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p–HYPONORMALITY IS NOT TRANSLATION–INVARIANT

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Abstract. In this note we provide an example of a semi-hyponormal Hilbert space operator T for which $T-\lambda$ is not p–hyponormal for some $\lambda \in \mathbb{C}$ and all $0 < p \leq \frac{1}{2}$.

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *p-hyponormal* if

$$(T^*T)^p - (TT^*)^p \geq 0.$$

If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is called semi-hyponormal. It is well known that q-hyponormal operators are p-hyponormal for $p \leq q$, by Löwner's Theorem. Throughout this note we assume $0 < p < 1$. The notion of semi-hyponormality was introduced by D. Xia [Xi] and the notion of p-hyponormal operators was introduced by A. Aluthge [Al]. The p-hyponormal operators have been studied by many authors (cf. [Al], [AlW], [Ch], [ChH], [ChI], [ChJ]). An operator $T \in B(\mathcal{H})$ is said to be *paranormal* if $||T^2x|| \geq ||Tx||^2$ for every unit vector $x \in \mathcal{H}$. It is well known (cf. [An], [ChJ], [FHM]) that

$$p\text{–hyponormal} \Longrightarrow \text{paranormal}. \tag{0.1}$$

The p–hyponormal operators share many properties with hyponormal operators: for example, if T is p–hyponormal, then (cf. [Al], [Ch], [ChH])

(i) T is normaloid (i.e., norm equals spectral radius);
(ii) T is reduced by its eigenspaces;
(iii) if T is invertible, then T^{-1} is also p–hyponormal.

However, it has remained open whether p–hyponormality is translation–invariant. For a long time researchers have guessed that this is not true. But no concrete example has been found. The purpose of this note is to provide an example which shows that p–hyponormality is *not* translation–invariant.

Concrete examples of non–hyponormal p–hyponormal are scarce in the literature before [AlW]. But the following result in [AlW] helps provide an abundancy of examples.

Theorem 1 ([AlW, Corollary 1])**.** *If $T \in B(\mathcal{H})$ is p–hyponormal and n is a positive integer, then T^n is $\frac{p}{n}$–hyponormal for every $0 < p \leq 1$.*

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To see how Theorem 1 gives rise to many examples of operators which are semi–hyponormal but not hyponormal, recall the definition of Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$. Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \cdots\}$. An element $f \in L^2(\mathbb{T})$ is said to be analytic if $f \in H^2(\mathbb{T})$, and co-analytic if $f \in L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. If P denotes the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$ the operator T_φ defined by

$$T_\varphi g := P(\varphi g) \qquad (g \in H^2(\mathbb{T}))$$

is called the *Toeplitz operator* with symbol φ. If φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero, then T_φ is called a *trigonometric* Toeplitz operator.

The following theorem gives a useful necessary condition for hyponormality and normality of trigonometric Toeplitz operators.

Theorem 2 ([FaL]). *Suppose that φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero.*

(i) *If T_φ is hyponormal, then $m \le N$ and $|a_{-m}| \le |a_N|$.*
(ii) *If T_φ is normal, then $m = N$ and $|a_{-m}| = |a_N|$.*

The following theorem together with Theorem 1 gives abundant examples which are semi–hyponormal operators but not hyponormal.

Theorem 3 ([CuL, Theorem 3.2]). *Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic.*

For example, if φ is a trigonometric polynomial of the form

$$\varphi(z) \equiv \sum_{n=-m}^{N} a_n z^n \quad (a_{-m} \neq 0 \text{ and } a_N \neq 0)$$

and satisfies

(i) $m < N$,
(ii) $|a_{-m}| = |a_N|$,
(iii) $\overline{a_N} \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \overline{a_{N-m+1}} \\ \overline{a_{N-m+2}} \\ \vdots \\ \overline{a_N} \end{pmatrix}$,

then T_φ is hyponormal by an argument of [FaL, Theorem 1.4], but by Theorem 3, T_φ^2 is not hyponormal since T_φ is neither analytic nor normal. However by Theorem 1, T_φ^2 is semi–hyponormal.

We now have

Theorem 4. *p–hyponormality is not translation–invariant; more precisely, there exists an operator T satisfying that T is semi–hyponormal, but $T - \lambda$ is not p–hyponormal for some $\lambda \in \mathbb{C}$ and any $p > 0$.*

Proof. If U is the unilateral shift on ℓ_2, define

$$S \equiv 4\,U^2 + U^{*2} + 2\,UU^* + 2.$$

We now claim that (i) S is semi–hyponormal, and (ii) $S - 4$ is not p–hyponormal for any $p > 0$.

(i) Define $\varphi(z) = 2\,z + z^{-1}$. Then it is easy to see that T_φ is hyponormal, but T_φ^2 is not hyponormal by Theorem 2. A straightforward calculation shows that $T_\varphi^2 = S$. Thus, by Theorem 1, S is semi–hyponormal.

(ii) To show that $S-4$ is not p–hyponormal for any $p>0$, in view of (0.1) it suffices to show that $S-4$ is not paranormal. Indeed,

$$\begin{aligned} ||(S-4)\,e_0||^2 &= ||(4U^2 + U^{*2} + 2UU^* - 2)\,e_0||^2 \\ &= ||4\,e_2 - 2\,e_0||^2 = 20 \end{aligned}$$

and

$$\begin{aligned} ||(S-4)^2\,e_0|| &= ||(4U^2 + U^{*2} + 2UU^* - 2)(4e_2 - 2e_0)|| \\ &= ||8e_0 - 8e_2 + 16e_4|| = \sqrt{384}, \end{aligned}$$

which shows that $||(S-4)\,e_0||^2 > ||(S-4)^2\,e_0||$. This implies that $S-4$ is not paranormal. □ □

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