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DISCRETE SPECTRA OF C^*-ALGEBRAS AND COMPLEMENTED SUBMODULES IN HILBERT C^*-MODULES

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Abstract. Let A be a C^*-algebra and let X be a full (right) Hilbert A-module. It is shown that if the spectrum $\widehat{A}$ of A is discrete, then every closed $\mathcal{K}(X)$-A-submodule of X is complemented in X, and conversely that if $\widehat{A}$ is a T_1-space and if every closed $\mathcal{K}(X)$-A-submodule of X is complemented in X, then $\widehat{A}$ is discrete.

1. Introduction

Hilbert C^*-modules first appeared in the work of Kaplansky [3]. His idea was to generalize Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. In the sequel, about twenty years later, an outstanding work in which Hilbert C^*-modules played a crucial role in the use of them was Rieffel's work [10] on induced representations of C^*-algebras. Now Hilbert C^*-modules are getting to become a more important, standard tool in various research areas in C^*-algebras, for example, KK-theory and Morita equivalence (see [2] for KK-theory, [7] for the C^*-algebraic approach to quantum groups, and [9] for Morita equivalence).

Although a Hilbert C^*-module is a generalization of a Hilbert space, we cannot, a priori, expect that Hilbert C^*-modules behave like Hilbert spaces, and in some ways they do. In fact, as is mentioned below, there is an essential way in which Hilbert C^*-modules differ from Hilbert spaces. Let X be a Hilbert C^*-module equipped with inner product $\langle \cdot\ ,\ \cdot \rangle$ taking values in the C^*-algebra and let Y be a closed subspace Y of X. We denote by $Y^\perp$ the orthogonally complemented subspace of Y in X, i.e.,

$$Y^\perp = \{\ x \in X \mid \langle x\ ,\ y \rangle = 0 \text{ for all } y \in Y\ \}.$$

Then $Y^\perp$ is also a closed subspace of X, and $(Y^\perp)^\perp$ is usually larger than Y. In general, X is not equal to the direct sum $Y \oplus Y^\perp$ of Y and $Y^\perp$. We say that a closed subspace Y of a Hilbert C^*-module X is *complemented* in X if X coincides with $Y \oplus Y^\perp$. Throughout this paper, by a complemented subspace of X we mean one that is *orthogonally* complemented. Since the whole theory of Hilbert spaces and their linear operators is based on the use of the orthogonally complemented

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subspaces, it is clear that there will be obstacles in developing a theory of Hilbert C^*-modules analogous to that of Hilbert spaces. Hence, when it is necessary to obtain an analogous theory which works for Hilbert C^*-modules, what we must first know is when *every* closed submodule of a Hilbert C^*-module is complemented. At present, as for related results, there are only a few results on what kind of closed submodule is complemented in a Hilbert C^*-module (see, for example, [7, Chapter 3]). As far as we know, however, there are no answers to the problem of when every closed submodule of a Hilbert C^*-module is complemented. The aim of this paper is to provide a few sufficiently satisfactory answers to such a problem.

On the other hand, when some property in C^*-algebras A and B is preserved under Morita equivalence, it is natural to expect that an A-B-imprimitivity bimodule X must contain sufficient information on that property of both A and B (for example, [5, Theorem 2.3], [6, Theorem 3.3]). Hence it would be very interesting to attempt to characterize X in terms of some property in both A and B. In fact, in connection with the problem mentioned in the preceding paragraph, for a *full* (right) Hilbert C^*-module X over a C^*-algebra A, we show that if the spectrum $\widehat{A}$ of A is discrete in the Jacobson topology, then every closed $\mathcal{K}(X)$-A-submodule of X is complemented, and conversely that if $\widehat{A}$ is a T_1-space and if every closed $\mathcal{K}(X)$-A-submodule of X is complemented in X, then $\widehat{A}$ is discrete in the Jacobson topology.

2. Results

Recall the definition of a Hilbert C^*-module. Let A be a C^*-algebra. By a *left Hilbert A-module*, we mean a left A-module X equipped with an A-valued pairing $\langle\cdot\ ,\ \cdot\rangle$, called an A-valued inner product, satisfying the following conditions:

(a) $\langle\cdot\ ,\ \cdot\rangle$ is sesquilinear. (We make the convention that $\langle\cdot\ ,\ \cdot\rangle$ is linear in the first variable and is conjugate-linear in the second variable.)

(b) $\langle x\ ,\ y\rangle = \langle y\ ,\ x\rangle^*$ for all $x, y \in X$.

(c) $\langle ax\ ,\ y\rangle = a\langle x\ ,\ y\rangle$ for all $x, y \in X$ and $a \in A$.

(d) $\langle x\ ,\ x\rangle \geq 0$ for all $x \in X$, and $\langle x\ ,\ x\rangle = 0$ implies that $x = 0$.

(e) X is complete with respect to the norm $\|x\| = \|\langle x\ ,\ x\rangle\|^{\frac{1}{2}}$.

We remark that the Hilbert A-module is always assumed to be a vector space over the field of complex numbers. Furthermore, X is said to be *full* if X satisfies an additional condition:

(f) the closed linear span of $\{\langle x\ ,\ y\rangle \mid x, y \in X\}$ coincides with A.

Let B be a C^*-algebra. Right Hilbert B-modules are defined similarly except that we require that B should act on the right of X, that $\langle\cdot\ ,\ \cdot\rangle$ should be conjugate-linear in the first variable, and that $\langle x\ ,\ yb\rangle = \langle x\ ,\ y\rangle b$ for all $x, y \in X$ and $b \in B$.

Let A and B be C^*-algebras. We denote by ${}_A\langle\cdot\ ,\ \cdot\rangle$ the A-valued inner product on the left Hilbert A-module and by $\langle\cdot\ ,\ \cdot\rangle_B$ the B-valued inner product on the right Hilbert B-module, respectively. By an *A-B-imprimitivity bimodule* X, we mean a full left Hilbert A-module and full right Hilbert B-module X satisfying

(g) ${}_A\langle xb\ ,\ y\rangle = {}_A\langle x\ ,\ yb^*\rangle$ and $\langle ax,\ y\rangle_B = \langle x\ ,\ a^*y\rangle_B$ for all $x, y \in X, a \in A$ and $b \in B$;

(h) ${}_A\langle x\ ,\ y\rangle\cdot z = x\cdot\langle y\ ,\ z\rangle_B$ for all $x, y, z \in X$.

Throughout this paper, by an *A-B-Hilbert module* we mean a left Hilbert A-module and right Hilbert B-module.

Two C^*-algebras A and B are said to be *Morita equivalent* if there exists an A-B-imprimitivity bimodule. We remark that in this paper, Morita equivalence means strong Morita equivalence in the sense of Rieffel (cf. [9, Remark 3.15]). The reader is referred to [9] for Hilbert C^*-modules and Morita equivalence.

We denote by $\mathrm{Prim}(A)$ the primitive spectrum of A, that is, the set of primitive ideals of A equipped with the Jacobson topology, which is defined as the topology for which the class $\{\mathrm{hull}(I)|\ I \text{ is a closed ideal of } A\}$ form the closed sets, where

$$\mathrm{hull}(I) = \{t \in \mathrm{Prim}(A)|\ t \supset I\}$$

(see [8, 4.1] for details). The primitive spectrum is always locally compact and is a T_0-space. In general, however, it is not necessarily a T_1-space. We denote by $\widehat{A}$ the spectrum of A, that is, the set of (unitary) equivalence classes of nonzero irreducible representations of A equipped with the Jacobson topology, where the Jacobson topology on $\widehat{A}$ is defined as the topology for which the canonical map from $\widehat{A}$ onto $\mathrm{Prim}(A)$ is open and continuous. We note that $\widehat{A}$ is a locally compact space, not necessarily a T_0-space. We remark that $\widehat{A}$ is a T_0-space if and only if the canonical map from $\widehat{A}$ onto $\mathrm{Prim}(A)$ is injective. The reader is referred to [1], [8] for the spectrum of a C^*-algebra.

Lemma 2.1. *Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. If a closed A-B-submodule Y of X is complemented in X with respect to ${}_A\langle\cdot\ ,\ \cdot\rangle$ and with respect to $\langle\cdot\ ,\ \cdot\rangle_B$ respectively, then we have*

$$\begin{aligned} Y^\perp &= \{\ x \in X \mid {}_A\langle x\ ,\ y\rangle = 0 \textit{ for all } y \in Y\ \} \\ &= \{\ x \in X \mid \langle x\ ,\ y\rangle_B = 0 \textit{ for all } y \in Y\ \}. \end{aligned}$$

Proof. By symmetry, it suffices to show that if ${}_A\langle x\ ,\ y\rangle = 0$ for all $y \in Y$, then $\langle x\ ,\ y\rangle_B = 0$. Let $x = y_0 \oplus y'$ for $y_0 \in Y$ and $y' \in X$ with $\langle y'\ ,\ y\rangle_B = 0$ for all $y \in Y$. Then we have only to show $\langle y_0\ ,\ y\rangle_B = 0$. Since $\langle y'\ ,\ y\rangle_B = 0$, we have

$$\begin{aligned} &\langle y_0\ ,\ y\rangle_B{}^* \cdot \langle y_0\ ,\ y\rangle_B = \langle y_0\ ,\ y\rangle_B{}^* \cdot \langle y_0 + y'\ ,\ y\rangle_B = \langle y_0\ ,\ y\rangle_B{}^* \cdot \langle x\ ,\ y\rangle_B \\ &= \langle x \cdot \langle y_0\ ,\ y\rangle_B\ ,\ y\rangle_B = \langle {}_A\langle x\ ,\ y_0\rangle \cdot y\ ,\ y\rangle_B = \langle 0 \cdot y\ ,\ y\rangle_B = 0. \end{aligned}$$

Thus we see that $\langle y_0\ ,\ y\rangle_B = 0$, so that $\langle x\ ,\ y\rangle_B = \langle y_0\ ,\ y\rangle_B = 0$. □

Let A be a C^*-algebra and let X be a left Hilbert A-module or a right Hilbert A-module. When we do not need to specify exactly whether X is a left Hilbert A-module or a right one, we shall say that X is a *Hilbert C^*-module over A*, or, simply a *Hilbert A-module*.

Lemma 2.2. *Let A be a C^*-algebra and let X be a Hilbert C^*-module over A. For any subset Y of X, $Y^\perp$ is a closed A-submodule of X.*

Proof. Without loss of generality, we may assume that X is a left A-module with A-valued inner product $\langle\cdot\ ,\ \cdot\rangle$. It follows from continuity of $\langle\cdot\ ,\ \cdot\rangle$ that $Y^\perp$ is a closed subset, and it is routine to verify that $Y^\perp$ is a vector subspace of X. For $a \in A$, $x \in Y^\perp$ and $y \in Y$, we have $\langle ax\ ,\ y\rangle = a\langle x\ ,\ y\rangle = 0$, from which it follows that $AY^\perp \subset Y^\perp$. Thus $Y^\perp$ is a left A-submodule. □

Now we recall the Rieffel correspondence (see [9, Theorem 3.22]). Let two C^*-algebras A and B be Morita equivalent and let X be an A-B-imprimitivity bimodule. We denote by $\mathcal{I}(A)$ (resp. $\mathcal{I}(B)$) the set of all closed (two-sided) ideals of A

(resp. B), and by $\mathcal{S}(X)$ the set of closed A-B-submodules of X. Note that $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$ can be partially ordered by inclusion, and are then lattices. Then there are natural lattice isomorphisms among $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$ given by

$$\begin{gathered}\mathcal{I}(A) \ni I \longmapsto {}_I X \in \mathcal{S}(X), \text{ where } {}_I X = \{y \in X |_A \langle y\ ,\ x\rangle \in I \text{ for all } x \in X\ \} \\ \mathcal{S}(X) \ni Y \longmapsto I_Y \in \mathcal{I}(A),\ {}_Y I \in \mathcal{I}(B), \\ \text{where } I_Y = \overline{\text{span}}\{{}_A\langle y\ ,\ x\rangle \mid y \in Y \text{ and } x \in X\ \} \\ \text{and } {}_Y I = \overline{\text{span}}\{\langle x\ ,\ y\rangle_B \mid y \in Y \text{ and } x \in X\ \}; \\ \mathcal{I}(B) \ni J \longmapsto X_J \in \mathcal{S}(X), \text{ where } X_J = \{y \in X \mid \langle x\ ,\ y\rangle_B \in J \text{ for all } x \in X\ \}.\end{gathered}$$

We refer to such lattice isomorphisms among $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$ as the *Rieffel correspondences* (see [9, 3.3]).

Theorem 2.3. *Let two C^*-algebras A and B be Morita equivalent and let X be an A-B-imprimitivity bimodule. Consider the following conditions:*

(1) *The spectrum $\widehat{A}$ of A is discrete in the Jacobson topology.*

(2) *The spectrum $\widehat{B}$ of B is discrete in the Jacobson topology.*

(3) *Every closed A-B-submodule of X is complemented in X.*

Then we have (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). *If either $\widehat{A}$ or $\widehat{B}$ is a T_1-space, then conditions* (1) $-$ (3) *are equivalent.*

Proof. (1) $\Longleftrightarrow$ (2). Since $\widehat{A}$ and $\widehat{B}$ are homeomorphic by the Rieffel homeomorphism ([9, Corollary 3.33]), the equivalence of (1) and (2) is obvious.

(1) $\Longrightarrow$ (3). Let Y be any closed A-B-submodule of X. Put

$$Y^\perp = \{\ x \in X \mid {}_A\langle x\ ,\ y\rangle = \langle x\ ,\ y\rangle_B = 0 \text{ for all } y \in Y\ \}.$$

Then Lemma 2.2 shows that $Y^\perp$ is also a closed A-B-submodule of X. By the Rieffel correspondence ([9, Theorem 3.22]), there corresponds a closed ideal I_Y of A to Y. Then condition (1) yields that $A = I_Y \oplus (I_Y)^\perp$ for some closed ideal $(I_Y)^\perp$ of A with $I_Y(I_Y)^\perp = \{0\}$. In fact, let p be an open projection in the second dual A^{**} with $A^{**}p \cap A = I_Y$ ([8, 3.11.9]). Since $\widehat{A}$ is discrete, it hence follows from [4, Theorem 2.3] that p is a multiplier for A, so that $Ap = I_Y$. Put $q = 1 - p$. Then q is a multiplier for A, and so it is also an open projection ([8, 3.12.9, 3.11.9]). Hence we see that $Aq = (I_Y)^\perp$.

We claim that $(I_Y)^\perp = I_{Y^\perp}$. Take any ${}_A\langle y\ ,\ x\rangle \in I_{Y^\perp}$ with $y \in Y^\perp$ and $x \in X$. For any $x' \in X$ and $y' \in Y$, ${}_A\langle y'\ ,\ x'\rangle$ belongs to I_Y ([9, Theorem 3.22]) and ${}_A\langle y\ ,\ y'\rangle = \langle y\ ,\ y'\rangle_B = 0$ by the definition of $Y^\perp$ above. Then we have

$$\begin{aligned}{}_A\langle y\ ,\ x\rangle^*{}_A\langle y'\ ,\ x'\rangle &= {}_A\langle x\ ,\ y\rangle{}_A\langle y'\ ,\ x'\rangle = {}_A\langle {}_A\langle x\ ,\ y\rangle \cdot y'\ ,\ x'\rangle \\ &= {}_A\langle x \cdot \langle y\ ,\ y'\rangle_B\ ,\ x'\rangle = {}_A\langle x \cdot 0\ ,\ x'\rangle = 0,\end{aligned}$$

which shows that ${}_A\langle y\ ,\ x\rangle \in (I_Y)^\perp$. Since $I_{Y^\perp} = \overline{\text{span}}\{\ {}_A\langle y\ ,\ x\rangle \mid y \in Y^\perp \text{ and } x \in X\ \}$, we see that $I_{Y^\perp} \subset (I_Y)^\perp$. Next we must show that $(I_Y)^\perp \subset I_{Y^\perp}$. Let Y' be the closed A-B-submodule of X corresponding to the closed ideal $(I_Y)^\perp$ of A under the Rieffel correspondence. Then we see that $(I_Y)^\perp = I_{Y'}$ and that $Y' = \{y \in X|_A\langle y\ ,\ x\rangle \in (I_Y)^\perp \text{ for all } x \in X\ \}$ ([9, Theorem 3.22]). Thus $(I_Y)^\perp = I_{Y'} = \overline{\text{span}}\{{}_A\langle y'\ ,\ x\rangle \mid y' \in Y' \text{ and } x \in X\ \}$. Take any $y' \in Y'$ and fix it. We assert that $y' \in Y^\perp$. Take any $x, z \in X$ and any $y \in Y$. We then see that

${}_A\langle y' \ , \ x\rangle \in I_{Y'} = (I_Y)^\perp$ and ${}_A\langle y \ , \ z\rangle \in I_Y$. We therefore have

$$\begin{aligned} 0 &= {}_A\langle y' \ , \ x\rangle^*{}_A\langle y \ , \ z\rangle = {}_A\langle x \ , \ y'\rangle{}_A\langle y \ , \ z\rangle \\ &= {}_A\langle {}_A\langle x \ , \ y'\rangle \cdot y \ , \ z\rangle = {}_A\langle x \cdot \langle y' \ , \ y\rangle_B \ , \ z\rangle. \end{aligned}$$

Since z is arbitrary in X, we see that $x \cdot \langle y' \ , \ y\rangle_B = 0$ for all $x \in X$. Since X is full, we obtain that $\langle y' \ , \ y\rangle_B = 0$. On the other hand, since we have ${}_A\langle y' \ , \ y\rangle \in I_Y \cap I_{Y'} = I_Y \cap (I_Y)^\perp = \{0\}$, we obtain that ${}_A\langle y' \ , \ y\rangle = 0$ for all $y \in Y$. Hence we conclude that $y' \in Y^\perp$, so that $Y' \subset Y^\perp$. We thus see that $I_{Y'} \subset I_{Y^\perp}$, i.e., $(I_Y)^\perp = I_{Y^\perp}$.

Since the Rieffel correspondences are natural lattice isomorphisms among $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{S}(X)$, a least upper bound $Y \vee Y^\perp (= Y \oplus Y^\perp)$ of Y and $Y^\perp$ corresponds to a least upper bound $I_Y \vee I_{Y^\perp} (= I_Y \vee (I_Y)^\perp = I_Y \oplus (I_Y)^\perp = A)$ of I_Y and $I_{Y^\perp}$. Since A and X correspond by the Rieffel correspondence, we see that $X = Y \oplus Y^\perp$.

(3) $\Longrightarrow$ (1). Suppose that $\widehat{A}$ is a T_1-space. Take any open central projection p in the second dual A^{**} of A. In order to obtain condition (1), by [4, Theorem 2.3] it suffices to show that p is a multiplier for A. For this, we shall show that p is also a closed projection in A^{**} ([8, 3.12.9, 3.11.9 and 3.11.10]), hence, equivalently, that $q = 1-p$ is open in A^{**}. Put $I = A^{**}p \cap A$ and let ${}_IX$ be the closed A-B submodule of X corresponding to the closed ideal I of A under Rieffel correspondence. Thus condition (3) yields that $X = {}_IX \oplus ({}_IX)^\perp$, and, taking into account Lemma 2.1, Lemma 2.2 shows that $({}_IX)^\perp$ is also a closed A-B-submodule of X. Then $({}_IX)^\perp = {}_JX$ for some closed ideal J of A by the Rieffel correspondence, and we see that $A = I \vee J$ because A corresponds to X.

We now assert that $IJ = I \cap J = \{0\}$, that is, I and J are mutually orthogonal. In fact, since a greatest lower bound $I \wedge J$ of I and J correponds to a greatest lower bound ${}_{I\wedge J}X (= {}_IX \wedge {}_JX)$ and since ${}_IX \wedge {}_JX = {}_IX \cap {}_JX = {}_IX \cap ({}_IX)^\perp = \{0\}$, we see that $I \cap J = I \wedge J = \{0\}$. Thus we obtain that $A = I \oplus J$. Then we conclude that $J = A^{**}q \cap A$, hence q is open. Thus we see that p is a multiplier for A, from which it follows that $\widehat{A}$ is discrete. □

Remark 2.4. (1) In the proof of the implication (1) $\Longrightarrow$ (3) in Theorem 2.3 above, we have seen therein that $A = I_Y \oplus I_{Y'}$ and $I_Y \wedge I_{Y'} = \{0\}$, which shows that $Y \vee Y' = X$ and $Y \wedge Y' = \{0\}$. Then it is not trivial that $Y \wedge Y' = \{0\}$ immediately implies $Y' = Y^\perp$. In fact, this implication is an essential part of the proof.

(2) In the implication (3) $\Longrightarrow$ (1) in Theorem 2.3, the assumption that $\widehat{A}$ or $\widehat{B}$ be a T_1-space is necessary. In addition, in general, condition (3) does not necessarily imply that $\widehat{A}$ or $\widehat{B}$ is a T_1-space. For example, let A be a UHF C^*-algebra which is not of type I. Then $\widehat{A}$ is not a T_1-space, hence, not discrete. If it were so, $\widehat{A}$ and $\mathrm{Prim}(A)$ must be homeomorphic. Since A is separable, it must be of type I, which is a contradiction. Let X be a Hilbert A-module. Then X is full because A is simple. By the Rieffel correspondence, X does not have any non-trivial closed A-B-submodules because A is simple. Hence every closed A-B-submodule of X is complemented in X. But, $\widehat{A}$ is not a T_1-space.

Let A be a C^*-algebra and let X be a right Hilbert A-module with A-valued inner product $\langle \cdot \ , \ \cdot \rangle$. Define $\langle X \ , \ X\rangle$ to be the linear span of the set $\{ \ \langle x \ , \ y \ \rangle | \ x, y \in X \ \}$. We define the linear operator $\theta_{x,y}$ by $\theta_{x,y}(z) = x \cdot \langle y \ , \ z\rangle$ for all $x, y, z \in X$. We denote by $\mathcal{K}(X)$ the C^*-algebra generated by the set $\{\theta_{x,y} \mid x\, y \in X \ \}$ (see [9, Proposition 2.21 and Lemma 2.25]). Now we are in a position to establish the main

result as a corollary to Theorem 2.3. From now on, without loss of generality, we will assume that a Hilbert A-module X is a right module.

Corollary 2.5. *Let A be a C^*-algebra and let X be a Hilbert C^*-module over A. We denote by I the closed ideal of A generated by $\langle X\ ,\ X\rangle$. Consider the following conditions:*

(1) *The spectrum $\widehat{A}$ of A is discrete in the Jacobson topology.*

(2) *The spectrum $\widehat{I}$ of I is discrete in the Jacobson topology.*

(3) *Every closed $\mathcal{K}(X)$-I-submodule of X is complemented in X.*

Then we have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). *If $\widehat{I}$ is a T_1-space, then conditions* (2) *and* (3) *are equivalent.*

Proof. (1) $\Longrightarrow$ (2). Since I is a closed ideal of A, $\widehat{I}$ is identified with an open subset of $\widehat{A}$ ([8, 4.1.11]). Thus $\widehat{I}$ is discrete.

(2) $\Longrightarrow$ (3). Since I is the closed ideal of A generated by $\langle X\ ,\ X\rangle$, X can be considered as a full Hilbert C^*-module over I. Then X becomes a $\mathcal{K}(X)$-I-imprimitivity bimodule (see [9, Proposition 3.8]). Hence condition (3) follows from Theorem 2.3.

Assume that $\widehat{I}$ is a T_1-space. Then the implication (3) $\Longrightarrow$ (2) follows from Theorem 2.3. □

The use of [4, Theorem 2.3] is essential in the proof of Theorem 2.3. The following theorem can be proved along lines similar to those in the proof of Theorem 2.3 above, using [4, Theorem 2.5] instead of [4, Theorem 2.3]. We will leave the details to the reader.

Theorem 2.6. *Let two C^*-algebras A and B be Morita equivalent and let X be an A-B-imprimitivity bimodule. Consider the following conditions:*

(1) *The primitive spectrum Prim(A) of A is discrete in the Jacobson topology.*

(2) *The primitive spectrum Prim(B) of B is discrete in the Jacobson topology.*

(3) *Every closed A-B-submodule of X is complemented in X.*

Then we have (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). *If either Prim(A) or Prim(B) is a T_1-space, then conditions* (1) $-$ (3) *are equivalent.*

Corollary 2.7. *Let A be a C^*-algebra and let X be a Hilbert C^*-module over A. We denote by I the closed ideal of A generated by $\langle X\ ,\ X\rangle$. Consider the following conditions:*

(1) *The primitive spectrum Prim(A) of A is discrete in the Jacobson topology.*

(2) *The primitive spectrum Prim(I) of I is discrete in the Jacobson topology.*

(3) *Every closed $\mathcal{K}(X)$-I-submodule of X is complemented in X.*

Then we have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). *If Prim(I) is a T_1-space, then conditions* (2) *and* (3) *are equivalent.*

Proof. Note that Prim(I) is an open subset in Prim(A) ([8, 4.1.10 or 4.1.11]). We assume that X is a right Hilbert A-module. Since X is then a $\mathcal{K}(X)$-I-imprimitivity bimodule, the corollary follows from Theorem 2.6. □

Example 2.8. If a C^*-algebra A is finite-dimensional, then it is isomorphic to the direct sum of some matrix algebras. Hence the spectrum $\widehat{A}$ of A is discrete. Let X be a full right Hilbert A-module. Then, by Corollary 2.5, every closed $\mathcal{K}(X)$-A-submodule of X is complemented in X.

More generally, if A is a dual C^*-algebra, then the spectrum $\widehat{A}$ of A is discrete. Recall here that a C^*-algebra is said to be *dual* if it is isomorphic to a C^*-subalgebra of the C^*-algebra of all compact operators on some Hilbert space (cf. [1, 4.7.20]). Note that a separable C^*-algebra is dual if and only if its spectrum is discrete in the Jacobson topology ([4, p. 706]). Let A be a dual C^*-algebra and let X be a full right Hilbert A-module. Then every closed $\mathcal{K}(X)$-A-submodule of X is complemented in X.

We end by stating a remark. Let A be a C^*-algebra. Then A can be regarded as an A-A-imprimitivity bimodule, for the bimodule structure given by the multiplication in A, with ${}_A\langle a\ ,\ b\rangle = ab^*$ and $\langle a\ ,\ b\rangle_A = a^*b$ for $a, b \in A$ (see [9, Example 3.5]). Hence Theorem 2.3 (resp. Theorem 2.6) generalizes [4, Theorem 2.3] (resp. [4, Theorem 2.5]).

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