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CONVEXITY OF MOMENT POLYTOPES OF ALGEBRAIC VARIETIES

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ABSTRACT. We consider the situation of a compact irreducible subvariety of a smooth compact complex variety equipped with a Kähler form preserved by a torus action. We study the image of that subvariety under the moment map of the Kähler form.

1. MAIN RESULT

The moment map of a Hamiltonian T-symplectic/Kähler manifold has been one of the main interests of study for mathematical as well as physical reasons. In the early 80's, Atiyah in [A] considered the following situation:

Let M be a compact finite dimensional Kähler manifold on which a real torus acts in a Hamiltonian fashion. There is then a natural extension of the real torus action to the complexified torus action by applying the almost complex structure induced by the complex structure to the infinitesimal real torus action and then integrating to obtain the action of the "imaginary part" of the complexified torus.

He proved the following:

Theorem 1.1 ([A]). *Let f be a moment map of M with respect to the torus action. Let Y be an orbit of the complexified torus, and let $Z_1, \cdots, Z_p$ be the components of the set of all fixed points of the torus lying in the closure of Y. Then the image of the closure of Y under the moment map is equal to the convex hull of the images of $c_1, \cdots, c_p$ where $c_i = f(Z_i)$ for each i.*

The convex hull in the theorem is sometimes called the *moment polytope*.

One can ask if the above theorem holds for more general subspaces than a single orbit of M. The answer turns out to be positive in the following setting:

Suppose that M is a compact finite dimensional smooth complex variety equipped with a Kähler form ω. Let T be a real torus acting on (M, ω) that preserves ω and the complex structure such that the moment map exists. Denote its Lie algebra by $\mathbf{t}$. Let $T_{\mathbb{C}}$ be the complexified torus which also acts on M. Let X be a compact

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irreducible subvariety, possibly singular, of M invariant under the $T_{\mathbb{C}}$ action. We then have:

Theorem 1.2. *Let $\mu : M \longrightarrow \mathbf{t}^*$ be the moment map of ω. Then the image of X under μ coincides with the convex hull of the images of all $T_{\mathbb{C}}$-fixed points in X under μ.*

Remarks 1.3. (1) A simple but important observation is that a point in M fixed by T is also fixed by $T_{\mathbb{C}}$ and vice versa. This fact applies to all the spaces we shall consider in this paper.

(2) Using a bi-invariant inner product on $\mathbf{t}$, we shall freely identify $\mathbf{t}$ with its dual $\mathbf{t}^*$.

Proof of Theorem 1.2. We shall prove that

(a) the image of X is contained in the convex hull, and
(b) the image contains the convex hull.

Proof of (a). To prove part (a), we need the following two simple and well-known lemmas:

Lemma 1.4. *Let $D \subset \mathbb{R}^n$, $\theta_1, \cdots, \theta_p \in D$. If for any $v \in \mathbb{R}^n$, the image of D under the orthogonal projection (denoted by μ) on the line $\mathbb{R} \cdot v$ equals an interval of the form $[\mu(\theta_i), \mu(\theta_j)]$ for some i, j, then D is contained in the convex hull of $\theta_1, \cdots, \theta_p$.*

Proof. Suppose there exists an element x in D which is not in the convex hull of $\theta_1, \cdots, \theta_p$. Then x must be on the opposite side to the convex hull of a hyperplane defined by a face with vertices, say, $\theta_1, \cdots, \theta_k$. Choose a vector $v \in \mathbb{R}^n$ which is perpendicular to that face. Then one can easily see that the image of x under the projection on the straight line $\mathbb{R}v$ is outside $[\mu(\theta_i), \mu(\theta_j)]$ for any i, j, which leads to a contradiction. □

Remark 1.5. Notice that the above lemma still holds if "for any v" is replaced by "for almost all v".

Lemma 1.6. *Let G be a compact Lie group and (M, ω, G) a Hamiltonian G-space with moment map $\mu : M \to \mathbf{g}$ where $\mathbf{g}$ is the Lie algebra of G. Let H be a subgroup of G which is not necessarily closed and $\mathbf{h}$ its Lie algebra. Then (M, ω, H) is a Hamiltonian H-space with moment map given by the composite*

$$M \xrightarrow{\mu} \mathbf{g} \xrightarrow{pr} \mathbf{h}$$

where pr is the orthogonal projection with respect to the bi-invariant metric on G used to identify $\mathbf{g}$ and $\mathbf{g}^*$.

We can now carry out the proof of part (a). Regarding $\mathbf{t}$ as a Euclidean space, we show that for any $\alpha \in \mathbf{t}$ with irrational coordinates (so that the subgroup $exp\,(\mathbb{R} \cdot \alpha)$ is dense in T), the image of $\mu(X)$ under the orthogonal projection on the line $\mathbb{R} \cdot \alpha$ equals $[\mu(\theta_i), \mu(\theta_j)]$ for some fixed points θ_i, θ_j in X. Consider the composite map

$$\tilde{\mu} : X \xrightarrow{\mu} \mathbf{t} \longrightarrow \mathbb{R} \cdot \alpha$$

where the second map is the orthogonal projection. Since X is compact connected, its image under $\tilde{\mu}$ is a closed interval. The maximum and minimum of that interval will be images of some critical points of $\tilde{\mu}$. Notice that $\mathbb{R} \cdot \alpha$ is just the Lie algebra of the subgroup $exp(\mathbb{R} \cdot \alpha)$ of T. Therefore by Lemma 1.6, $\tilde{\mu}$ is the restriction of the moment map with respect to that subgroup. Hence the critical points of this map are the same as the fixed points of the subgroup, which are precisely the fixed points of the whole torus T because the subgroup is dense in T. The desired result now follows from Lemma 1.4. □

Proof of (b). Following an idea used in Atiyah-Pressley [A-P], one could first try to show that there exists an element y in X such that the closure of the complexified torus orbit $T_{\mathbb{C}} \cdot y$ contains all fixed points of $T_{\mathbb{C}}$ in X and then apply Theorem 1.1. However, there may be some fixed points $x \in X$ such that $\mu(x)$ is not a vertex of the convex hull. We shall prove a weaker statement that there exists y in X such that the image of the closure of $T_{\mathbb{C}} \cdot y$ under μ contains all vertices of the convex hull of the images of all fixed points of $T_{\mathbb{C}}$ in X under μ, and then apply Theorem 1.1. This weaker statement is sufficient for proving part (b).

Let $c_1, \dots, c_p$ be the vertices of the convex hull of the image of all T-fixed points in X under μ. For each c_i, take a hyperplane which touches the convex hull only at c_i. Let v_i^* be the unit vector normal to the hyperplane and points into the convex hull, and let $L_i = \mathbb{R}v_i^*$ be the vector subspace spanned by v_i^*. Consider the composite

$$f_i : M \xrightarrow{\mu} \mathbf{t}^* \xrightarrow{\pi_i} L_i \xrightarrow{s_i} \mathbb{R}$$

where π_i is the orthogonal projection onto L_i and s_i is a linear map sending v_i^* to $1 \in \mathbb{R}$, which is an isomorphism. For technical reasons, we choose v_i^* such that its dual vector $v_i \in \mathbf{t}$ generates a dense subgroup $G = \{exp(tv_i) \in T \mid t \in \mathbb{R}\}$ of T. Then f_i is just the moment map of the action of G on M. Notice that G is isomorphic to $\mathbb{R}$, and we have:

Lemma 1.7. *This action extends to a $\mathbb{C}^*$ action on M.*

There is a well-known result:

Lemma 1.8 ([A], [F])**.** *The function f_i is Morse-Bott.*

This lemma can be proved by a straightforward calculation in local normal coordinates.

Now, notice that $s_i \circ \pi_i(c_i)$ is the absolute minimal value of $f_i|_X$. Let

$$F_i^1, \dots, F_i^k \subset M$$

be all the critical submanifolds that intersect X. Consider the stable manifolds

$$M_i^s = \{m \in M \mid m \text{ flows down to } F_i^s \text{ along some gradient flow line of } f_i\}$$

for $s = 1, \cdots, k$. Here a gradient flow line is a smooth curve $x(t)$ on M satisfying the differential equation

$$\frac{d}{dt}x(t) = -\text{grad } f_i\Big|_{x(t)}.$$

The condition defining M_i^s means that m lies on some gradient flow line $x(t)$ and

$$\lim_{x \to \infty} x(t) \in F_i^s.$$

Recall that in Lemma 1.7, we defined a $\mathbb{C}^*$ action on M. We have

Lemma 1.9. *Gradient flow lines of f_i are generated by this $\mathbb{C}^*$ action.*

Proof. Fix any point $m \in M$. Let $x(t),\ t \in \mathbb{R}$, be the gradient flow line such that $x(0) = m$. Recall that in the definition of f_i, there is a vector $v_i^* \in \mathbf{t}^*$ and its dual $v_i \in \mathbf{t}$ which generates G. Then $\sqrt{-1}v_i$ is in the complexification $\mathbb{C}^* \subset \mathbf{t}_{\mathbb{C}}$. Consider the curve $a(t) = \exp(t\sqrt{-1}v_i)m$ on M. Notice that $exp(\sqrt{-1}v_i)$ is a curve on $\mathbb{C}^*$. Hence $a(t)$ is generated by the $\mathbb{C}^*$ action. We claim that it coincides with $x(t)$. For any t, write $x = a(t)$, and for any $X \in T_xM$, we have

$$\begin{aligned} \langle X, \frac{d}{dt}a(t)\rangle_M &= \langle X, J(v_i \cdot x)\rangle_M \\ &= \omega(X, J^2(v_i \cdot x)) \\ &= -\omega(X, v_i \cdot x) \\ &= -\langle d\mu(X), v_i\rangle \\ &= -\pi_i \circ d\mu(X) \\ &= \langle X, -grad\, f_i\rangle_M \end{aligned}$$

where $\langle , \rangle : \mathbf{t}^* \times \mathbf{t} \to \mathbb{R}$ is the canonical pairing. Hence we have

$$\frac{d}{dt}a(t) = -grad\, f_i,$$

that is, $a(t) = x(t)$. □

Therefore, we have an alternative description of M_i^s:

$$M_i^s = \left\{ m \in M \;\middle|\; \lim_{t\to\infty} exp(t\sqrt{-1}v_i) \cdot m \in F_i^s \right\}$$

for $s = 1, \cdots, k$. By a result of [C-S] (§Ic), $\overline{M_i^s}$ is a subvariety of M, and M_i^s is Zariski open in $\overline{M_i^s}$ for each s.

Every point of X must flow down to some critical submanifold. Hence X is contained in the union $\bigcup_{s=1}^k M_i^s$.

Lemma 1.10. *The set X is contained in $\overline{M_i^s}$ for some s.*

Proof. Observe that

$$X \subset \bigcup_{s=1}^k M_i^s \subset \bigcup_{s=1}^k \overline{M_i^s}.$$

Therefore $X = \bigcup_{s=1}^k \left(X \cap \overline{M_i^s} \right)$.

Since $\overline{M_i^s}$ is a subvariety of M, the intersection $X \cap \overline{M_i^s}$ is a subvariety of X for $s = 1, \cdots, k$. By the irreducibility of X, we must have

$$X = X \cap \overline{M_i^s}$$

for some s. This implies that $X \subset \overline{M_i^s}$. □

Lemma 1.11. $f_i(F_i^s) = c_i$.

Proof. Since c_i is the absolute minimal value of $f_i|_X$, $f_i(F_i^s) \geq c_i$. Notice that the restriction of f_i to a gradient flow line $a(t)$ must be decreasing with respect to t. Moreover, by Morse theory, the closure of a stable manifold of a critical submanifold F precisely consists of points which are initial points of (downward) broken gradient flow lines that go down to F. A (downward) broken gradient flow line is a piecewise smooth curve such that each smooth segment is a gradient flow line.

By these two facts, we can see that the value of each point $x \in \overline{M_i^s}$ under f_i must be greater than or equal to $f_i(F_i^s)$. And since $c_i \in f_i(X)$, we must have $c_i \geq f_i(F_i^s)$. □

Since M_i^s is Zariski open in $\overline{M_i^s}$ and $X \cap M_i^s \neq \emptyset$, $X \cap M_i^s$ must be Zariski open in X. Therefore the finite intersection

$$\bigcap_{i=1}^{k} X \cap M_i^{s_i}$$

is also Zariski open in X. In particular it is non-empty. Take any element y in that intersection and consider the complex torus orbit $T_{\mathbb{C}} \cdot y$.

Lemma 1.12. *The image of the closure of the orbit $T_{\mathbb{C}} \cdot y$ under μ contains all vertices $c_1, \ldots, c_p$ of the convex hull.*

Proof. For each i, since $y \in X \cap M_i \subset M_i$, there is a flow line $x(t)$ with respect to f_i that contains y and $x(t)$ flows down to F_i^s for some s. But we saw that gradient flow lines can be obtained by the $\mathbb{C}^*$ action. In particular, $x(t)$ is entirely contained in the orbit $T_{\mathbb{C}} \cdot y$. Hence the image of the closure of $T_{\mathbb{C}} \cdot y$ under μ contains c_i. This completes the proof of the lemma. □

By Theorem 1.1, the image of the closure of $T_{\mathbb{C}} \cdot y$ under μ is equal to the convex hull of the images of all T-fixed points in $T_{\mathbb{C}} \cdot y$. Hence by the above lemma, it contains the convex hull of all c_i, and we have

$$\mu(X) \supset \mu(\text{the closure of } T_{\mathbb{C}} \cdot y) \supset \textit{ convex hull } \langle c_1, \ldots, c_p \rangle,$$

which concludes the proof of part (b) of the theorem. □

Hence the proof of Theorem 1.2 is complete. □

Remark 1.13. The irreducibility condition of the subvariety X in Theorem 1.2 is necessary. Consider the following example:

Let $M = \mathbb{C}P^2$ be equipped with the standard Kähler structure. Let $T = \mathbf{S}^1 \times \mathbf{S}^1$ act on M by

$$(a, b) \cdot [z_0; z_1; z_2] = [z_0; az_1; bz_2].$$

This torus action defines a moment map $\mu : M \to \mathbb{R}^2$.

Let $X = \{[z_0; z_1; z_2] \in \mathbb{C}P^2 \mid z_1 z_2 = 0\}$. It is a compact, *reducible* algebraic subvariety on $\mathbb{C}P^2$. By simple calculation, one can see that $\mu(X)$ is a union of two line segments meeting at one point, whereas there are three fixed points in X: $[1; 0; 0], [0; 1; 0]$ and $[0; 0; 1]$ such that the convex hull of their images under μ is a triangle.

2. Applications

2.1. **Convexity theorem.** In [A] and [G-S], a theorem, which is now usually referred to as the *convexity theorem* in symplectic geometry, is proved:

Theorem 2.1 (Convexity Theorem [A], [G-S])**.** *Let (M, ω) be a finite dimensional symplectic manifold with a real torus T acting on M that preserves ω such that a moment map μ exists. Then the image of M under the moment map is equal to the convex hull of the images of all T-fixed points under μ.*

In [A], it is suggested that one could use the idea of Theorem 1.1 to prove a Kähler version of Theorem 2.1. We shall now realize this suggestion by using the idea in the proof of Theorem 1.2 to prove the following Kähler version of Theorem 2.1:

Theorem 2.2. *Let (M, ω, J) be a compact Kähler manifold where ω is the Kähler (symplectic) form invariant under the complex structure J. Let T be a real torus acting on M which preserves ω and J such that a moment map μ exists. Then the image of M under μ is equal to the convex hull of the images of all T-fixed points in M.*

Proof. We show that

(a) the image of M is contained in the convex hull, and
(b) the image contains the convex hull.

The proof of part (a) is again the same as in Theorem 1.2. For part (b), let $c_1, \cdots, c_p$ be the vertices of the convex hull. For each i, construct the Morse-Bott function

$$f_i : M \xrightarrow{\mu} \mathbf{t}^* \xrightarrow{\pi_i} L_i \xrightarrow{s_i} \mathbb{R}$$

as in the proof of Theorem 1.2. Let $m_1^i, \ldots, m_k^i \in M$ be the preimages of c_i under μ. They are the absolute minima of f_i. Let M_i be the union of the stable manifolds of the $m_j^i,\ j = 1, \ldots, k$. Notice that these stable manifolds are open and dense in M. Hence the finite intersection $\bigcap_{i=1}^p M_i$ is also open and dense in M. In particular, it is non-empty. Take an element y in $\bigcap_{i=1}^p M_i$. Then as in the proof of Lemma 1.12, the image of the closure of the orbit $T_{\mathbb{C}} \cdot y$ under μ contains all vertices $c_1, \ldots, c_p$ of the convex hull. Hence we have

$$\mu(X) \supset \mu(\text{the closure of } T_{\mathbb{C}} \cdot y) \supset \textit{ convex hull} \left\langle c_1, \ldots, c_p \right\rangle$$

where the second inclusion follows from Theorem 1.1. This concludes the proof of part (b). □

2.2. **Loop groups.** In this section, we shall generalize Theorem 1.2 to an infinite dimensional situation, replacing M by an "infinite dimensional subvariety" (which will be defined later) of the loop group ΩU_n which is the set of all smooth functions (called "loops") from the unit circle $\mathbf{S}^1 \subset \mathbb{C}$ into the unitary group U_n which send 1 to the identity matrix.

We first recall briefly various structures and properties of ΩU_n relevant to this paper, following [A-P]. For details, see [L] or [P-S].

With pointwise multiplication, ΩU_n is a group. In fact, it is a Lie group, modeled on the infinite dimensional Lie algebra Ωu_n which is the set of all smooth maps from $\mathbf{S}^1$ to the Lie algebra u_n of U_n which send 1 to $0 \in u_n$.

1. There is a left invariant complex structure and a left invariant symplectic structure on ΩU_n defined as follows:

For any $u(s) \in \Omega u_n$, consider the Laurent expansion:

$$u(s) = \sum_{k \neq 0} A_k \, e^{iks},$$

where each A_k is a complex $n \times n$ matrix. Define

$$J\, u(s) = \sum_{k<0} iA_k e^{iks} - \sum_{k>0} iA_k e^{iks}.$$

It is easy to see that this gives a linear map $J : \Omega u_n \to \Omega u_n$ such that $J^2 = -1$. By left translation using the group structure of ΩU_n, we have an almost complex structure J on ΩU_n.

For any two tangent vectors $X(\lambda), Y(\lambda)$ in the Lie algebra Ωu_n, set

$$\omega_{AP}(X,Y) = \frac{1}{2\pi}\int_{\mathbf{S}^1} \langle X(s), Y'(s)\rangle ds$$

where $\langle \cdot, \cdot \rangle$ is the bi-invariant inner product on u_n defined by

$$\langle A, B\rangle = Tr(A^* B).$$

Here $\lambda = e^{is}$ is the loop parameter. By left translation again, we get a left invariant 2-form ω_{AP} on ΩU_n.

It turns out that J is integrable, and that ω_{AP} is a symplectic form such that $(\Omega U_n, \omega_{AP}, J)$ becomes a Kähler manifold.

2. There is an action of a real torus $T \times \mathbf{S}^1$ on ΩU_n, where T is the maximal torus consisting of diagonal matrices in U_n, defined as follows: T acts on U_n by conjugation, which induces an action on ΩU_n; and $\mathbf{S}^1$ action on ΩU_n by "rotating the loops". More precisely, for any $z \in \mathbf{S}^1$ and $\gamma(\lambda) \in \Omega U_n$, the action is defined to be

$$z \cdot \gamma(\lambda) = \gamma(z \cdot \lambda)\gamma(z)^{-1}.$$

It is not hard to see that this torus action preserves J and ω_{AP}. It also extends to an action of the complexified torus $(T \times \mathbf{S}^1)_{\mathbb{C}} = T_{\mathbb{C}} \times \mathbb{C}^*$.

3. A moment map exists, which can be defined explicitly as follows:

For any $f(s)$ in ΩU_n, define the energy function

$$E : \Omega U_n \to \mathbb{R} : f \mapsto \frac{1}{4\pi}\int_{\mathbf{S}^1} \|f'(s) f(s)^{-1}\|^2 ds$$

and the momentum function

$$p : \Omega U_n \to \mathbf{t} : f \mapsto pr_{\mathbf{t}} \frac{1}{2\pi}\int_{\mathbf{S}^1} f'(s) f(s)^{-1} ds$$

where $pr_{\mathbf{t}}$ is the orthogonal projection to $\mathbf{t}$. Then the map

$$\mu_{AP} = p \oplus E : \Omega U_n \longrightarrow \mathbf{t} \oplus \mathbb{R}$$

is a moment map for the action of $T \times \mathbf{S}^1$ on ΩU_n.

4. Consider the set of "algebraic loops" in U_n:

$$\Omega_{alg} U_n = \Big\{ \gamma \in \Omega U_n \;\Big|\; \gamma(\lambda) = \sum_{i=-N}^{N} A_i \lambda^i \text{ for some } N \in \mathbf{Z} \text{ and } n \times n \text{ matrices } A_i \Big\}.$$

It is a subgroup of ΩU_n. Moreover, it has a structure of an "infinite dimensional" algebraic variety defined as follows:

First, consider its identity component $(\Omega_{alg} U_n)_0 = \Omega_{alg} SU_n$. Let

$$M_k = \{ \gamma \in \Omega_{alg} U_n \mid \gamma \text{ is a polynomial in } \lambda^{-1}, deg\, det\, \gamma = -k \}.$$

Observe that there is a filtration

$$M_0 \subset \lambda M_n \subset \lambda^2 M_{2n} \subset \cdots \subset \bigcup_{k \geq 0} \lambda^k M_{kn} = \Omega_{alg} SU_n$$

where

$$\lambda M_n = \{ \hat{\gamma} \in \Omega_{alg} U_n \mid \hat{\gamma}(\lambda) = \lambda \cdot \gamma(\lambda) \text{ for some } \gamma \in M_n \},$$

etc. This filtration is called the Mitchell-Segal filtration [M, S].

Each space M_{kn}, as well as $\lambda^k M_{kn}$, has a structure of finite dimensional algebraic variety that we now define. Consider the Hilbert subspace of the space of all L^2-functions from $\mathbf{S}^1$ to $\mathbb{C}^n$:

$$H_+ = \overline{Span\{\lambda^i e_j : i \geq 0, j = 1, \ldots, n\}} \subset L^2(\mathbf{S}^1; \mathbb{C}^n)$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$ and the closure is taken with respect to the L^2-norm. Notice that for any $\gamma \in M_{kn}$, since it is a polynomial in λ^{-1}, γ^* is a polynomial in λ. So $\gamma^* H_+ \subset H_+$. This implies that $H_+ \subset \gamma H_+$. On the other hand, since $deg\, det\, \gamma = -kn$, we have $\lambda^{kn}\gamma \in H_+$, which implies $\gamma H_+ \subset \lambda^{-kn} H_+$. In sum, we have

$$H_+ \subset \gamma H_+ \subset \lambda^{-kn} H_+.$$

It can be shown that, since $deg\, det\, \gamma = -kn$, γH_+ has codimension kn in $\lambda^{-k} H_+$. (See, for example, [P-S].) Equivalently, the quotient space $(\gamma H_+)/H_+$ is a kn-dimensional vector subspace of $\lambda^{-kn} H_+/H_+$. Hence there is a map

$$\varphi : M_{kn} \to Gr_{kn}(\mathbb{C}^{kn^2}) : \gamma \mapsto (\gamma H_+)/H_+ \subset \lambda^{-kn} H_+/H_+.$$

Here we identify $\lambda^{-kn} H_+/H_+ \cong \langle \lambda^i e_j : j = 1, \cdots, n;\, i = -1, \cdots, -kn \rangle$ with $\mathbb{C}^{kn^2}$.

It is easy to see that φ is an injection under which the image of M_{kn} is a compact irreducible algebraic subvariety of the Grassmannian.

As for $\lambda^k M_{kn}$, notice that for any $\gamma \in M_{kn}$, we have

$$\lambda^k H_+ \subset \lambda^k \gamma H_+ \subset \lambda^{k-kn} H_+.$$

Similar to above, the quotient $(\lambda^k \gamma H_+)/(\lambda^k H_+)$ is a kn-dimensional vector subspace of

$$\lambda^{k-kn} H_+/(\lambda^k H_+) \cong \langle \lambda^i e_j : j = 1, \cdots, n;\, i = k - kn, \cdots, k-1 \rangle \cong \mathbb{C}^{kn^2},$$

and that this defines an embedding of $\lambda^k M_{kn}$ into a Grassmannian of kn-planes in $\mathbb{C}^{kn^2}$ as a compact irreducible algebraic subvariety. We shall denote this Grassmannian by $Gr[k]$ and denote this embedding of $\lambda^k M_{kn}$ by φ_k.

Therefore, the Mitchell-Segal filtration is in fact a filtration by algebraic varieties with $\Omega_{alg} SU_n$ being the inverse limit. By a similar argument, one can show that other components of $\Omega_{alg} U_n$ also are inverse limits of finite dimensional algebraic varieties.

Using Theorem 1.2, we shall prove the following

Theorem 2.3. *Let X be a compact irreducible subvariety of $\Omega_{alg} U_n$ which is invariant under the action of $T_{\mathbb{C}} \times \mathbb{C}^*$. The image of X under the moment map μ_{AP} is the convex hull of the images of the fixed points of the torus action which lie inside X.*

Proof. Since X is irreducible, it must be connected. Without loss of generality, we can assume that it is contained in the identity component $\Omega_{alg} SU_n$. By compactness of X, we see that it is contained in $\lambda^k M_{kn}$ for some k, which is regarded as a subvariety of $Gr[k]$.

Notice that for each k, $\lambda^k M_{kn}$ is invariant under the action of $T_{\mathbb{C}} \times \mathbb{C}^*$.

We now define an action of $T_{\mathbb{C}} \times \mathbb{C}^*$ (and hence of $T \times \mathbf{S}^1$) on the Grassmannian $Gr[k]$. As described above, $\mathbb{C}^{kn^2}$ has a basis

$$\{\lambda^i e_j : j = 1, \cdots, n;\, i = k - kn, \cdots, k-1\}.$$

Clearly $T_{\mathbb{C}}$ acts on e_i by matrix multiplication. This action induces an action on $Gr[k]$. On the other hand, let $\mathbb{C}^*$ act on λ^i by

$$z \cdot \lambda^i = (z\lambda)^i$$

which induces an action on $Gr[k]$.

As can be easily checked, this torus action, when restricted to X, coincides with the torus action that comes from the loop group.

On the other hand, the Grassmannian $Gr[k]$ is a Kähler manifold with the standard symplectic structure, which shall be denoted by ω_k. It is easily seen that the torus $T \times \mathbf{S}^1$ preserves all these structures and a moment map $\mu_k : Gr[k] \to \mathbf{t} \oplus \mathbb{R}$ exists.

Lemma 2.4. *There is a commutative diagram:*

$$\begin{array}{ccccccccc} X & \subset & \lambda^k M_{kn} & \subset & \Omega U_n & \xrightarrow{\mu_{AP}} & \mathbf{t} \oplus \mathbb{R} \\ & & & \searrow_{\varphi_k} & & \nearrow_{\mu_k} & \\ & & & & Gr[k] & & \end{array}$$

Proof. First notice that $Gr[k]$ can be identified with the following set of vector subspaces:

$$\{W \mid \lambda^k H_+ \subset W \subset \lambda^{k-kn} H_+; \;\; W \text{ has codimension } kn \text{ in } \lambda^{k-kn} H_+\}.$$

It is easy to check that this set is a subset of an "infinite dimensional Grassmannian" $\mathbf{Gr}$ defined as the set of all closed vector subspaces W of $L^2(\mathbf{S}^1; \mathbb{C}^n)$ such that the orthogonal projections $pr_\pm : W \to H_\pm$ are Fredholm and Hilbert-Schmidt operators respectively. Here H_- is the orthogonal complement of H_+ in $L^2(\mathbf{S}^1; \mathbb{C}^n)$. It is proved in [P-S] that $\mathbf{Gr}$ is a smooth infinite dimensional manifold such that the tangent space at each $W \in \mathbf{Gr}$ is $\mathcal{I}(W, W^\perp)$, the Hilbert space of all Hilbert-Schmidt operators $W \to W^\perp$.

Second, the complexified torus $T_{\mathbb{C}} \times \mathbb{C}^*$ acts on $\mathbf{Gr}$ in a similar way as its action on $Gr[k]$. Moreover, $\mathbf{Gr}$ has a symplectic structure Θ defined in the same fashion as the standard invariant symplectic structure on a finite dimensional Grassmannian. More precisely, for any f and g in $T_W\mathbf{Gr} = \mathcal{I}(W, W^\perp)$, define

$$\Theta(f, g) = -iTr(f^* g - g^* f).$$

Clearly, the restriction of Θ to $Gr[k]$ (regarded as a subset of $\mathbf{Gr}$) is the same as ω_k. It is easy to check that Θ is preserved by the real torus $T \times \mathbf{S}^1$ and there is a moment map $\mu : \mathbf{Gr} \to \mathbf{t} \oplus \mathbb{R}$.

Third, just like the embedding φ of M_{kn} into a finite dimensional Grassmannian, ΩU_n can be embedded into $\mathbf{Gr}$ by the map

$$\delta : \Omega U_n \to \mathbf{Gr} : \gamma \mapsto \gamma H_+.$$

Hence we have a diagram:

$$\begin{array}{cccccc} \lambda^k M_{kn} & \subset & \Omega U_n & \xrightarrow{\mu_{AP}} & \mathbf{t} \oplus \mathbb{R} \\ & \searrow_{\varphi_k} & & \searrow_{\delta} & \uparrow_{\mu} \\ & & Gr[k] & \subset & \mathbf{Gr} \end{array}$$

It is clear that μ, when restricted to $Gr[k]$, is the same as μ_k.

We now claim that $\delta^*(\Theta) = \omega_{AP}$. First, it is easy to check that Θ is "invariant under multiplication of loops", in the sense that, for any γ in ΩU_n, $L_\gamma^* \Theta = \Theta$ where

$L_\gamma^* : \mathbf{Gr} \to \mathbf{Gr}$ is left multiplication by γ. Together with the fact that ω_{AP} is left invariant, we see that we just need to check the claim for $T_I \Omega U_n = \Omega u_n$. Now, it is a straightforward matter to check that the differential $d\delta : T_I \Omega U_n = \Omega u_n \to T_{H_+}\mathbf{Gr} = \mathcal{I}(H_+, H_-)$ at the identity loop $I \in \Omega U_n$ satisfies

$$d\delta(X) = pr_- \circ L_X \in \mathcal{I}(H_+, H_-)$$

for any X in Ωu_n. Here $pr_- : L^2(\mathbf{S}^1; \mathbb{C}^n) \to H_-$ is the orthogonal projection and L_X is left multiplication by X.

For any X and Y in Ωu_n, write them in Laurent expansions $X = \sum A_k e^{iks}$ and $Y = \sum B_m e^{ims}$. By straightforward calculations, one gets

$$\omega_{AP}(X, Y) = i \sum_k Tr(A_k^* B_k).$$

Passing to $T_{H_+}\mathbf{Gr} = \mathcal{I}(H_+, H_-)$, let us denote the vectors $d\delta(X) = pr_- \circ L_X$ and $d\delta(Y) = pr_- \circ L_Y$ by f_X and f_Y respectively. We have

$$\begin{aligned} \Theta(f_X, f_Y) &= -iTr(f_X^* f_Y - f_Y^* f_X) \\ &= -i \sum_{j=1}^n \sum_{k \geq 0} \langle e_j \lambda^k, (f_X^* f_Y - f_Y^* f_X)(e_j \lambda^k) \rangle_{L^2} \\ &= -i \sum_{k \geq 0} \sum_{j=1}^n \{ \langle f_X(e_j \lambda^k), f_Y(e_j \lambda^k) \rangle_{L^2} \\ &\qquad - \langle f_Y(e_j \lambda^k), f_X(e_j \lambda^k) \rangle_{L^2} \} \end{aligned}$$

where $\lambda = e^{is}$ and $\langle \cdot, \cdot \rangle_{L^2}$ is the L^2-norm. Again, by straightforward calculations, one gets

$$\Theta(f_X, f_y) = i \sum_k Tr(A_k^* B_k)$$

which is the same as $\omega_{AP}(X, Y)$.

This claim implies that $\mu \circ \delta = \mu_{AP}$. So the above diagram is in fact commutative, from which the lemma follows. □

Hence we see that the image $\mu_{AP}(X) = \mu_k \circ \varphi_k(X)$, by Theorem 1.2, coincides with the convex hull of the images of the fixed points of the torus action which lie inside X under $\mu_{AP} = \mu_k \circ \varphi_k$. □

The following is proved in [A-P]:

Theorem 2.5 ([A-P])**.** *The image of μ_{AP} on each component of ΩSU_n is the convex hull of the images of the fixed points of the $T \times \mathbf{S}^1$ action lying in that component.*

The proof in [A-P] uses the group structure of ΩSU_n.

We now show how Theorem 2.5 is in fact an easy consequence of Theorem 2.3 for which the group structure of ΩSU_n is not needed.

Proof of Theorem 2.5. Recall that we have

$$\bigcup_{k \geq 0} \lambda^k M_{kn} = \Omega_{alg} SU_n \subset \Omega U_n \xrightarrow{\mu_{AP}} \mathbf{t} \oplus \mathbb{R}.$$

By Theorem 2.3, we have, for each k,

$$\begin{aligned}\mu_{AP}(\lambda^k M_{kn}) = &\ \textit{convex hull}\,\big\langle \mu_{AP}(m) : m \text{ is a fixed point of } T_{\mathbb{C}} \times \mathbb{C}^* \text{ in} \\ &\lambda^k M_{kn}\big\rangle \\ \subset &\ \textit{convex hull}\,\big\langle \mu_{AP}(m) : m \text{ is a fixed point of } T_{\mathbb{C}} \times \mathbb{C}^* \text{ in} \\ &\Omega_{alg} SU_n\big\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\mu_{AP}(\Omega_{alg} SU_n) = &\lim_{i\to\infty} \mu_{AP}(\lambda^k M_{kn}) \\ \subset &\ \textit{convex hull}\,\big\langle \mu_{AP}(m) : m \text{ is a fixed point of } T_{\mathbb{C}} \times \mathbb{C}^* \text{ in} \\ &\Omega_{alg} SU_n\big\rangle.\end{aligned}$$

On the other hand, for any c in the convex hull, there exist some $m_1, \cdots, m_q$ in $\Omega_{alg} SU_n$ such that c is in *convex hull* $\big\langle \mu_{AP}(m_1), \cdots, \mu_{AP}(m_q)\big\rangle$. Clearly the points $m_1, \cdots, m_q$ must be contained in a single $\lambda^k M_{kn}$ for large enough k. Hence we have

$$\begin{aligned}c \in &\ \textit{convex hull}\,\big\langle \mu_{AP}(m_1), \cdots, \mu_{AP}(m_q)\big\rangle \\ &\subset \textit{convex hull}\,\big\langle \mu_{AP}(m) : m \text{ is a fixed point of } T_{\mathbb{C}} \times \mathbb{C}^* \text{ in } \lambda^k M_{kn}\big\rangle \\ &= \mu_{AP}(M_{kn}) \\ &\subset \mu_{AP}(\Omega_{alg} SU_n).\end{aligned}$$

Thus we have proved that

$$\begin{aligned}\mu_{AP}(\Omega_{alg} SU_n) = &\ \textit{convex hull}\,\big\langle \mu_{AP}(m) : m \text{ is a fixed point of } T_{\mathbb{C}} \times \mathbb{C}^* \text{ in} \\ &\Omega_{alg} SU_n\big\rangle.\end{aligned}$$

It can be easily seen from the above argument that $\mu_{AP}(\Omega_{alg} SU_n)$ is a closed set.

We now claim that all $T_{\mathbb{C}} \times \mathbb{C}^*$-fixed points in ΩSU_n must be in $\Omega_{alg} SU_n$. For any $\gamma(\lambda) \in \Omega SU_n$ which is fixed by $T_{\mathbb{C}} \times \mathbb{C}^*$, we have

$$A \cdot \gamma(\lambda) \cdot A^{-1} = \gamma(\lambda)$$

for all $A \in T$. This implies that $\gamma(\lambda)$ must be a diagonal matrix for all $\lambda \in \mathbf{S}^1$. For such a loop to be fixed by the $\mathbf{S}^1$ action, it must have the form

$$\begin{pmatrix} \lambda^{k_1} & & & \\ & \lambda^{k_2} & & \\ & & \ddots & \\ & & & \lambda^{k_n} \end{pmatrix}$$

where each k_i is an integer. These fixed points are easily seen to be in $\Omega_{alg} SU_n$.

The above claim shows that the convex hull

$$\textit{convex hull}\,\big\langle \mu_{AP}(m) : m \text{ is a fixed point of } T_{\mathbb{C}} \times \mathbb{C}^* \text{ in } \Omega_{alg} SU_n\big\rangle$$

is the same as

$$\textit{convex hull}\,\big\langle \mu_{AP}(m) : m \text{ is a fixed point of } T_{\mathbb{C}} \times \mathbb{C}^* \text{ in } \Omega SU_n\big\rangle.$$

It is proved in [A-P] (§5) that $\Omega_{alg}SU_n$ is dense in ΩSU_n. Hence we have

$$\mu_{AP}(\Omega SU_n) = \mu_{AP}(\overline{\Omega_{alg}SU_n}).$$

The theorem now follows from the fact that $\mu_{AP}(\Omega_{alg}SU_n)$ is closed. $\square$

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