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TRACES OF CONVEX DOMAINS

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Abstract. Diederich and Ohsawa proved that in $\mathbb{P}^5$ there exists a locally hyperconvex, Stein open subset which is not hyperconvex. In this paper we generalize their results.

Introduction

In [2] Diederich and Ohsawa proved that if M is a complex manifold and N is a complex submanifold, then any locally hyperconvex, Stein open subset of N is the trace of a locally hyperconvex, Stein open subset of M. In [4] it was proved that if Y is a closed complex subspace of X and Y is Stein, then Y has a Stein neighborhood. Also, it has been proved in [5] that if Y is hyperconvex, then Y has a hyperconvex neighborhood.

Using the methods of Demailly [1] we will set up a general framework for the above theorems and we will generalize Diederich and Ohsawa's results for reduced complex spaces.

The results

If M is a topological space we will denote by $\mathcal{C}(M)$ the set of continuous real functions defined on M, and by $\mathit{Open}(M)$ the set of open subsets of M.

Let $\mathcal{A}$ be the class of reduced complex spaces and let $\mathcal{B} \subset \mathcal{A}$ be such that for every $M \in \mathcal{B}$, $\mathit{Open}(M) \subset \mathcal{B}$. We assume also that for every $x \in M$, $\{x\} \in \mathcal{B}$. For each $M \in \mathcal{B}$ we consider a subset $\mathcal{P}(M)$ of $\mathcal{C}(M)$ such that the following conditions are satisfied:

1) For every $M \in \mathcal{B}$ and U an open subset of M, if $\phi \in \mathcal{P}(M)$, then $\phi_{|U} \in \mathcal{P}(U)$. Furthermore $\mathcal{P}_{|\mathit{Open}(M)}$ is a subsheaf of sets of $\mathcal{C}_{|\mathit{Open}(M)}$ and if M is just a point, then $\mathcal{P}(M) = \mathbb{R}$.

2) For every $f, g \in \mathcal{P}(M)$, a real number $a > 0$, U an open subset of $\mathbb{R}$ containing $g(M)$ and $\chi : U \to \mathbb{R}$ a smooth, convex, and non-decreasing function, we have $af + \chi \circ g \in \mathcal{P}(M)$ and $\max\{f, g\} \in \mathcal{P}(M)$.

3) For every $N \subset M$ a closed subspace, $N, M \in \mathcal{B}$, every continuous function $\lambda : N \to (0, \infty)$, and every $f \in \mathcal{P}(N)$ there exists an open neighborhood $V \subset M$ of N and $\tilde{f} \in \mathcal{P}(V)$ such that $|\tilde{f}_{|V} - f| < \lambda$.

4) For every $N \subset M$ a closed subspace, $N, M \in \mathcal{B}$, there exists an open neighborhood V of N and a continuous function $f : V \to [-\infty, \infty)$ such that $f^{-1}(-\infty) = N$

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and f has the following property: for every $x \in V$ and every $\phi \in \mathcal{P}_x$ there exist $k > 0$ and an open neighborhood U of x, contained both in V and in the domain of ϕ, such that $f + k\phi \in \mathcal{P}(U \setminus N)$. A function with this property will be called almost $\mathcal{P}$.

Definition 1. Let $M \in \mathcal{B}$

1. M is said to be $\mathcal{P}$-complete if there exists $\phi \in \mathcal{P}(M)$ such that for every $c \in \mathbb{R}$, $\{x \in M : \phi(x) < c\} \subset\subset M$.

2. M is said to be hyper $\mathcal{P}$-complete if there exists $\phi \in \mathcal{P}(M)$, $\phi : M \to (-\infty, 0)$, such that for every $c < 0$, $\{x \in M : \phi(x) < c\} \subset\subset M$.

3. An open subset D of M is said to be locally hyper $\mathcal{P}$-complete if for every $x \in \partial D$ there exists an open neighborhood B of x such that $B \cap D$ is hyper $\mathcal{P}$-complete.

Observation. It follows from the properties of $\mathcal{P}$ that every point of $M \in \mathcal{B}$ has a hyper $\mathcal{P}$-complete neighborhood.

We consider $N, M \in \mathcal{B}$, N a closed subspace of M. The proofs of the following two propositions are similar to the proof of Theorem 1 in [1]. Only the proof of Proposition 2 will be given here.

Proposition 1. *If N is $\mathcal{P}$-complete, then N has a $\mathcal{P}$-complete neighborhood in M.*

Proposition 2. *If N is hyper $\mathcal{P}$-complete, then N has a hyper $\mathcal{P}$-complete neighborhood in M.*

Proof. Let U be an open neighborhood of N and $v : U \to [-\infty, \infty)$ a continuous function such that $v^{-1}(-\infty) = N$ and v is almost $\mathcal{P}$ on U. Shrinking U we may suppose that there exists $\phi \in \mathcal{P}(U)$ such that $\phi < 0$ and for every $c \in \mathbb{R}$, $\{x \in N : \phi(x) < c\} \subset\subset N$. Let W be an open subset of M such that $\partial W \setminus N \subset U$, $N \subset W$ and for every $c < 0$, $\{x \in \overline{W} : \phi(x) \le c\}$ is compact.

Let $\tilde{v} = v + \chi \circ \phi$ where $\chi : (-\infty, 0) \to \mathbb{R}$ is a smooth, convex and increasing function. If χ increases fast enough $\tilde{v} \in \mathcal{P}(W \setminus N)$. To see that one sets $F_n := \phi^{-1}([\frac{-1}{n}, \frac{-1}{n+1}])$ and $F_0 := \phi^{-1}(-\infty, -1]$. For every $j \in \mathbb{N}$, F_j is compact and therefore there exists a neighborhood U_j of F_j and $k_j > 0$ such that $v + k_j\phi \in \mathcal{P}(U_j \setminus N)$. We then require that $\chi'_{|[-1/n, -1/n+1]} > k_j$. The condition $\tilde{v} \in \mathcal{P}(W \setminus N)$ is a local condition and $\bigcup F_j \supset W$. On a neighborhood of F_j, $\tilde{v} = v + k_j\phi + \chi_j \circ \phi$ where $\chi_j(t) = \chi(t) - k_j t$ is a convex and increasing function on a neighborhood of $\phi(F_j)$. This implies that $\tilde{v} \in \mathcal{P}(W \setminus N)$. In the same way we can choose χ such that $\tilde{v}_{|\partial W \setminus N} > 0$.

We set $V := \{x \in W : \tilde{v}(x) < 0\}$. Then $V \supset N$ and $\psi := \max\{\phi, \tilde{v}\}$ is a negative exhaustion for V. Also, since $\psi = \phi$ in a neighborhood of N, we have $\psi \in \mathcal{P}(V)$. $\square$

Proposition 3. *If D is an open, $\mathcal{P}$-complete, and locally hyper $\mathcal{P}$-complete subset of N, then there exists an open, $\mathcal{P}$-complete, and locally hyper $\mathcal{P}$-complete subset Ω of M such that $\Omega \cap N = D$.*

Proof. We denote by ∂D the boundary of D in N. For every $x \in \partial D$ we consider an open neighborhood Q_x of x in N such that $W_x := Q_x \cap D$ is hyper $\mathcal{P}$-complete.

Let $\{Q_j : j \in \mathbb{N}\}$ be a countable subset of $\{Q_x : x \in \partial D\}$ such that $\bigcup Q_j \supset \partial D$. Using Proposition 2 we choose $\widetilde{W_j}$ open hyper $\mathcal{P}$-complete subsets of M such that

$\widetilde{W}_j \cap N = W_j$ and we set $\widetilde{W} = \bigcup \widetilde{W}_j$. Let D_1 be an open subset of D such that $\overline{D}_1 \subset D$ and $(D \setminus \overline{D}_1) \subset \widetilde{W}$. For every $j \in \mathbb{N}$ let R_j be an open subset of M such that $\overline{R}_j \cap N \subset Q_j$, $\bigcup R_j \supset \partial D$, $\overline{R}_j \cap D_1 = \emptyset$ and $\{R_j\}$ is locally finite. For each $z \in D \setminus D_1$ we choose an open neighborhood I_z in M such that $I_z \cap N \subset D$ and for each $j \in \mathbb{N}$ we have:

if $z \in \overline{R}_j$, then $I_z \subset \widetilde{W}_j$,

if $z \notin \overline{R}_j$, then $I_z \cap \overline{R}_j = \emptyset$.

This is possible because $\{R_j\}$ is locally finite and $\overline{R}_j \cap D \subset \widetilde{W}_j$. Note that because $\{R_j\}$ is locally finite $\bigcup \overline{R}_j$ is closed. For $z \in D_1$ we choose an open neighborhood J_z in M such that $J_z \cap (\bigcup \overline{R}_j) = \emptyset$ and $J_z \cap N \subset D$.

Put

$$U := (\bigcup_{z \in D \setminus D_1} I_z) \cup (\bigcup_{z \in D_1} J_z).$$

Then U is open in M and $U \cap N = D$. Also for every $j \in \mathbb{N}$, $(R_j \cap U) \subset (\widetilde{W}_j \cap U)$. Using Proposition 1 we choose a $\mathcal{P}$-complete open subset $U_1 \subset U$ of U such that $U_1 \cap N = D$ and $\partial U_1 \setminus N \subset U$. Let $\phi \in \mathcal{P}(U_1)$ be an exhaustion function and $v : U_1 \to [-\infty, \infty)$ an almost $\mathcal{P}$ function such that $v^{-1}(-\infty) = D$. We can assume of course that $\phi > 1$. Set $h = v + \chi \circ \phi$ where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth, convex and increasing function such that $h \in \mathcal{P}(U_1 \setminus D)$. We define $\Omega := \{x \in U_1 : h(x) < 0\}$. Then Ω is $\mathcal{P}$-complete since $\max\{\phi, (1 - e^h)^{-1}\}$ is an exhaustion.

Ω is also hyper $\mathcal{P}$-complete. Indeed: if $x \in \partial\Omega \setminus N$, it follows directly from the definition that Ω is hyper $\mathcal{P}$-complete around x. If $x \in \partial\Omega \cap N = \partial D$, we choose $j \in \mathbb{N}$ such that $x \in R_j$ and let $B := R_j \cup \tilde{W}_j$. Then B is an open neighborhood of x and $B \cap \Omega = \tilde{W}_j \cap \Omega$. If $\rho : \tilde{W}_j \to (-\infty, 0)$ is an exhaustion function for $\tilde{W}_j$ and $\rho \in \mathcal{P}(\tilde{W}_j)$, then $\psi := \max\{h, \rho\}$ is a bounded exhaustion function for B and $\psi \in \mathcal{P}(U)$. ☐

Two examples. I) $\mathcal{B} = \mathcal{A}$ and, for a reduced complex space X, $\mathcal{P}(X) =$ the set of strictly plurisubharmonic functions. It follows from [1] that $\mathcal{P}$ satisfies all the required properties.

In this case Proposition 1 is the Main Theorem in [4] and Proposition 2 is Theorem 4 in [5]. Proposition 3 becomes:

Proposition 4. *Let Y be a complex subspace of X. If D is an open, Stein, locally hyperconvex subset of X, then there exists an open, Stein, locally hyperconvex subset Ω of X such that $\Omega \cap Y = D$.*

It was proved in [2] that for every $n \geq 5$ there exists an open subset Ω of $\mathbb{P}^n$ which is Stein, locally hyperconvex but not hyperconvex.

If Y is a projective algebraic variety of dimension $n \geq 5$ let $\pi : Y \to \mathbb{P}^n$ be a proper, surjective and finite holomorphic map and let $\Omega_1 := \pi^{-1}(\Omega)$. Since π is finite, Ω_1 is Stein and locally hyperconvex (see in this sense [5]).

Suppose Ω_1 is hyperconvex. Let $\phi : \Omega_1 \to (-\infty, 0)$ be a plurisubharmonic exhaustion function and let $\pi_*\phi$ be the unique continuous function that outside the branching set satisfies: $\pi_*\phi(x') = \sum_{\pi(x)=x'} \phi(x)$.

By Varouchas [6], $\pi_*\phi$ is plurisubharmonic. If p is the maximal number of points in the fiber of π and $\epsilon > 0$, then $\{\pi_*\phi \leq -\epsilon\} \subset \pi(\{\phi \leq \frac{-\epsilon}{p}\})$. Therefore $\pi_*\phi$ is a bounded plurisubharmonic exhaustion function for Ω. But a Stein domain that has a bounded plurisubharmonic exhaustion function is hyperconvex and this is a contradiction. Combining this last observation and Proposition 4 we obtain:

Theorem 1. *Let X be a complex space. If X has a complex subspace Y which is a projective algebraic variety with $dim(Y) \geq 5$, then there exists an open Stein subset $\Omega \subset X$ which is locally hyperconvex but not hyperconvex.*

II) Let (X, ω) be a Kähler manifold and set $\mathcal{B} := \bigcup\{Open(Y) : Y$ is a closed submanifold of $X\}$. Every $M \in \mathcal{B}$ is a Kähler manifold with the induced metric from X. Definition 2 and Proposition 5 are due to H. Wu [7].

Definition 2. 1) Let M be a Kähler manifold, $x \in M$ and G the hermitian inner product on T_xM given by the Kähler metric. A set of q vectors $\{Z_1, Z_2, ..., Z_q\}$ is ϵ-normal, for $\epsilon > 0$, if $|G(Z_i, Z_j) - \delta_{ij}| < \epsilon$ for $i, j = 1, 2, ..., q$.

2) If f is a continuous function defined near x and L is a 1-dimensional complex submanifold of M passing though x we choose a coordinate system $\{z_1, z_2, ..., z_n\}$ such that $z_i(x) = 0$, $i = 1, ..., n$, and such that near x, $L = \{z_2 = z_3 = ... = z_n = 0\}$. Furthermore, assume $|\partial/\partial z_1|(x) = 1$. Then we define

$$Pf(x, L) = \liminf_{r \to 0} \frac{2}{\pi r^2} \Big(\int_0^{2\pi} f(re^{i\theta}, 0, ..., 0) d\theta - 2\pi f(0) \Big).$$

If $Z \in T_xM$, also define $Pf(x, Z) = |Z|^2 \inf_L Pf(x, L)$ where L runs through all the 1-dimensional complex submanifolds of M tangent to $span_{\mathbb{R}}\{Z, JZ\}$ and defined near x.

3) If U is an open subset of M we define $\Psi(q; U)$ to be those continuous functions f defined on U with the following property: for every $x_0 \in U$ there exists a neighborhood W of x_0 and positive constants ϵ and η such that if $x \in W$ and $\{Z_1, ..., Z_q\}$ is an ϵ-orthonormal set in T_xM, then

$$\sum_{j=1}^{q} Pf(x, Z_j) \geq \eta.$$

If f is a C^2 function we denote by Lf the Levi form of f.

Proposition 5. *On a Kähler manifold M the class $\Psi_q(M)$ enjoys the following properties:*

a) *A real-valued C^2 function, f, belongs to $\Psi_q(M)$ if and only if for each set of vector fields $\{Z_1, ..., Z_q\}$ which are orthonormal with respect to G we have: $\sum_{i=1}^q Lf(Z_i, Z_j) > 0$.*

b) *$\Psi_q(M)$ is a cone in the space of continuous functions, i.e., if f_1, f_2 are in $\Psi_q(M)$, then so is any positive combination thereof.*

c) *$\Psi_q(M)$ has the maximum-closure property, i.e., if f_1, f_2 are in $\Psi_q(M)$, then so is* $\max\{f_1, f_2\}$.

d) *$C^\infty \cap \Psi_q(M)$ is dense in $\Psi_q(M)$ in the C^o topology.*

Then $\mathcal{P} := \Psi_q$ satisfies properties 1), 2) and 4). $\mathcal{P}$ also satisfies 3). This follows from the density of $C^\infty \cap \Psi_q(M)$ in $\Psi_q(M)$ and from the next proposition:

Proposition 6. *Let M be an m-dimensional Kähler manifold, $N \subset M$ a closed n-dimensional complex submanifold, and $\phi \in \Psi_q(N) \cap C^\infty$. Then there exist a neighborhood V of N in M and $\tilde{\phi} \in \Psi_q(V) \cap C^\infty$ such that $\tilde{\phi}_{|Y} = \phi$.*

Proof. We consider ϕ' an arbitrary C^∞ extension of ϕ to a neighborhood of N and $\{\Omega_\lambda, z_\lambda\}$ a locally finite covering of N with coordinate patches $z_\lambda : \Omega_\lambda \to \mathbb{C}^m$ in which $N \cap \Omega_\lambda$ is given by $z'_\lambda = (z_{\lambda,n+1}, ..., z_{\lambda,m}) = 0$. Let $\{\theta_\lambda\}$ be C^∞ functions with compact support in Ω_λ such that $\sum \theta_\lambda = 1$ on N. Set

$$\tilde{\phi} = \phi'(x) + \sum \theta_\lambda \log(1 + \epsilon_\lambda^{-1}|z'_\lambda|^2).$$

Then $\tilde{\phi}_{|N} = \phi$ and for $x \in N \cap supp(\theta_\mu)$, $L\tilde{\phi} \geq L\phi' + \theta_\mu {\epsilon_\mu}^{-1} L|z'_\mu|^2$ and therefore if we choose $\{\epsilon_\lambda\}$ to be small enough it follows that $\tilde{\phi} \in \Psi_q(V)$ for some neighborhood V of N. □

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References

[1] J. P. Demailly, *Cohomology of q-convex spaces in top degrees*, Math. Z. **204** (2) (1990), 283–295. MR **91e:**32014

[2] K. Diederich; T. Ohasawa, *On pseudoconvex domains in* $\mathbb{P}^n$, Tokyo J. Math. **21** (1998), 353-358. MR **99k:**32024

[3] T. Ohsawa, *A Stein domain with smooth boundary which has a product structure*, Publ. Res. Inst. Math. Sci. **18** (1982), 1185–1186. MR **84i:**32022

[4] Y. T. Siu, *Every Stein subvariety admits a Stein neighborhood*, Invent. Math. **38** (1976/77), 89–100. MR **55:**8407

[5] V. Vâjâitu, *On locally hyperconvex morphisms*, C. R. Acad. Sci. Paris Ser. I Math. **322** (9) (1996), 823-828. MR **97b:**32014

[6] J. Varouchas, *Stabilité de la classe des variétés käleriénnes par certains morphismes propres*, Invent. Math. **77** (1) (1984), 117–127. MR **86a:**32026

[7] H. Wu, *On certain Kähler manifolds which are q-complete*, Complex analysis of several variables (Madison, Wis., 1982), Proc. Sympos. Pure Math., vol. 41., pp. 253–276. MR **85j:**32031

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