} \in l^2$ such that $\sum_{k=1}^{\infty} \frac{a_k}{k} v_k = \sum_{k=1}^{\infty} \frac{b_k}{k} v_k$ in X, gives the quotient space identified with a closed subspace H of l^2 needed in Corollary 2.3.

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