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L^p ESTIMATES FOR A CLASS OF OSCILLATORY INTEGRALS

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ABSTRACT. We obtain a simpler proof of Theorem 3.1 of *The complete* (L^p, L^p) *mapping properties of some oscillatory integrals in several dimensions,* by G. Sampson and P. Szeptycki (Canad. Math. J. **53** (5) (2001), 1031–1056).

1. INTRODUCTION

The purpose of this article is to obtain simpler arguments to the $L^p(\mathbb{R}^2_+)$-estimates of some of the operators that appear in [SS]. To be more precise, we found a simpler proof of Theorem 3.1 of [SS] (also see Theorem 1.2 of [SS]). For these purposes it suffices to consider, with $a_1, a_2, b_1, b_2 \geq 1$, the operator given by

$$(1.1)\quad Tf(x) = \sum_{m,n\geq \bar{1}} K_{mn}f(x), \text{ where } K_{mn}f(x) = \int_{\mathbb{R}^2_+} k_{mn}(x,y)f(y)dy, x \in \mathbb{R}^2_+,$$

with the kernel $k_{mn}(x,y) = \psi_m(x)\psi_n(y)e^{ix^a\cdot y^b}\varphi(x,y)\beta(x-y)$, $x^a \cdot y^b = {x_1}^{a_1}{y_1}^{b_1} + {x_2}^{a_2}{y_2}^{b_2}$, $\psi_n(x) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$ $(n = (n_1, n_2))$, $\psi_{n_l}(x_l) = \psi(2^{-n_l}x_l), l = 1, 2,$ where $0 \leq \psi \in C^\infty(\mathbb{R})$, supp $\psi(t) \subset [\frac{1}{4}, 1]$ is so chosen that $\sum_{r=1}^\infty \psi(2^{-r}t) = 1$ for $t > 1$. Thus here $n_1, n_2 \geq 1$. Also we suppose that $0 \leq \beta(x) \leq 1, \beta(x) = 1$ if $|x| \geq 2$, and $\beta(x) = 0$ if $|x| \leq 1$ and $\beta(x) \in C^\infty(\mathbb{R}^2)$.

We suppose that $\varphi(x,y)$ in (1.1) satisfies

$$(1.2)\qquad |\partial_x^\alpha \partial_y^\beta \varphi(x,y)| \leq C_{\alpha\beta}|x-y|^{-|\alpha|-|\beta|}, \text{ for all } \alpha, \beta.$$

Note the φ's that satisfy (1.2) are bounded and so it follows that our L^p result will also hold if in (1.1) we replace $\beta(x-y)$ by 1.

We improve Theorem 3.1 of [SS], which we showed in the case $\varphi(x,y) = |x-y|^{i\tau}, \tau$ real, but here, in Theorem 1.1, we obtain this result for any $\varphi(x,y)$ that satisfies (1.2).

Theorem 1.1. *Assume $\varphi(x,y)$ satisfies (1.2). If $1 \leq b_l \leq a_l, l = 1,\ 2,\ \frac{b_1}{a_1} = \frac{b_2}{a_2}$, then*

$$(1.3)\qquad \|Tf\|_p \leq C\|f\|_p, \quad \textit{for } p = \frac{a_1 + b_1}{a_1}.$$

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For real numbers s, we define $\bar{s} = (s, s)$, and also for any $a, b \in \mathbb{R}^2$ we say that $a \geq b$ if $a_1 \geq b_1$ and $a_2 \geq b_2$. In this paper, we always assume that $a, b \geq \bar{1}$. For $y = (y_1, y_2) \in \mathbb{R}^2_+ = [0, \infty) \times [0, \infty)$, we set $y^b = {y_1}^{b_1} {y_2}^{b_2}$, while for a given measurable function f, we set $f(y^{\frac{1}{b}}) = f({y_1}^{\frac{1}{b_1}}, {y_2}^{\frac{1}{b_2}})$. Also for $1 \leq p < \infty$, we take $\frac{1}{p} + \frac{1}{p'} = 1$. Finally, we denote by C, indexed if needed, positive constants depending only on the kernel k, and C in different places may stand for different constants.

2. The main result

We begin with the operators given by

$$S_{mn} f(x) = \int_{\mathbb{R}^2_+} k_{mn}(x^{\frac{1}{a}}, y^{\frac{1}{b}}) f(y) dy \tag{2.1}$$

where $k_{mn}(x, y)$ is defined in (1.1).

We begin by obtaining L^2-estimates of the operators in (2.1); we apply the Cotlar and Stein theorem that appears on p. 278 of [St].

Lemma 2.1. *Let $a, b \geq \bar{1}, a_l \cdot b_l > 1, l = 1, 2$ and $\varphi(x, y)$ satisfy (1.2). Then for some $\delta > 0$,*

$$\|S_{mn} f\|_2 \leq C d_{mn} \|f\|_2, \tag{2.2}$$

with $d_{mn} = \frac{1}{2^{\delta(n_1+m_1+n_2+m_2)}}$, $n_1, m_1, n_2, m_2 \geq 1$.

Proof. We argue as on pp. 282-285 of [St] (applied to the operator S_{mn}) and we consider the kernel

$$s_{\vec{i}\vec{j}}(\xi, \eta; m, n) = \int a_{\vec{i}}(\eta, x) a_{\vec{j}}(\xi, x) e^{2\pi i x \cdot (\eta - \xi)} dx; \tag{2.3}$$

suppose $n_1 = max(n_1, n_2, m_1, m_2) = m^*$ with $\epsilon = (\epsilon_1, 0), \epsilon_1 > 0$,

$$a_{\vec{j}}(\xi, x) = \psi_n(x^{\frac{1}{b}}) \psi_m(\xi^{\frac{1}{a}}) \varphi(\xi^{\frac{1}{a}}, x^{\frac{1}{b}}) \beta(x^{\frac{1}{b}} - \xi^{\frac{1}{a}}) \phi(x^{\bar{1}-\epsilon} - j') \phi(\xi^{\bar{1}-\epsilon} - j),$$

$\vec{j} = (j, j') \in \mathbb{Z}^4$. Because of the support restriction imposed by $\psi_n(x^{\frac{1}{b}}) \psi_m(y^{\frac{1}{a}})$ to our operator S_{mn}, we get that $2^{(m-\bar{2})a(\bar{1}-\epsilon)} \leq j \leq 2^{(m+\bar{4})a(\bar{1}-\epsilon)}$, $2^{(n-\bar{2})b(\bar{1}-\epsilon)} \leq j' \leq 2^{(n+\bar{4})b(\bar{1}-\epsilon)}$.

Case 1. $|\xi_1 - \eta_1| \geq 2^{-\frac{m^* \epsilon_1}{2}}$.

$$(i\partial_{x_1})^{2N} e^{i2\pi x \cdot (\eta - \xi)} = ((2\pi)^2 |\eta_1 - \xi_1|^2)^N e^{i2\pi x \cdot (\eta - \xi)}; \tag{2.4}$$

therefore from (2.3) and (2.4)

$$s_{\vec{i}\vec{j}} = \int \frac{(i\partial_{x_1})^{2N} [a_{\vec{i}}(\eta, x) a_{\vec{j}}(\xi, x)]}{((2\pi)|\eta_1 - \xi_1|)^{2N}} e^{i2\pi x \cdot (\eta - \xi)} dx. \tag{2.5}$$

First suppose that $b_1 > 1$ and take $\epsilon_1 = 1 - \frac{1}{b_1}$. It follows that

$$|(\partial_{x_1})^{2N} [a_{\vec{i}}(\eta, x) a_{\vec{j}}(\xi, x)]| \leq C_N x_1^{-2N\epsilon_1} \psi_n(x^{\frac{1}{b}}) \leq C_N 2^{m^*(-2Nb_1\epsilon_1)}. \tag{2.6}$$

Next suppose that $b_1 = 1$ and take $\epsilon_1 = 1 - \frac{1}{a_1}$. Also assume that $|n_1 - m_1| \geq 3$; therefore $\beta(x^{\frac{1}{b}} - y^{\frac{1}{a}}) = 1$. For the differentiated term in the integrand of (2.5) we notice that

(a) $|(\partial_{x_1})^{2N} \psi_n(x^{\frac{1}{b}})| \leq \frac{C_N}{2^{2Nm^*}}$,

(b) $|(\partial_{x_1})^{2N} \varphi(\xi^{\frac{1}{a}}, x^{\frac{1}{b}})| \leq C_N [\xi_1^{\frac{1}{a_1}} - x_1]^{-2N}$, and

(c) $|(\partial_{x_1})^{2N} \phi(x^{\bar{1}-\epsilon} - j')| \leq C_N |x_1|^{-2N\epsilon_1}$.

Since $n_1 \geq m_1 + 3$, we get from (a), (b), (c) that for $b_1 = 1$

$$|(\partial_{x_1})^{2N}[a_{\vec{i}}(\eta, x)a_{\vec{j}}(\xi, x)]| \leq \frac{C_N}{2^{m^*(2N\epsilon_1)}}. \tag{2.7}$$

Thus, if $|\eta_1 - \xi_1| \geq 2^{-\epsilon_1 \frac{m^*}{2}}$, it follows from (2.5), (2.6), and (2.7) that

$$|s_{\vec{i}\vec{j}}| \leq C_N \phi(\xi^{\bar{1}-\epsilon} - j)\phi(\eta^{\bar{1}-\epsilon} - i)\frac{2^{N\epsilon_1 m^*}}{2^{m^*(2N\epsilon_1)}}. \tag{2.8}$$

If instead $n_1 \leq m_1 + 3$ and $b_1 = 1$, then $m^* - 3 \leq m_1 \leq m^*$ and we note that $\|S_{mn}\|_{2,2} = \|S^*_{mn}\|_{2,2}$. We apply our method to the operator S^*_{mn}; here $a_1 > 1$ plays the role of b_1 and so we argue as we did in (2.6).

Case 2. $|\xi_1 - \eta_1| \leq 2^{-\epsilon_1 \frac{m^*}{2}}$.

Using (2.3) with the standard arguments we get

$$|s_{\vec{i}\vec{j}}| \leq \frac{\chi(|\xi_1 - \eta_1| \leq 2^{-\epsilon_1 \frac{m^*}{2}})\psi_m(\xi^{\frac{1}{a}})\psi_m(\eta^{\frac{1}{a}})\phi(\xi^{\bar{1}-\epsilon} - j)\phi(\eta^{\bar{1}-\epsilon} - i)}{(1 + |\eta - \xi|^2)^{N_1}}$$

in case that $i' - j' \in Q_1$, and we get that

$$\int |s_{\vec{i}\vec{j}}| \leq C_N \frac{1}{2^{\epsilon_1 \frac{m^*}{2}}(1 + |\vec{i} - \vec{j}|^2)^{N_1}}. \tag{2.9}$$

Therefore from (2.8) and (2.9) and the restriction on j, j' we get that

$$\sum_{\vec{j}}(\gamma_1 + \gamma_2)(\vec{j}) \leq C_N \frac{2^{\frac{1}{2}(n_1 b_1 + m_1 a_1)\epsilon_1} 2^{\frac{1}{2}m^*(a_1(1-\epsilon_1) + a_2 + b_1(1-\epsilon_1) + b_2)}}{2^{m^* \frac{N\epsilon_1}{2}}} + C_N \frac{1}{2^{\frac{\epsilon_1}{4}m^*}},$$

and select N so that $N\epsilon_1 = 2(b_1 + b_2 + a_1 + a_2) + 8$, and therefore

$$\|S_{mn}\|_{2,2} \leq \frac{C}{2^{4\delta m*}} \leq \frac{C}{2^{\delta(m_1 + n_1 + m_2 + n_2)}}, \tag{2.10}$$

which completes our proof. □

Next we set

$$U_{mn}g(x) = \int_{\mathbb{R}^2_+} k_{mn}(x^{\frac{b}{a}}, y)g(y)dy \tag{2.11}$$

and we study the operator

$$Ug(x) = \sum_{m,n \geq \bar{1}} U_{mn}g(x). \tag{2.12}$$

Lemma 2.2. *With the hypothesis as in Lemma 2.1 we get*

$$\int_{\mathbb{R}^2_+} |U_{mn}f(x)|^2 dx \leq C d^2_{mn} \int_{\mathbb{R}^2_+} |f(y)|^2 dy, \tag{2.13}$$

where d_{mn} is defined below (2.2).

Proof. Now since $b \geq \bar{1}$ and $S_{mn}g(x)$ is supported in $x \geq \frac{\bar{1}}{2}$, we get that

$$\int_{\mathbb{R}^2_+} x^{\frac{1}{b} - \bar{1}}|S_{mn}g(x)|^2 dx \leq \int_{\mathbb{R}^2_+} |S_{mn}g(x)|^2 dx$$
$$\leq C d^2_{mn} \int_{\mathbb{R}^2_+} |\psi_n(y^{\frac{1}{b}})g(y)|^2 dy,$$

and this last estimate follows from Lemma 2.1. We write out the left side of this estimate to get

$$\int_{\mathbb{R}_+^2} x^{\frac{1}{b}-\bar{1}}|\psi_m(x^{\frac{1}{a}})\int_{\mathbb{R}_+^2}\psi_n(y^{\frac{1}{b}})e^{ix\cdot y}\varphi(x^{\frac{1}{a}},y^{\frac{1}{b}})\beta(x^{\frac{1}{a}}-y^{\frac{1}{b}})g(y)dy|^2dx$$
$$\le Cd_{mn}^2\int_{\mathbb{R}_+^2}|\psi_n(y^{\frac{1}{b}})g(y)|^2dy,$$

and set $x=v^b, y=u^b$. We then get that

$$\int_{\mathbb{R}_+^2}|\psi_m(x^{\frac{b}{a}})\int_{\mathbb{R}_+^2}\psi_n(y)e^{ix^b\cdot y^b}\varphi(x^{\frac{b}{a}},y)\beta(x^{\frac{b}{a}}-y)g(y^b)y^{b-\bar{1}}dy|^2dx$$
$$\le Cd_{mn}^2\int_{\mathbb{R}_+^2}|\psi_n(y^{\frac{1}{b}})g(y)|^2dy.$$

Next set $f(y)=g(y^b)y^{b-\bar{1}}$; therefore it follows that

$$\int_{\mathbb{R}_+^2}|\int_{\mathbb{R}_+^2}k_{mn}(x^{\frac{b}{a}},y)f(y)dy|^2dx$$
$$\le Cd_{mn}^2\int_{\mathbb{R}_+^2}|\psi_n(y^{\frac{1}{b}})f(y^{\frac{1}{b}})y^{\frac{1}{b}-\bar{1}}|^2dy.$$

Setting $v=y^{\frac{1}{b}}, dy=v^{b-\bar{1}}dv$, we get that

$$\le Cd_{mn}^2\int_{\mathbb{R}_+^2}|\psi_n(v)f(v)v^{\bar{1}-b}|^2v^{b-\bar{1}}dv$$
$$\le Cd_{mn}^2\int_{\mathbb{R}_+^2}|\psi_n(y)f(y)|^2y^{\bar{1}-b}dy,$$

and since here $b\ge\bar{1}$ and $y\ge\frac{\bar{1}}{2}$, we obtain our result (2.13). □

Next we form the operators

$$J_{mn}f(x)=x_1x_2U_{mn}f(x) \text{ with the measure } d\nu(x)=\frac{dx_1dx_2}{{x_1}^2{x_2}^2}. \tag{2.14}$$

Lemma 2.3. *With the hypothesis as in Lemma 2.1, for the operators defined in (2.14) we get that*

$$\|J_{mn}f\|_{p,\,\nu}^p=\int_{\mathbb{R}_+^2}x^{\bar{p}-\bar{2}}|U_{mn}f(x)|^pdx\le Cd_{mn}^{2(p-1)}\|f\|_p^p, \tag{2.15}$$

for $1<p\le 2$.

Proof. We first show that

$$\nu(\{x:|J_{mn}f(x)|>\lambda\})\le\frac{C}{\lambda}\|f\|_1. \tag{2.16}$$

Without any loss of generality, we can suppose that $f(y)\ge 0$ and $\|f\|_1>0$. Also set $E_\lambda=\{x:|J_{mn}f(x)|>\lambda\}$ and so obtaining (2.16) we need to estimate $\nu(E_\lambda)$.

Set for $l=1$ and 2, $I_{m_l}=[(\frac{2^{m_l}}{4})^{\frac{a_l}{b_l}},(2\cdot 2^{m_l})^{\frac{a_l}{b_l}}]$, and notice that $J_{mn}f(x)$ is supported in $I_{m_1}\times I_{m_2}$. Also, $\lambda<|x_1x_2U_{mn}f(x)|\le\psi_m(x^{\frac{b}{a}})\|f\|_1x_1x_2$. Thus, $x_2>\frac{\lambda}{\|f\|_1x_1}$ and $x_2\in I_{m_2}$. Hence if $\frac{\lambda}{\|f\|_1x_1}>(2\cdot 2^{m_2})^{\frac{a_2}{b_2}}$, then the set $E_\lambda=\emptyset$ and so $\nu(E_\lambda) = 0$. We need only concern ourselves with the cases where $\frac{\lambda}{\|f\|_1 x_1} \leq (2 \cdot 2^{m_2})^{\frac{a_2}{b_2}}$.

Then we get

$$(2.17) \qquad \begin{cases} \text{(a) } \int_{\frac{\lambda}{\|f\|_1 x_1}}^{(2\cdot 2^{m_2})^{\frac{a_2}{b_2}}} \frac{1}{{x_2}^2} dx_2 \leq \frac{\|f\|_1 x_1}{\lambda}, \text{ if } \frac{\lambda}{\|f\|_1 x_1} \in I_{m_2}, \\ \text{ and} \\ \text{(b) } \int_{I_{m_2}} \frac{1}{{x_2}^2} dx_2 \leq \frac{C}{2^{m_2 \frac{a_2}{b_2}}}, \text{ if } \frac{\lambda}{\|f\|_1 x_1} < (\frac{2^{m_2}}{4})^{\frac{a_2}{b_2}}. \end{cases}$$

Therefore we get from (2.17) that

$$\begin{aligned} \nu(E_\lambda) &= \int_{I_{m_1}} (\int_{\frac{\lambda}{\|f\|_1 x_1}}^{(2\cdot 2^{m_2})^{\frac{a_2}{b_2}}} \frac{1}{{x_2}^2} dx_2) \frac{dx_1}{{x_1}^2} \\ &\quad + \int_{\frac{\lambda}{\|f\|_1} (\frac{4}{2^{m_2}})^{\frac{a_2}{b_2}}}^{(2\cdot 2^{m_1})^{\frac{a_1}{b_1}}} (\int_{I_{m_2}} \frac{1}{{x_2}^2} dx_2) \frac{dx_1}{x_1^2} = I + II, \\ I &\leq \frac{\|f\|_1}{\lambda} \int_{I_{m_1}} \frac{1}{x_1} dx_1 \leq C \frac{\|f\|_1}{\lambda}, \text{ and} \\ II &\leq C(\frac{4}{2^{m_2}})^{\frac{a_2}{b_2}} (2^{m_2})^{\frac{a_2}{b_2}} \frac{\|f\|_1}{\lambda} \leq C \frac{\|f\|_1}{\lambda}. \end{aligned}$$

Thus we obtain our estimate for (2.16).

Next it follows from Lemma 2.2 that

$$(2.18) \qquad \|J_{mn} f\|_{2,\nu}^2 = \int_{\mathbb{R}_+^2} |U_{mn} f(x)|^2 dx \leq C d_{mn}^2 \int_{\mathbb{R}_+^2} |f(y)|^2 dy.$$

Now we shall prove that (2.15) follows from (2.16) and (2.18).

To complete this proof, we take

$$f_1^\lambda(y) = \begin{cases} f(y), \text{ if } 0 \leq f(y) \leq \lambda\alpha, \\ 0, \text{ elsewhere,} \end{cases}$$

and $f(y) = f_1^\lambda(y) + f_2^\lambda(y)$, and we take $\alpha = \frac{1}{d_{mn}^2},\ d_{mn}$ defined below (2.2).

Thus (2.16) and (2.18) imply that

$$(2.19) \qquad \begin{cases} \text{(a) } \nu(\{|J_{mn} f_1^\lambda(x)| > \lambda\}) \leq \frac{C d_{mn}^2}{\lambda^2} \|f_1^\lambda\|_2^2, \\ \text{ and} \\ \text{(b) } \nu(\{|J_{mn} f_2^\lambda(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f_2^\lambda\|_1. \end{cases}$$

Now for $1 < p < 2$ we get that

$$\begin{aligned} \|J_{mn} f\|_{p,\nu}^p &= 2^p p \int_0^\infty \lambda^{p-1} \nu(\{|J_{mn} f(x)| > 2\lambda\}) d\lambda \\ &\leq C_p \int_0^\infty \lambda^{p-1} [\nu(\{|J_{mn} f_1^\lambda(x)| > \lambda\}) + \nu(\{|J_{mn} f_2^\lambda(x)| > \lambda\})] d\lambda \\ &= I + II. \end{aligned}$$

Thus (2.19) implies that

$$\begin{aligned}
I+II &\le C_p(\int_0^\infty \lambda^{p-1}[\frac{d_{mn}^2}{\lambda^2}(\int_{\mathbb{R}_+^2}|f_1^\lambda(y)|^2dy)+\frac{1}{\lambda}(\int_{\mathbb{R}_+^2}|f_2^\lambda(y)|dy)]d\lambda)\\
&= C_p(\int_{\mathbb{R}_+^2}[d_{mn}^2(\int_0^\infty \lambda^{p-3}|f_1^\lambda(y)|^2d\lambda)+(\int_0^\infty \lambda^{p-2}|f_2^\lambda(y)|d\lambda)]dy)\\
&\le C_p(\int_{\mathbb{R}_+^2}[d_{mn}^2(\int_{\{y:0<f(y)\le\lambda\alpha\}}\lambda^{p-3}|f(y)|^2d\lambda)\\
&\quad +(\int_{\{y:f(y)>\lambda\alpha\}}\lambda^{p-2}|f(y)|d\lambda)]dy)\\
&\le C_p(\int_{\mathbb{R}_+^2}d_{mn}^2|f(y)|^2(\int_{\frac{f(y)}{\alpha}}^\infty \lambda^{p-3}d\lambda)dy+\int_{\mathbb{R}_+^2}|f(y)|(\int_0^{\frac{f(y)}{\alpha}}\lambda^{p-2}d\lambda)dy)\\
&\le C_p(\frac{d_{mn}^2}{2-p}\int_{\mathbb{R}_+^2}\frac{|f(y)|^p}{\alpha^{p-2}}dy+\frac{1}{p-1}\int_{\mathbb{R}_+^2}\frac{|f(y)|^p}{\alpha^{p-1}}dy)\\
&\le C_p(d_{mn}^2d_{mn}^{2(p-2)}+d_{mn}^{2(p-1)})\|f\|_p^p \le C_pd_{mn}^{2(p-1)}\|f\|_p^p,
\end{aligned}$$

which completes our proof (note $\alpha=\frac{1}{d_{mn}^2}$). □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We notice that in case $a_l=b_l\ge 1, l=1$ or 2, our result follows from Proposition 4.3 of [SS]. Now to do the remaining cases. It follows from (2.12) and Lemma 2.3 that

$$\begin{aligned}
&(\int_{\mathbb{R}_+^2}x^{\bar{p}-\bar{2}}|Uf(x)|^pdx)^{\frac{1}{p}} \le \sum_{m,n\ge\bar{1}}(\int_{\mathbb{R}_+^2}x^{\bar{p}-\bar{2}}|U_{mn}f(x)|^pdx)^{\frac{1}{p}}\\
&\le C\|f\|_p\sum_{m,n\ge\bar{1}}d_{mn}^{\frac{2}{p'}}\le C\|f\|_p\ ,
\end{aligned}\tag{2.20}$$

with d_{mn} defined below (2.2).

Since $Tf(x)=Uf(x^{\frac{a}{b}})$, then

$$\|Tf\|_p=(\int_{\mathbb{R}_+^2}x^{\frac{b}{a}-\bar{1}}|Uf(x)|^pdx)^{\frac{1}{p}},$$

and if we take $\bar{p}-\bar{2}=\frac{b}{a}-\bar{1}$ or $p=\frac{b_1}{a_1}+1=\frac{b_2}{a_2}+1$, then we get the desired result. □

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