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DIVERGENT CESÀRO AND RIESZ MEANS OF JACOBI AND LAGUERRE EXPANSIONS

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Abstract. We show that for δ below certain critical indices there are functions whose Jacobi or Laguerre expansions have almost everywhere divergent Cesàro and Riesz means of order δ.

1. Introduction

1.1. **Orthogonal expansions.** Suppose that (X,μ) is a positive measure space, $(\varphi_n)_{n=0}^{\infty}$ is an orthogonal subset of $L^2(X,\mu)$, and $h_n = \|\varphi_n\|_2^2$ for all $n \geq 0$. If f is a function on X for which all the products $f.\overline{\varphi_n}$ are μ-integrable, then f has an orthogonal expansion

$$\sum_{n=0}^{\infty} c_n(f) h_n^{-1} \varphi_n(x) \tag{1}$$

with coefficients

$$c_n(f) = \int_X f(x)\overline{\varphi_n(x)}\, d\mu(x), \quad \forall n \geq 0. \tag{2}$$

1.2. **Cesàro means.** As described in Zygmund's book [16, pp. 76–77], the Cesàro means of order δ of the expansion (1) are defined by

$$\sigma_N^\delta f(x) = \sum_{n=0}^{N} \frac{A_{N-n}^\delta}{A_N^\delta} c_n(f) h_n^{-1} \varphi_n(x), \tag{3}$$

where $A_n^\delta = \binom{n+\delta}{n}$. Theorem 3.1.22 in [16] says that if the Cesàro means converge, then the terms of the series have controlled growth.

Lemma 1.1. *Suppose that* $\lim_{N\to\infty} \sigma_N^\delta f(x)$ *exists for some* $x \in X$ *and* $\delta > -1$. *Then*

$$\left|c_N(f) h_N^{-1} \varphi_N(x)\right| \leq C_\delta\, N^\delta \max_{0\leq n\leq N} \left|\sigma_n^\delta f(x)\right|, \qquad \forall N \geq 0.$$

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1.3. **Riesz means.** Hardy and Riesz [6] had proved a similar result for Riesz means. Recall that the Riesz means of order $\delta \geq 0$ are defined for each $r > 0$ by

$$S_r^\delta f(x) = \sum_{0 \leq n < r} \left(1 - \frac{n}{r}\right)^\delta c_n(f) h_n^{-1} \varphi_n(x). \tag{4}$$

Theorem 21 of [6] tells us how the convergence of $S_r^\delta f(x)$ controls the size of the partial sums $S_r^0 f(x)$.

Lemma 1.2. *Suppose that f has an orthogonal expansion and for some $\delta > 0$ and $x \in X$ its Riesz means $S_r^\delta f(x)$ converges to c as $r \to \infty$. Then*

$$\left|S_r^0 f(x) - c\right| \leq A_\delta \, r^\delta \sup_{0 < t \leq r+1} \left|S_t^\delta f(x)\right|.$$

Note that we can write

$$c_n(f) h_n^{-1} \varphi_n(x) = \left(S_n^0 f(x) - c\right) - \left(S_{n-1}^0 f(x) - c\right) = \mathbf{O}(n^\delta)$$

and obtain the same growth estimates as in Lemma 1.1.

Gergen [5] wrote formulae relating the Riesz and Cesàro means of order $\delta \geq 0$, from which the equivalence of the two methods of summation follows.

1.4. **Uniform boundedness.** Suppose there is a $1 < q \leq \infty$ for which $\varphi_n \in L^q(X, \mu)$ for all n. In addition, suppose that there is some positive number λ with

$$\|\varphi_n\|_q \geq c n^\lambda, \qquad \forall n \geq 1.$$

The formation of the coefficient $f \mapsto c_n(f)$ is then a bounded linear functional on the dual of $L^q(X, \mu)$ with norm bounded below by a constant multiple of n^λ. The uniform boundedness principle implies that for p conjugate to q and each $0 \leq \varepsilon < \lambda$ there is an $f \in L^p(X, \mu)$ so that

$$c_n(f)/n^\varepsilon \to \infty \text{ as } n \to \infty. \tag{5}$$

1.5. **Cantor-Lebesgue Theorem.** The following argument is based on [16, Section IX.1]. Suppose we have a sequence of functions F_n on an interval in the real line with the asymptotic property

$$F_n(\theta) = c_n \left(\cos(M_n\theta + \gamma_n) + \mathbf{o}(1)\right), \qquad \forall n \geq 0,$$

uniformly on a set E of finite positive measure, and with $M_n \to \infty$ as $n \to \infty$. Integrating $|F_n|^2$ over E gives

$$\begin{aligned} \int_E |F_n(\theta)|^2 \, d\theta &= |c_n|^2 \left(\int_E \cos^2(M_n\theta + \gamma_n) \, d\theta + \mathbf{o}(1)\right) \\ &= |c_n|^2 \left(\frac{|E|}{2} + \frac{e^{2i\gamma_n}}{4} \widehat{\chi_E}(2M_n) + \frac{e^{-2i\gamma_n}}{4} \widehat{\chi_E}(-2M_n) + \mathbf{o}(1)\right). \end{aligned}$$

The Riemann-Lebesgue Theorem [16, Thm. II.4.4] says that the Fourier transforms $\widehat{\chi_E}(\pm 2M_n) \to 0$ as $M_n \to \infty$. If we know that there is some function G for which $|F_n(\theta)| \leq G(n)$ uniformly on E for all n, then there is an $n_0 > 0$ for which

$$\frac{|E|}{4} |c_n|^2 \leq \int_E |F_n(\theta)|^2 \, d\theta \leq G(n)^2 |E|, \quad \forall n \geq n_0.$$

This shows that $|c_n| \leq 2G(n)$ for all $n \geq n_0$.

2. Jacobi polynomials

2.1. **Notation.** Fix real numbers $\alpha \geq \beta \geq -1/2$, with $\alpha > -1/2$, and let μ denote the measure on $[-1,1]$ defined by

$$d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n associated to the pair (α,β) as in Szegő's book [15]. Then $\left(P_n^{(\alpha,\beta)}\right)_{n=0}^{\infty}$ is an orthogonal subset of $L^2([-1,1],\mu)$. Equation (4.3.3) in [15] shows that the normalization terms $h_n^{(\alpha,\beta)} = \left\|P_n^{(\alpha,\beta)}\right\|_2^2$ satisfy

$$h_n^{(\alpha,\beta)} \sim c_{\alpha,\beta} n^{-1} \text{ as } n \to \infty. \tag{6}$$

The Jacobi polynomial expansion of $f \in L^1(\mu)$ is

$$\sum_{n=0}^{\infty} c_n(f)\left(h_n^{(\alpha,\beta)}\right)^{-1} P_n^{(\alpha,\beta)}(x),$$

with coefficients $c_n(f) = \int_{-1}^{1} f(x)P_n^{(\alpha,\beta)}(x)\,d\mu(x)$. We take α and β as fixed and use $\sigma_N^{\delta} f(x)$ and $S_r^{\delta} f(x)$ to denote the Cesàro and Riesz means of this expansion, respectively.

2.2. **Asymptotics.** Theorem 8.21.8 in Szegő's book [15] gives the following asymptotic behaviour for the Jacobi polynomials.

Lemma 2.1. *For $\alpha \geq \beta \geq -1/2$ and $\varepsilon > 0$ the following estimate holds uniformly for all $\varepsilon \leq \theta \leq \pi - \varepsilon$ and $n \geq 1$:*

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2} k(\theta)\cos\left(M_n\theta + \gamma\right) + \mathbf{O}\left(n^{-3/2}\right). \tag{7}$$

Here $k(\theta) = \pi^{-1/2}\left(\sin(\theta/2)\right)^{-\alpha-1/2}\left(\cos(\theta/2)\right)^{-\beta-1/2}$, $M_n = n + (\alpha+\beta+1)/2$, and $\gamma = -(\alpha+1/2)\pi/2$.

From Egoroff's theorem and Lemma 1.1 we can say that if $\sigma_N^{\delta} f(x)$ converges on a set of positive measure in $[-1,1]$, then there is a set of positive measure E on which

$$\left|c_n n^{(1/2)-\delta}\left(\cos\left(M_n\theta+\gamma\right) + \mathbf{O}(n^{-1})\right)\right| \leq A \tag{8}$$

uniformly for $\cos\theta \in E$. The argument of subsection 1.5 shows that

$$\left|c_n n^{(1/2)-\delta}\right| \leq A, \qquad \forall n \geq 1. \tag{9}$$

2.3. **Norm estimates.** Next we recall the calculation of Lebesgue norms of Jacobi polynomials, according to Markett [10] and Dreseler and Soardi [4]. Equation (2.2) in [10] gives the following lower bounds on these norms.

Lemma 2.2. *For real numbers $\alpha \geq \beta \geq -1/2$, with $\alpha > -1/2$, $1 \leq q < \infty$, and $r > -1/q$,*

$$\left(\int_0^1 \left|P_n^{(\alpha,\beta)}(x)\,(1-x)^r\right|^q dx\right)^{1/q} \sim \begin{cases} n^{-1/2} & \text{if } r > \alpha/2 + 1/4 - 1/q, \\ n^{-1/2}\left(\log n\right)^{1/q} & \text{if } r = \alpha/2 + 1/4 - 1/q, \\ n^{\alpha-2r-2/q} & \text{if } r < \alpha/2 + 1/4 - 1/q. \end{cases}$$

2.4. **Main result.** There are critical indices, as used in [11],

$$p_c = \frac{4(\alpha+1)}{(2\alpha+3)} \quad \text{and its conjugate } p_c' = \frac{4(\alpha+1)}{(2\alpha+1)}.$$

Taking $r = \alpha/q$ in Lemma 2.2 we have that

$$\left\| P_n^{(\alpha,\beta)} \right\|_{L^q(\mu)} > \left(\int_0^1 \left| P_n^{(\alpha,\beta)}(x) \right|^q (1-x)^\alpha \, dx \right)^{1/q} \sim n^{\alpha - 2\alpha/q - 2/q} \tag{10}$$

for $\alpha/q < \alpha/2 + 1/4 - 1/q$. This last inequality can be rewritten as

$$q > \frac{4(\alpha+1)}{2\alpha+1} = p_c'. \tag{11}$$

We can now prove that below the critical index there are functions with almost everywhere divergent Cesàro and Riesz means.

Theorem 2.3. *For real numbers $\alpha \geq \beta \geq -1/2$, with $\alpha > -1/2$,*

$$1 \leq p < p_c = \frac{4(\alpha+1)}{(2\alpha+3)}, \text{ and } 0 \leq \delta < \frac{(2\alpha+2)}{p} - \frac{(2\alpha+3)}{2},$$

there is an $f \in L^p(\mu)$, supported in $[0,1]$, whose Cesàro means $\sigma_N^\delta f(x)$ and Riesz means $S_r^\delta f(x)$ are divergent almost everywhere on $[-1,1]$.

Proof. Let q be conjugate to p, so that $1/p = (q-1)/q$. Suppose that

$$\delta < \frac{(2\alpha+2)}{p} - \frac{(2\alpha+3)}{2} = \frac{(2\alpha+2)(q-1)}{q} - \frac{(2\alpha+3)}{2} = \alpha + \frac{1}{2} - \frac{(2\alpha+2)}{q}.$$

Then

$$\delta - \frac{1}{2} < \alpha - \frac{(2\alpha+2)}{q} = \alpha - \frac{2\alpha}{q} - \frac{2}{q},$$

which is the exponent of n in the inequality (10). Now apply the argument given in subsection 1.4. The norms of the Jacobi polynomials in Lemma 2.2 are calculated over $[0,1]$ and so we can find f in $L^p([-1,1],\mu)$, supported on $[0,1]$, for which the coefficients satisfy

$$c_n(f)/n^{\delta-1/2} \to \infty, \quad \text{as } n \to \infty.$$

Combine this with Lemmas 1.1 and 1.2 and the argument around inequality (9) to see that for this f both $\sigma_N^\delta f(x)$ and $S_r^\delta f(x)$ are divergent almost everywhere. This argument follows the methods used in [11, 8, 7, 12]. □

2.5. **Remarks.** Convergence results above the critical index are contained in the work of Bonami and Clerc [1], Colzani, Taibleson and Weiss [3], and Chanillo and Muckenhoupt [2]. In particular, in [2, Thm. 1.4] it is shown that for

$$1 \leq p < p_c = \frac{4(\alpha+1)}{(2\alpha+3)} \text{ and } \delta = \frac{(2\alpha+2)}{p} - \frac{(2\alpha+3)}{2},$$

the maximal operator $f \mapsto \sup_{N\geq 0} \left| \sigma_N^\delta f(x) \right|$ is of weak type (p,p).

For $\delta = 0$ and $p = p_c$, divergence was proved in [11].

3. Laguerre functions

3.1. **Notation.** For each $\alpha > -1$ let μ_α be the measure on $[0, \infty)$ defined by

$$d\mu_\alpha(x) = e^{-x} x^\alpha \, dx.$$

We denote by $L_n^{(\alpha)}(x)$ the Laguerre polynomial of degree n, as in [15, Chpt. 5]. The $L^2(\mu_\alpha)$-norms of these satisfy the identity

$$h_n^{(\alpha)} = \|L_n^{(\alpha)}\|_{L^2(\mu_\alpha)}^2 = \Gamma(\alpha+1)\binom{n+\alpha}{n} \sim n^\alpha.$$

Fejér's formula [15, Thm. 8.22.1] gives the asymptotic properties of these polynomials. For each $\alpha > -1$ and $0 < \varepsilon < \omega$,

$$L_n^{(\alpha)}(x) = \frac{e^{x/2}}{\pi^{1/2} x^{\alpha/2}} n^{\alpha/2-1/4} \cos\left(2(nx)^{1/2} - \alpha\pi/2 - \pi/4\right) + \mathbf{O}\left(n^{\alpha/2-3/4}\right), \tag{12}$$

uniformly in $x \in [\varepsilon, \omega]$. The corresponding normalized functions are

$$\mathcal{L}_n^\alpha(x) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}\, e^{-x/2} x^{a/2} L_n^{(\alpha)}(x), \qquad \forall x \geq 0, n \geq 0. \tag{13}$$

These provide an orthonormal subset of $L^2([0,\infty))$, where the half line carries Lebesgue measure.

3.2. **Norm estimates.** Markett [9, Lemma 1] has calculated the Lebesgue norms of the Laguerre functions, for $\alpha > -1/2$,

$$\|\mathcal{L}_n^\alpha\|_q \sim \begin{cases} n^{1/q-1/2}, & \forall \quad 1 \leq q < 4, \\ n^{-1/4}(\log n)^{1/4}, & \text{if } q = 4, \\ n^{-1/q}, & \forall \quad 4 < q \leq \infty. \end{cases} \tag{14}$$

3.3. **Divergence result.**

Theorem 3.1. *If $\alpha > -1/2$, $p > 4$ and $0 < \delta < 1/4 - 1/p$, then there is a function $f \in L^p(0,\infty)$ whose Laguerre expansion*

$$\sum_{n=0}^{\infty} c_n(f) \mathcal{L}_n^\alpha(x)$$

has Cesàro and Riesz means of order δ which diverge almost everywhere.

Proof. Suppose that the expansion $\sum_{n=0}^\infty c_n(f)\mathcal{L}_n^\alpha(x)$ is either Cesàro or Riesz summable of order δ on a set of positive measure in $[0,\infty)$. Then Lemma 1.1 or Lemma 1.2 implies that

$$c_n(f)\mathcal{L}_n^\alpha(x) = \mathbf{O}(n^\delta) \tag{15}$$

on a set of positive measure. When equations (12) and (13) are combined with the argument of subsection 1.5 we find that

$$c_n(f) = \mathbf{O}(n^{\delta+1/4}). \tag{16}$$

The case when $\delta = 0$ is Lemma 2.3 in Stempak's paper [14]. Suppose that

$$\frac{1}{q} - \frac{1}{2} > \delta + \frac{1}{4},$$

so that $\delta < \frac{1}{q} - \frac{3}{4} = \frac{4-3q}{q}$. If $\frac{1}{q} = 1 - \frac{1}{p}$, then this inequality is $\delta < \frac{1}{4} - \frac{1}{p}$. The argument of subsection 1.4 shows that if $p > 4$ and $\delta < 1/4 - 1/p$, then there is a function $f \in L^p(0,\infty)$ for which the inequality (16) fails,

$$c_n(f)/n^{\delta+1/4} \to \infty \text{ as } n \to \infty.$$

The Laguerre expansion of this function has Cesàro and Riesz means of order δ which diverge almost everywhere. □

3.4. **Remarks.** There is an extensive treatment of almost everywhere convergence results for Laguerre expansions in [13]. In particular, [13, Thm. 1.20] implies that if $p > 4$ and $\delta \geq 1/4 - 1/p$, then all $f \in L^p(0,\infty)$ have almost everywhere convergent Cesàro means of order δ.

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