MODULAR GROUP ALGEBRAS
OF \( \aleph_1 \)-SEPARABLE \( p \)-GROUPS

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Abstract. Under the assumptions of MA and \( \neg \text{CH} \), it is proved that if \( F \) is a field of prime characteristic \( p \) and \( G \) is an \( \aleph_1 \)-separable abelian \( p \)-group of cardinality \( \aleph_1 \), then an isomorphism of the group algebras \( FG \) and \( FH \) implies an isomorphism of \( G \) and \( H \).

In describing the unit groups of abelian group algebras, the algebras of separable \( p \)-groups over fields of characteristic \( p \) represent a crucial case. The well-known isomorphism and direct factor problems remain largely unresolved for this class once one gets beyond the case of totally projective \( p \)-groups. We wish to consider the isomorphism problem for \( \aleph_1 \)-separable \( p \)-groups of cardinality \( \aleph_1 \). In this setting, we show that isomorphism of the group algebras implies that the groups have equal \( \Gamma \)-invariants and are quotient equivalent; see [EM2] for the notions of \( \Gamma \)-invariant and quotient equivalence. It is known that this is not enough to guarantee isomorphism of the groups, but for cardinality \( \aleph_1 \) it is known that the groups are direct factors of the normalized unit groups with complements which are totally projective. Utilizing this together with a result of the authors [GM] on cancellation in direct sums (see Proposition 3) and a theorem of Eklof and Mekler [EM], we prove the following theorem.

Theorem (MA + \( \neg \text{CH} \)). Let \( F \) be a field of prime characteristic \( p \) and let \( FG \) and \( FH \) be isomorphic group algebras. Assume that \( G \) is an \( \aleph_1 \)-separable abelian \( p \)-group of cardinality \( \aleph_1 \). Then \( G \) and \( H \) are isomorphic.

To make the paper more nearly self-contained, we shall give a brief introduction to some basic facts on abelian group algebras.

1. Preliminaries

In this paper, \( F \) will always denote a field of characteristic \( p \) and all groups will be abelian. For background on \( \aleph_1 \)-separable \( p \)-groups, see [EM1, H]. Groups will be written multiplicatively as is customary when discussing group algebras. The augmentation, \( \text{aug}(\alpha) \), of an element \( \alpha \in FG \) is the sum of the coefficients of \( \alpha \), and the augmentation map \( \text{aug} : FG \to F \) \( (\alpha \mapsto \text{aug}(\alpha)) \) is a ring homomorphism. The group of units which have augmentation 1 is called the group of normalized units of \( FG \) and is denoted by \( V(FG) \), or simply \( V(G) \). If \( G \) is a \( p \)-group, then \( V(G) \) is a \( p \)-group and consists precisely of all elements of augmentation 1.
Lemma 1. For \( \Phi \) as above, \( I(G;G_1) \) is the ideal of \( FG \) generated by all elements of the form \( 1-g \) for \( g \in G_1 \). In fact, it is only necessary that \( g \) range over a set of generators of \( G_1 \).

Proof. The ideal described is certainly contained in the kernel of \( \Phi \). Let \( \{g_i|i<\mu\} \) be a set of coset representatives of \( G \) modulo \( G_1 \). Then the kernel clearly consists of all (finite) sums \( \sum_{i<\mu} g_i \beta_i \), where \( \beta_i \in I(G_1) \). But if \( \beta = \sum_{g \in G_1} r_g g \) has augmentation 0, then \( \beta = \sum_{g \in G_1} r_g (g-1) \), hence the kernel is contained in the ideal. The remark about generators follows from the simple observation that \( 1-xy = (1-x)y + (1-y) \).

Now let \( \Phi : FG \to FH \) be a homomorphism preserving augmentation and suppose that \( G_1 \) and \( H_1 \) are subgroups of \( G \) and \( H \), respectively, such that \( \Phi(G_1) \subseteq FH_1 \). Regarding \( I(G;G_1) \) as the kernel of the homomorphism induced by the natural homomorphism from \( G \) to \( G/G_1 \), and similarly for \( I(H;H_1) \), then Lemma 1 implies that there is a natural homomorphism \( \overline{\Phi} : F(G/G_1) \to F(H/H_1) \) such that we have a commutative diagram

\[
\begin{array}{ccc}
FG & \to & FH \\
\downarrow & & \downarrow \\
F(G/G_1) & \to & F(H/H_1)
\end{array}
\]

where the vertical maps are the natural ones. For use in the proof of Proposition 2 note that it is not strictly necessary for \( \Phi \) to preserve augmentation since the argument goes through if \( \Phi \) simply preserves augmentation 0. The following lemma is immediate.

Lemma 2. Let \( \Phi : FG \to FH \) be an isomorphism taking \( FG_1 \) to \( FH_1 \). Then there is an induced isomorphism \( \overline{\Phi} : F(G/G_1) \to F(H/H_1) \).

If one has a homomorphism \( \Phi : FG \to FH \) which does not preserve augmentation, then it can be “adjusted” to do so by an automorphism of \( FG \). Specifically, precede \( \Phi \) by the automorphism of \( FG \) given by \( g \mapsto \text{aug}(\Phi(g))^{-1}g \). Thus, if an isomorphism exists which does not preserve augmentation, then one exists which does preserve it. This has the following consequence. Suppose that \( G \) is a p-group and that \( FG \cong FH \) for some group \( H \). Since we may assume that the isomorphism preserves augmentation, we have \( V(G) \cong V(H) \). But we have observed that \( V(G) \) is a p-group, hence we may conclude that \( H \) is also a p-group.

To discuss p-height in group algebras, we shall assume that the field \( F \) is perfect. If \( G \) is a group, define \( G_\alpha \) for ordinals \( \alpha \) in the usual way: if \( \alpha = \beta + 1 \), then \( G_\alpha = G_\beta^p \), and if \( \alpha \) is a limit, then \( G_\alpha = \bigcap_{\beta<\alpha} G_\beta \). Define subalgebras \( (FG)_\alpha \) of \( FG \) in the same fashion, taking either \( (FG)_\alpha^p \), or \( \bigcap_{\beta<\alpha} (FG)_\beta \). It is easy to see that
(FG)α = F(Gα) since, for a perfect field, (FG)p = F(Gp). It is further clear that a homomorphism Φ : FG → FH will satisfy Φ(FGα) ⊆ FHα for every ordinal α.

To show the well-known invariance of Ulm invariants under isomorphism of group algebras, we need a lemma.

**Lemma 3.** Let A = ⟨u1⟩ × · · · × ⟨un⟩ × A1, where u1 is a nontrivial p-torsion element, let α2,...,αn+m ∈ FA, and let β1,...,βm ∈ I(A). Suppose that

\[ 1 - u_1 = \alpha_2(1 - u_2) + \cdots + \alpha_n(1 - u_n) + \alpha_{n+1}\beta_1 + \cdots + \alpha_{n+m}\beta_m. \]

Then for some i with n + 1 ≤ i ≤ n + m, αi has nonzero augmentation.

**Proof.** Suppose to the contrary that αn+1,...,αn+m ∈ I(A). Using the given decomposition of A, project onto ⟨u1⟩ and consider the induced projection of FA onto F⟨u1⟩. Let I1 denote the augmentation ideal of F⟨u1⟩. Since I(A) projects into I1, we conclude that 1 − u1 ∈ I12. But since Lemma 1 implies that I1 is a principal ideal generated by 1 − u1, we have 1 − u1 = γ(1 − u1)2 for some γ ∈ FA.

If pm is the order of u1, then we have \( 1 - u_1 = \gamma p^{m-1}(1 - u_1)p^m = \gamma p^{m-1}(1 - u_1) = 0 \), which is a contradiction. □

**Proposition 1 (Berman [3], Berman and Mollov [BM], May [M1]).** If G and H are abelian groups such that FG ≅ FH, then the Ulm-Kaplansky invariants of G and H for the prime p are equal.

**Proof.** We may assume that the isomorphism preserves augmentation. By taking an extension field of F which is perfect and tensoring the isomorphism over F, we may further assume that F itself is perfect. Since the isomorphism carries FGα to FHα for every ordinal α, it suffices to show the 0-th Ulm invariants are equal.

The p-th power endomorphism of G maps Gp into Gp2, thus inducing a homomorphism \( \phi : G/Gp \to G/Gp^2 \). Let \( \Phi : F(G/Gp) \to F(G/Gp^2) \) be induced by \( \phi \). If \( U \) denotes the kernel of \( \phi \), then \( U \) is an elementary p-group and the 0-th Ulm invariant of G is the rank of \( U \). The kernel of \( \Phi \) is the ideal \( I(G/Gp;U) \). Now consider the p-th power endomorphism of FG. This carries \( F(Gp) \) to \( F(Gp^2) \) and preserves augmentation 0, thus inducing a homomorphism \( \Psi : F(G/Gp) \to F(G/Gp^2) \) by the remark before Lemma 2. The kernel of \( \Psi \) is also \( I(G/Gp;U) \) since a sum of p-th powers of coefficients from F is 0 if and only if the sum of coefficients is 0. If we carry out a similar construction for FH, then the naturality of \( \Psi \) together with the isomorphism of FG with FH guarantees that we will have an isomorphism of \( F(G/Gp) \) with \( F(H/Hp) \) that relates the corresponding ideals. Thus, it will suffice to show that the rank of \( U \) is equal to the minimum number of ideal generators of \( I(G/Gp;U) \).

Let \( U = \bigcap_{\mu < p} \langle u_i \rangle \), where each \( u_i \) has order p. Then \{1 − u_i| i < μ\} is a minimal set of generators of the ideal \( I(G/Gp;U) \), as can be seen in the lemma by taking the higher α’s to be 0. Let \( S \) be another minimal set of generators of the ideal. If either set is infinite, then it is a simple argument that they have the same cardinality, so we may assume that they are finite. But then the lemma allows a Steinitz exchange argument to show that \( \mu \) does not exceed the cardinality of \( S \). In the argument, if 1 − u2,..., 1 − un have already replaced generators from \( S \) and the remaining generators from \( S \) are the β’s, then the lemma allows us to assume that \( \alpha_{n+1} \) has nonzero augmentation. Hence it is a unit and \( \beta_1 \) may be replaced by 1 − u1. Consequently, \( \mu \) is the minimum number of generators of the ideal. □
Applying the proposition to \( p \)-groups of cardinality \( \aleph_1 \), we conclude that isomorphism of the group algebras implies that the groups are quotient equivalent.

**Corollary 1.** Let \( G \) be a \( p \)-group of cardinality \( \aleph_1 \) and \( H \) a group such that \( FG \cong FH \). Then \( G \) and \( H \) are quotient equivalent \( p \)-groups.

**Proof.** By the remarks after Lemma \( \mathbb{2} \) we know that \( H \) must be a \( p \)-group. Its cardinality must be \( \aleph_1 \) from the dimension of the group algebra. Via the isomorphism we may write \( FG = FH \). Choose filtrations of \( G \) and \( H \) by countable subgroups, say \( G = \bigcup_{\alpha < \aleph_1} G_\alpha \) and \( H = \bigcup_{\beta < \aleph_1} H_\beta \). Let \( C \) be the subset of \( \aleph_1 \) on which \( FG_\alpha = FH_\beta \). Then \( C \) is clearly closed, and is unbounded by a countable back-and-forth argument. If \( i, j \in C \) with \( i < j \), then Lemma \( \mathbb{2} \) implies that \( F(G_j/G_i) \cong F(H_j/H_i) \), hence the equality of Ulm invariants and countability allow us to conclude that the quotients are isomorphic by Ulm’s theorem \([\mathbb{F}_1 \text{ Theorem 77.3}].\)

\( \square \)

An important fact about the group of normalized units \( V(G) \) when \( G \) is separable and the cardinality of \( G \) does not exceed \( \aleph_1 \) is that \( V(G) \) is the direct product of \( G \) and a direct sum of cyclic groups. We show this after the following lemma.

**Lemma 4 (May \( \mathbb{M} \)).** Let \( F \) be a perfect field of characteristic \( p \) and let \( G \) be a \( p \)-group with subgroups \( B \) and \( S \).

1. If \( \alpha \in V(S) \), then the coset of \( \alpha \) modulo \( GV(B) \) has a proper element whose \( p \)-height in \( V(G) \) is the \( p \)-height of some element of \( S \).
2. If \( B \) is pure in \( G \), then \( GV(B) \) is pure in \( V(G) \).

**Proof.** (1) We may assume that \( \alpha \notin GV(B) \). The augmentation of \( \alpha \) must be nonzero on some coset of \( B \cap S \), so suppose it is \( r \) on the coset of \( s \). Put \( \alpha_1 = s^{-1} \alpha \). Then \( \alpha_1 = s \beta + \gamma \), where \( \beta \in V(B \cap S) \) and \( \gamma \) has support in \( S \setminus B \). Put \( \alpha_2 = \beta^{-1} \alpha_1 \). Then \( \alpha_2 = s \gamma + \gamma_1 \), where \( \gamma_1 \) has support in \( S \setminus B \) and the \( p \)-height of \( \alpha_2 \) equals the \( p \)-height of \( \gamma_1 \). Note that \( \alpha_2 \) is in the same coset as \( \alpha \) modulo \( GV(B) \) and has the \( p \)-height of some element of \( S \). It will suffice to show that \( \alpha_2 \) is a proper element.

Let \( g \beta \in GV(B) \) \( (g \in G, \beta \in V(B)) \). Then \( g \beta \alpha_2 = sg \beta + g \beta \gamma_1 \). The supports of \( sg \beta \) and \( g \beta \gamma_1 \) are clearly disjoint; thus if the \( p \)-heights of \( sg \beta \) and \( \alpha_2 \) are equal, then the \( p \)-height of \( g \beta \alpha_2 \) cannot exceed that of \( g \beta \), that is, the \( p \)-height of \( \alpha_2 \). Thus, \( \alpha_2 \) is a proper element.

(2) Let \( g \beta \in V(G)^p \), where \( g \in G \) and \( \beta = \sum_{1 \leq i \leq n} r_i b_i \in V(B) \). Then we have \( r_i = \hat{r}_i^p \) and \( gb_i \in G^p \) for \( 1 \leq i \leq n \). Thus, \( gb_i = \hat{g}^p b_i \) for some \( \hat{g} \in G \) and \( b_i^{-1} = \hat{b}_i^p \) for some \( \hat{b}_i \in B \) \((1 \leq i \leq n)\), by purity. Since \( F \) is a characteristic \( p \)-field, \( \sum_{1 \leq i \leq n} \hat{r}_i = 1 \), thus the element \( \beta = \sum_{1 \leq i \leq n} \hat{r}_i \hat{b}_i \) lies in \( V(B) \). Consequently, \( g \beta = \sum_{1 \leq i \leq n} r_i gb_i = gb_i \sum_{1 \leq i \leq n} \hat{r}_i \hat{b}_i = (g \beta)^p \), finishing the proof. \( \square \)

**Proposition 2 (May \( \mathbb{M} \)).** Let \( F \) be a field of characteristic \( p \) and let \( G \) be a separable \( p \)-group of cardinality not exceeding \( \aleph_1 \). Then there is a direct sum of cyclic \( p \)-groups \( B \) such that \( V(G) = G \times B \).

**Proof.** First suppose that \( F \) is perfect. We may write \( G = \bigcup_{\alpha < \omega_1} G_\alpha \) as the union of a filtration of countable pure subgroups. Since \( G \) is pure in \( V(G) \) by Lemma \( \mathbb{4} \) it suffices to show that \( V(G)/G \) is a direct sum of cyclic groups. Putting \( V_\alpha = V(G_\alpha) \), \( \{GV_\alpha | \alpha < \omega_1 \} \) is a continuous chain of subgroups from \( G \) to \( V(G) \) with \( GV_\alpha \) pure in \( GV_{\alpha+1} \) by Lemma \( \mathbb{4} \). It therefore suffices to show that \( GV_{\alpha+1}/GV_\alpha \) is a direct sum of cyclic groups. We may express \( G_{\alpha+1} = \bigcup_{i < \omega} S_i \), where \( \{S_i | i < \omega \} \) is a
filtration by finite groups, thus \( V_{n+1} = \bigcup_{i < \omega} V(S_i) \). Consequently, \( GV_{n+1}/GV_n \) is the union of the images of the \( V(S_i) \). By Lemma 4, the \( p \)-heights of nonidentity elements in the image of \( V(S_i) \) are finite, hence the quotient group is a direct sum of cyclic groups by Kulikov's theorem \([11, Theorem 17.1]\).

Now suppose that \( F \) is not perfect, and let \( \hat{F} \) be an extension field which is perfect (e.g., an algebraic closure). We then have a projection \( V(\hat{F}G) \to G \) with a kernel which is a direct sum of cyclic groups. But then this restricts to a projection \( V(FG) \to G \) with the same result. \( \square \)

As a corollary, we apply the proposition to \( \aleph_1 \)-separable \( p \)-groups of cardinality \( \aleph_1 \). We recall that \( \aleph_1 \)-separable means that every countable subgroup is contained in a countable direct summand which is a direct sum of cyclics.

**Corollary 2.** If \( G \) and \( H \) are \( \aleph_1 \)-separable \( p \)-groups of cardinality \( \aleph_1 \) such that \( FG \cong FH \), then \( G \) and \( H \) have the same \( \Gamma \)-invariant.

**Proof.** Since \( V(G) \cong V(H) \), we have \( G \times B \cong H \times C \) for \( B \) and \( C \) direct sums of cyclic groups. We may further assume that the cardinality of \( B \) and \( C \) do not exceed \( \aleph_1 \). It now follows that \( G \) and \( H \) have the same \( \Gamma \)-invariant. \( \square \)

## 2. Proof of the theorem

We assume that \( G \) is an \( \aleph_1 \)-separable \( p \)-group of cardinality \( \aleph_1 \), and that \( H \) is an abelian group such that \( FG \cong FH \). By the discussion following Lemma 2, we may assume that \( V(G) = V(H) \) and that \( H \) is a \( p \)-group. By Proposition 2 there are direct sums of cyclic \( p \)-groups \( B \) and \( C \) such that \( G \times B = H \times C \). Thus, \( H \) is an \( \aleph_1 \)-separable \( p \)-group whose cardinality is \( \aleph_1 \) from the dimension of the group algebra.

We shall use the following special case of a cancellation theorem for direct sums of countable \( p \)-groups proved in \([16]\). Let us say that the Ulm-Kaplansky \( p \)-invariants of two groups are disjoint if whenever an invariant of one group is nonzero, then the corresponding invariant of the other group must be zero.

**Proposition 3.** Let \( M \times A_1 = N \times A_2 \), where \( M \cong N \) are direct sums of cyclic \( p \)-groups and \( A_1 \) and \( A_2 \) are arbitrary. Assume that the Ulm-Kaplansky \( p \)-invariants of \( M \) are disjoint from those of \( A_1 \) and \( A_2 \). Then there exists a subgroup \( L \) such that \( L \times A_1 = L \times A_2 \). In particular, \( A_1 \cong A_2 \).

In \( G \times B = H \times C \), by passing to the subgroup generated by \( G \) and \( H \) we may assume that \( B \) and \( C \) have cardinality not exceeding \( \aleph_1 \). Let \( I \) be the set of all finite ordinals at which the Ulm-Kaplansky invariant of \( G \) is \( \aleph_1 \). We know that the Ulm invariants of \( G \) and \( H \) are the same by Proposition 1. If \( I \) is a finite set, then \( G \) and \( H \) are direct sums of cyclic groups, thus equal Ulm invariants imply that they are isomorphic. Consequently, we may assume that \( I \) is an infinite set.

We wish to form four decompositions, \( G = G_0 \times G_1, H = H_0 \times H_1, B = B_0 \times B_1 \), and \( C = C_0 \times C_1 \), where subscript 1 indicates the nonzero Ulm invariants occur at ordinals in \( I \), and subscript 0 indicates the nonzero Ulm invariants occur at ordinals outside of \( I \). It is trivial that this can be done for \( B \) and \( C \). For \( G \) and \( H \), the Ulm invariants outside of \( I \) are countable, hence one can find a summand which is a direct sum of cyclics and contains that part of a basic subgroup representing precisely those Ulm invariants. If there are any cyclic summands corresponding to
ordinals from $I$, they can be moved into the subscript 1 part. Thus we obtain direct sums of cyclics $G_0$ and $H_0$.

We have that $G_0 \times B_0 \cong H_0 \times C_0$ since they are direct sums of cyclics with equal Ulm invariants. Since the factors indexed by 0 and 1 have disjoint Ulm invariants, Proposition 3 implies that $G_1 \times B_1 \cong H_1 \times C_1$. If we now take the direct product of each side with a direct sum of cyclics which have nonzero Ulm invariants of $N_1$ at the ordinals in $I$, then we may assume that $B_1$ and $C_1$ have similar Ulm invariants. Now assume Martin’s Axiom and negation of the Continuum Hypothesis. The result of Eklof and Mekler [EM1, Theorem 2.2] can be applied after examining the proof of that theorem to verify that the direct sum of cyclics which we now have will suffice in that proof. Note that $G_1$ and $H_1$ have final rank $N_1$ since $I$ is infinite. Thus, $G_1 \cong G_1 \times B_1$ and $H_1 \cong H_1 \times C_1$. Consequently, $G$ and $H$ are isomorphic and the theorem is proved.

References


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