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MODULAR GROUP ALGEBRAS OF $\aleph_1$-SEPARABLE p-GROUPS

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Abstract. Under the assumptions of MA and $\neg$ CH, it is proved that if F is a field of prime characteristic p and G is an $\aleph_1$-separable abelian p-group of cardinality $\aleph_1$, then an isomorphism of the group algebras FG and FH implies an isomorphism of G and H.

In describing the unit groups of abelian group algebras, the algebras of separable p-groups over fields of characteristic p represent a crucial case. The well-known isomorphism and direct factor problems remain largely unresolved for this class once one gets beyond the case of totally projective p-groups. We wish to consider the isomorphism problem for $\aleph_1$-separable p-groups of cardinality $\aleph_1$. In this setting, we show that isomorphism of the group algebras implies that the groups have equal Γ-invariants and are quotient equivalent; see [EM2] for the notions of Γ-invariant and quotient equivalence. It is known that this is not enough to guarantee isomorphism of the groups, but for cardinality $\aleph_1$ it is known that the groups are direct factors of the normalized unit groups with complements which are totally projective. Utilizing this together with a result of the authors [GM] on cancellation in direct sums (see Proposition 3) and a theorem of Eklof and Mekler [EM1], we prove the following theorem.

Theorem (MA + $\neg$ CH). *Let F be a field of prime characteristic p and let FG and FH be isomorphic group algebras. Assume that G is an $\aleph_1$-separable abelian p-group of cardinality $\aleph_1$. Then G and H are isomorphic.*

To make the paper more nearly self-contained, we shall give a brief introduction to some basic facts on abelian group algebras.

1. Preliminaries

In this paper, F will always denote a field of characteristic p and all groups will be abelian. For background on $\aleph_1$-separable p-groups, see [EM1, H]. Groups will be written multiplicatively as is customary when discussing group algebras. The augmentation, $aug(\alpha)$, of an element $\alpha \in FG$ is the sum of the coefficients of α, and the augmentation map $aug : FG \to F$ $(\alpha \mapsto aug(\alpha))$ is a ring homomorphism. The group of units which have augmentation 1 is called the group of normalized units of FG and is denoted by $V(FG)$, or simply $V(G)$. If G is a p-group, then $V(G)$ is a p-group and consists precisely of all elements of augmentation 1. We

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note for later use that when G is a p-group, any element of nonzero augmentation is a unit.

The elements of FG of augmentation 0 form an ideal of FG, called the augmentation ideal and denoted by $I(G)$. Unless otherwise specified, all homomorphisms of group algebras will be algebra homomorphisms and we shall assume that they preserve augmentation. As an example, a homomorphism $\phi : G \to H$ of arbitrary groups induces a natural homomorphism $\Phi : FG \to FH$ of the group algebras. If the kernel of ϕ is G_1, then the kernel of Φ will be denoted by $I(G;G_1)$. The following lemma will make clear that $I(G;G_1)$ depends only on G and G_1. Note that the augmentation ideal of FG is just $I(G;G)$.

Lemma 1. *For Φ as above, $I(G;G_1)$ is the ideal of FG generated by all elements of the form $1-g$ for $g \in G_1$. In fact, it is only necessary that g range over a set of generators of G_1.*

Proof. The ideal described is certainly contained in the kernel of Φ. Let $\{g_i | i < \mu\}$ be a set of coset representatives of G modulo G_1. Then the kernel clearly consists of all (finite) sums $\sum_{i<\mu} g_i\beta_i$, where $\beta_i \in I(G_1)$. But if $\beta = \sum_{g\in G_1} r_g\, g$ has augmentation 0, then $\beta = \sum_{g\in G_1} r_g(g-1)$, hence the kernel is contained in the ideal. The remark about generators follows from the simple observation that $1-xy = (1-x)y + (1-y)$. $\square$

Now let $\Phi : FG \to FH$ be a homomorphism preserving augmentation and suppose that G_1 and H_1 are subgroups of G and H, respectively, such that $\Phi(FG_1) \subseteq FH_1$. Regarding $I(G;G_1)$ as the kernel of the homomorphism induced by the natural homomorphism from G to G/G_1, and similarly for $I(H;H_1)$, then Lemma 1 implies that there is a natural homomorphism $\overline{\Phi} : F(G/G_1) \to F(H/H_1)$ such that we have a commutative diagram

$$\begin{array}{ccc} FG & \to & FH \\ \downarrow & & \downarrow \\ F(G/G_1) & \to & F(H/H_1) \end{array}$$

where the vertical maps are the natural ones. For use in the proof of Proposition 1, note that it is not strictly necessary for Φ to preserve augmentation since the argument goes through if Φ simply preserves augmentation 0. The following lemma is immediate.

Lemma 2. *Let $\Phi : FG \to FH$ be an isomorphism taking FG_1 to FH_1. Then there is an induced isomorphism $\overline{\Phi} : F(G/G_1) \to F(H/H_1)$.*

If one has a homomorphism $\Phi : FG \to FH$ which does not preserve augmentation, then it can be "adjusted" to do so by an automorphism of FG. Specifically, precede Φ by the automorphism of FG given by $g \mapsto aug(\Phi(g))^{-1}g$. Thus, if an isomorphism exists which does not preserve augmentation, then one exists which does preserve it. This has the following consequence. Suppose that G is a p-group and that $FG \cong FH$ for some group H. Since we may assume that the isomorphism preserves augmentation, we have $V(G) \cong V(H)$. But we have observed that $V(G)$ is a p-group, hence we may conclude that H is also a p-group.

To discuss p-height in group algebras, we shall assume that the field F is perfect. If G is a group, define G_α for ordinals α in the usual way: if $\alpha = \beta + 1$, then $G_\alpha = G_\beta^p$, and if α is a limit, then $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$. Define subalgebras $(FG)_\alpha$ of FG in the same fashion, taking either $(FG)_\alpha^p$, or $\bigcap_{\beta<\alpha}(FG)_\beta$. It is easy to see that

$(FG)_\alpha = F(G_\alpha)$ since, for a perfect field, $(FG)^p = F(G^p)$. It is further clear that a homomorphism $\Phi : FG \to FH$ will satisfy $\Phi(FG_\alpha) \subseteq FH_\alpha$ for every ordinal α.

To show the well-known invariance of Ulm invariants under isomorphism of group algebras, we need a lemma.

Lemma 3. *Let $A = \langle u_1 \rangle \times \cdots \times \langle u_n \rangle \times A_1$, where u_1 is a nontrivial p-torsion element, let $\alpha_2, \ldots, \alpha_{n+m} \in FA$, and let $\beta_1, \ldots, \beta_m \in I(A)$. Suppose that*

$$1 - u_1 = \alpha_2(1 - u_2) + \cdots + \alpha_n(1 - u_n) + \alpha_{n+1}\beta_1 + \cdots + \alpha_{n+m}\beta_m.$$

Then for some i with $n + 1 \leq i \leq n + m$, α_i has nonzero augmentation.

Proof. Suppose to the contrary that $\alpha_{n+1}, \ldots, \alpha_{n+m} \in I(A)$. Using the given decomposition of A, project onto $\langle u_1 \rangle$ and consider the induced projection of FA onto $F\langle u_1 \rangle$. Let I_1 denote the augmentation ideal of $F\langle u_1 \rangle$. Since $I(A)$ projects into I_1, we conclude that $1 - u_1 \in I_1^2$. But since Lemma 1 implies that I_1 is a principal ideal generated by $1 - u_1$, we have $1 - u_1 = \gamma(1 - u_1)^2$ for some $\gamma \in FA$. If p^k is the order of u_1, then we have $1 - u_1 = \gamma^{p^k-1}(1 - u_1)^{p^k} = \gamma^{p^k-1}(1 - u_1^{p^k}) = 0$, which is a contradiction. □

Proposition 1 (Berman [B], Berman and Mollov [BM], May [M1])**.** *If G and H are abelian groups such that $FG \cong FH$, then the Ulm-Kaplansky invariants of G and H for the prime p are equal.*

Proof. We may assume that the isomorphism preserves augmentation. By taking an extension field of F which is perfect and tensoring the isomorphism over F, we may further assume that F itself is perfect. Since the isomorphism carries FG_α to FH_α for every ordinal α, it suffices to show the 0-th Ulm invariants are equal.

The p-th power endomorphism of G maps G^p into G^{p^2}, thus inducing a homomorphism $\phi : G/G^p \to G/G^{p^2}$. Let $\Phi : F(G/G^p) \to F(G/G^{p^2})$ be induced by ϕ. If U denotes the kernel of ϕ, then U is an elementary p-group and the 0-th Ulm invariant of G is the rank of U. The kernel of Φ is the ideal $I(G/G^p; U)$. Now consider the p-th power endomorphism of FG. This carries $F(G^p)$ to $F(G^{p^2})$ and preserves augmentation 0, thus inducing a homomorphism $\Psi : F(G/G^p) \to F(G/G^{p^2})$ by the remark before Lemma 2. The kernel of Ψ is also $I(G/G^p; U)$ since a sum of p-th powers of coefficients from F is 0 if and only if the sum of coefficients is 0. If we carry out a similar construction for FH, then the naturality of Ψ together with the isomorphism of FG with FH guarantees that we will have an isomorphism of $F(G/G^p)$ with $F(H/H^p)$ that relates the corresponding ideals. Thus, it will suffice to show that the rank of U is equal to the minimum number of ideal generators of $I(G/G^p; U)$.

Let $U = \coprod_{i<\mu} \langle u_i \rangle$, where each u_i has order p. Then $\{1 - u_i | i < \mu\}$ is a minimal set of generators of the ideal $I(G/G^p; U)$, as can be seen in the lemma by taking the higher α's to be 0. Let S be another minimal set of generators of the ideal. If either set is infinite, then it is a simple argument that they have the same cardinality, so we may assume that they are finite. But then the lemma allows a Steinitz exchange argument to show that μ does not exceed the cardinality of S. In the argument, if $1 - u_2, \ldots, 1 - u_n$ have already replaced generators from S and the remaining generators from S are the β's, then the lemma allows us to assume that α_{n+1} has nonzero augmentation. Hence it is a unit and β_1 may be replaced by $1 - u_1$. Consequently, μ is the minimum number of generators of the ideal. □

Applying the proposition to p-groups of cardinality $\aleph_1$, we conclude that isomorphism of the group algebras implies that the groups are quotient equivalent.

Corollary 1. *Let G be a p-group of cardinality $\aleph_1$ and H a group such that $FG \cong FH$. Then G and H are quotient equivalent p-groups.*

Proof. By the remarks after Lemma 2, we know that H must be a p-group. Its cardinality must be $\aleph_1$ from the dimension of the group algebra. Via the isomorphism we may write $FG = FH$. Choose filtrations of G and H by countable subgroups, say $G = \bigcup_{i<\aleph_1} G_i$ and $H = \bigcup_{i<\aleph_1} H_i$. Let C be the subset of $\aleph_1$ on which $FG_i = FH_i$. Then C is clearly closed, and is unbounded by a countable back-and-forth argument. If $i, j \in C$ with $i < j$, then Lemma 2 implies that $F(G_j/G_i) \cong F(H_j/H_i)$, hence the equality of Ulm invariants and countability allow us to conclude that the quotients are isomorphic by Ulm's theorem [Fu, Theorem 77.3]. $\square$

An important fact about the group of normalized units $V(G)$ when G is separable and the cardinality of G does not exceed $\aleph_1$ is that $V(G)$ is the direct product of G and a direct sum of cyclic groups. We show this after the following lemma.

Lemma 4 (May [M])**.** *Let F be a perfect field of characteristic p and let G be a p-group with subgroups B and S.*

(1) *If $\alpha \in V(S)$, then the coset of α modulo $GV(B)$ has a proper element whose p-height in $V(G)$ is the p-height of some element of S.*
(2) *If B is pure in G, then $GV(B)$ is pure in $V(G)$.*

Proof. (1) We may assume that $\alpha \notin GV(B)$. The augmentation of α must be nonzero on some coset of $B \cap S$, so suppose it is r on the coset of s. Put $\alpha_1 = s^{-1}\alpha$. Then $\alpha_1 = s\beta + \gamma$, where $\beta \in V(B \cap S)$ and γ has support in $S \setminus B$. Put $\alpha_2 = \beta^{-1}\alpha_1$. Then $\alpha_2 = se + \gamma_1$, where γ_1 has support in $S \setminus B$ and the p-height of α_2 equals the p-height of γ_1. Note that α_2 is in the same coset as α modulo $GV(B)$ and has the p-height of some element of S. It will suffice to show that α_2 is a proper element.

Let $g\beta \in GV(B)$ $(g \in G, \beta \in V(B))$. Then $g\beta\alpha_2 = sg\beta + g\beta\gamma_1$. The supports of $sg\beta$ and $g\beta\gamma$ are clearly disjoint; thus if the p-heights of $g\beta$ and α_2 are equal, then the p-height of $g\beta\alpha_2$ cannot exceed that of $g\beta$, that is, the p-height of α_2. Thus, α_2 is a proper element.

(2) Let $g\beta \in V(G)^{p^k}$, where $g \in G$ and $\beta = \sum_{1 \le i \le n} r_i b_i \in V(B)$. Then we have $r_i = \widehat{r}_i^{\,p^k}$ and $gb_i \in G^{p^k}$ for $1 \le i \le n$. Thus, $gb_1 = \widehat{g}^{p^k}$ for some $\widehat{g} \in G$ and $b_1^{-1}b_i = \widehat{b}_i^{\,p^k}$ for some $\widehat{b}_i \in B$ $(1 \le i \le n)$, by purity. Since F is a characteristic p field, $\sum_{1 \le i \le n} \widehat{r}_i = 1$, thus the element $\widehat{\beta} = \sum_{1 \le i \le n} \widehat{r}_i \widehat{b}_i$ lies in $V(B)$. Consequently, $g\beta = \sum_{1 \le i \le n} r_i g b_i = gb_1 \sum_{1 \le i \le n} \widehat{r}_i^{\,p^k} \widehat{b}_i^{\,p^k} = (\widehat{g}\widehat{\beta})^{p^k}$, finishing the proof. $\square$

Proposition 2 (May [M])**.** *Let F be a field of characteristic p and let G be a separable p-group of cardinality not exceeding $\aleph_1$. Then there is a direct sum of cyclic p-groups B such that $V(G) = G \times B$.*

Proof. First suppose that F is perfect. We may write $G = \bigcup_{\alpha<\omega_1} G_\alpha$ as the union of a filtration of countable pure subgroups. Since G is pure in $V(G)$ by Lemma 4, it suffices to show that $V(G)/G$ is a direct sum of cyclic groups. Putting $V_\alpha = V(G_\alpha)$, $\{GV_\alpha | \alpha < \omega_1\}$ is a continuous chain of subgroups from G to $V(G)$ with GV_α pure in $GV_{\alpha+1}$ by Lemma 4. It therefore suffices to show that $GV_{\alpha+1}/GV_\alpha$ is a direct sum of cyclic groups. We may express $G_{\alpha+1} = \bigcup_{i<\omega} S_i$, where $\{S_i | i < \omega\}$ is a filtration by finite groups, thus $V_{\alpha+1} = \bigcup_{i<\omega} V(S_i)$. Consequently, $GV_{\alpha+1}/GV_\alpha$ is the union of the images of the $V(S_i)$. By Lemma 4, the p-heights of nonidentity elements in the image of $V(S_i)$ are finite, hence the quotient group is a direct sum of cyclic groups by Kulikov's theorem [Fu, Theorem 17.1].

Now suppose that F is not perfect, and let $\widehat{F}$ be an extension field which is perfect (e.g., an algebraic closure). We then have a projection $V(\widehat{F}G) \to G$ with a kernel which is a direct sum of cyclic groups. But then this restricts to a projection $V(FG) \to G$ with the same result. □

As a corollary, we apply the proposition to $\aleph_1$-separable p-groups of cardinality $\aleph_1$. We recall that $\aleph_1$-separable means that every countable subgroup is contained in a countable direct summand which is a direct sum of cyclics.

Corollary 2. *If G and H are $\aleph_1$-separable p-groups of cardinality $\aleph_1$ such that $FG \cong FH$, then G and H have the same Γ-invariant.*

Proof. Since $V(G) \cong V(H)$, we have $G \times B \cong H \times C$ for B and C direct sums of cyclic groups. We may further assume that the cardinality of B and C do not exceed $\aleph_1$. It now follows that G and H have the same Γ-invariant. □

2. Proof of the theorem

We assume that G is an $\aleph_1$-separable p-group of cardinality $\aleph_1$, and that H is an abelian group such that $FG \cong FH$. By the discussion following Lemma 2, we may assume that $V(G) = V(H)$ and that H is a p-group. By Proposition 2, there are direct sums of cyclic p-groups B and C such that $G \times B = H \times C$. Thus, H is an $\aleph_1$-separable p-group whose cardinality is $\aleph_1$ from the dimension of the group algebra.

We shall use the following special case of a cancellation theorem for direct sums of countable p-groups proved in [GM]. Let us say that the Ulm-Kaplansky p-invariants of two groups are disjoint if whenever an invariant of one group is nonzero, then the corresponding invariant of the other group must be zero.

Proposition 3. *Let $M \times A_1 = N \times A_2$, where $M \cong N$ are direct sums of cyclic p-groups and A_1 and A_2 are arbitrary. Assume that the Ulm-Kaplansky p-invariants of M are disjoint from those of A_1 and A_2. Then there exists a subgroup L such that $L \times A_1 = L \times A_2$. In particular, $A_1 \cong A_2$.*

In $G \times B = H \times C$, by passing to the subgroup generated by G and H we may assume that B and C have cardinality not exceeding $\aleph_1$. Let I be the set of all finite ordinals at which the Ulm-Kaplansky invariant of G is $\aleph_1$. We know that the Ulm invariants of G and H are the same by Proposition 1. If I is a finite set, then G and H are direct sums of cyclic groups, thus equal Ulm invariants imply that they are isomorphic. Consequently, we may assume that I is an infinite set.

We wish to form four decompositions, $G = G_0 \times G_1, H = H_0 \times H_1, B = B_0 \times B_1$, and $C = C_0 \times C_1$, where subscript 1 indicates the nonzero Ulm invariants occur at ordinals in I, and subscript 0 indicates the nonzero Ulm invariants occur at ordinals outside of I. It is trivial that this can be done for B and C. For G and H, the Ulm invariants outside of I are countable, hence one can find a summand which is a direct sum of cyclics and contains that part of a basic subgroup representing precisely those Ulm invariants. If there are any cyclic summands corresponding to ordinals from I, they can be moved into the subscript 1 part. Thus we obtain direct sums of cyclics G_0 and H_0.

We have that $G_0 \times B_0 \cong H_0 \times C_0$ since they are direct sums of cyclics with equal Ulm invariants. Since the factors indexed by 0 and 1 have disjoint Ulm invariants, Proposition 3 implies that $G_1 \times B_1 \cong H_1 \times C_1$. If we now take the direct product of each side with a direct sum of cyclics which have nonzero Ulm invariants of $\aleph_1$ at the ordinals in I, then we may assume that B_1 and C_1 have similar Ulm invariants. Now assume Martin's Axiom and negation of the Continuum Hypothesis. The result of Eklof and Mekler [EM1, Theorem 2.2] can be applied after examining the proof of that theorem to verify that the direct sum of cyclics which we now have will suffice in that proof. Note that G_1 and H_1 have final rank $\aleph_1$ since I is infinite. Thus, $G_1 \cong G_1 \times B_1$ and $H_1 \cong H_1 \times C_1$. Consequently, G and H are isomorphic and the theorem is proved.

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