ON VON NEUMANN’S PROBLEM IN EXTENSION THEORY OF NONNEGATIVE OPERATORS

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(Communicated by Joseph A. Ball)

Abstract. The solution of von Neumann’s problem about parametrization of all nonnegative selfadjoint extensions of a nonnegative densely defined operator in terms of his formulas is obtained.

1.

John von Neumann [18] formulated a problem about existence and description (parametrization) of all selfadjoint extensions preserving the lower bound of a given densely defined symmetric operator bounded from below acting on some Hilbert space. The existing descriptions of all nonnegative self-adjoint extensions of a nonnegative densely defined operator are not in terms of von Neumann’s classical formulas. They have been obtained indirectly by M.Krein [15], [16] in terms of his theory of self-adjoint contractive extensions of non-densely defined Hermitian contractions and by M.Birman [6] for a symmetric operator with strictly positive lower bound using the M.Vishik approach [20] (see [1]). In terms of abstract boundary conditions [13] and the Weyl-Titchmarsh functions [8], [11], [12], descriptions of the domains of all nonnegative self-adjoint and proper $m$-sectorial extensions were obtained in [9].

In this paper, taking into account methods and approaches in [3], [4], [17], we establish new formulas (Theorem 2) which provide a parametrization of all nonnegative self-adjoint extensions of a nonnegative symmetric operator with, generally speaking, zero lower bound under the assumption that the so-called Friedrichs extension [15] of this operator is known. As a result of this approach we obtain a solution of von Neumann’s problem about parametrization of all nonnegative self-adjoint extensions in terms of his formulas. Since the Friedrichs extension can be found independently of the above-mentioned methods, it turns out that the presented approach is efficient. An example of an operator with zero lower bound is considered. We plan to consider applications to canonical resolvents, point-interactions in $\mathbb{R}^3$ with $m$-points of interaction, perturbation theory and the theory of Krein-Langer $Q$-functions in a future paper.

We shall use the following notations: $\mathcal{L}(H_1, H_2)$ denotes the Banach space of all continuous linear operators acting from the Hilbert space $H_1$ into the Hilbert...
Let $H$ be a complex Hilbert space and let $H^2$ be the Hilbert space of all pairs $\langle u_1, u_2 \rangle$, $u_1, u_2 \in H$, with the inner product defined by

$$\langle (u_1, u_2), (v_1, v_2) \rangle = (u_1, v_1) + (u_2, v_2).$$

As is well known [7], a closed subspace $\mathbf{T} \subseteq H^2$ is called a linear relation (l.r.) in $H$. We denote by $\mathcal{D}(\mathbf{T})$ the domain of a linear relation $\mathbf{T}$ and by $\mathbf{T}(u)$ the set of all $v$ such that $\langle u, v \rangle \in \mathbf{T}$. By definition

$$\mathcal{D}(\mathbf{T}) = \{u_1 \in H : \text{ there exists } u_2 \in H \text{ such that } \langle u_1, u_2 \rangle \in \mathbf{T}\},$$

$$\mathcal{R}(\mathbf{T}) = \{u_2 \in H : \text{ there exists } u_1 \in H \text{ such that } \langle u_1, u_2 \rangle \in \mathbf{T}\},$$

$$\mathbf{T}^{-1} = \{\langle u_2, u_1 \rangle : \langle u_1, u_2 \rangle \in \mathbf{T}\}.$$

An arbitrary l.r. $\mathbf{T}$ has the decomposition $\mathbf{T} = \text{Gr}(\mathbf{T}) \oplus \{0, \mathbf{T}(0)\}$, where $T$ is a linear operator (the operator part of $\mathbf{T}$), $\text{Gr}(\mathbf{T}) = \langle u, Tu \rangle$, $u \in \mathcal{D}(T)$, and $\mathcal{D}(T) = \mathcal{D}(\mathbf{T})$. It is evident that $\mathbf{T}(u) = Tu \oplus \mathbf{T}(0)$ for all $u \in \mathcal{D}(\mathbf{T})$ and $\mathcal{R}(\mathbf{T}) = \mathcal{R}(T) \oplus \{0\}$. An l.r. $\mathbf{T}$ is called self-adjoint if it is Hermitian and has no Hermitian extensions. In this case the operator part $T$ [2] is self-adjoint in the subspace $\mathcal{D}(\mathbf{T})$ and $\mathbf{T}(0) \oplus \mathcal{D}(\mathbf{T}) = H$. An l.r. $\mathbf{T}$ is called nonnegative if $\langle \mathbf{T}(u), u \rangle \geq 0$ for all $u \in \mathcal{D}(\mathbf{T})$. Let $\tau[\cdot, \cdot]$ be a sesquilinear, symmetric and nonnegative form in a Hilbert space $H$ defined on a linear manifold $\mathcal{D}[\tau]$, i.e. $\tau[u, v] = \overline{\tau[v, u]}$ and $\tau[u] := \tau[u, u] \geq 0$ for all $u, v \in \mathcal{D}[\tau]$. A sequence $\{u_n\}$ is called $\tau$-converging to the vector $u \in H$ if

$$\lim_{n \to \infty} u_n = u \text{ and } \lim_{n,m \to \infty} \tau[u_n - u_m] = 0.$$
where \( \hat{T}^{-1/2} = (T^{1/2} \mathcal{R}(T^{1/2}))^{-1} \). We will denote by \( \mathcal{R}[T] \) the linear manifold \( \mathcal{R}(T^{1/2}) \parallel T(0) \). The form \( \tau \) is called closable if it has a closed extension; in this case the closure of \( \tau \) is the smallest closed extension of \( \tau \). If \( S \) is a nonnegative Hermitian operator \((Su, u) \geq 0 \) for all \( u \in D(S) \)), then the form \( \tau[u, v] := (Su, v) \) is closable. Following the notations of M.G. Krein we denote by \( S[\cdot, \cdot] \) the closure of the form \( \tau \) and by \( D[S] \) its domain. By definition \( S[u] = S[u, u] \) for all \( u \in D[S] \).

We use the same notations for the case of a nonnegative linear relation. Note that if \( T \) is a nonnegative linear relation with the operator part \( T \) and \( T[u, v] = T[u, v] \) for all \( u, v \in D[T] \).

Let \( T_1 \) and \( T_2 \) be two nonnegative self-adjoint linear relations. We shall write \( T_1 \geq T_2 \) if \( D[T_1] \subseteq D[T_2] \) and \( T_1[u] \geq T_2[u] \) for all \( u \in D[T_1] \). In order to prove our main results we need the following propositions which can be proved by passing to the operator part of the l.r.

**Proposition 1.** Let \( T_1 \) and \( T_2 \) be two nonnegative self-adjoint linear relations in a Hilbert space \( H \). Then the following conditions are equivalent:

1. \( T_1 \geq T_2 \);
2. \( T_1^{-1} \leq T_2^{-1} \);
3. \( \mathcal{R}(T_2) \subseteq \mathcal{R}(T_1) \) and \( T_1^{-1}T_2(u) \leq (T_2(u), u) \) for all \( u \in D(T_2) \).

Note that the equivalence 1. \( \iff \) 2. is well known for linear operators [14].

**Proposition 2.** Let \( T_1 \) and \( T_2 \) be two nonnegative self-adjoint l.r.’s such that \( T_1 \geq T_2 \). Then the closure of the form \( (T_1(f), g) - T_2(f, g), f, g \in D(T_1) \), in the Hilbert space \( D[T_2] \) coincides with the form \( T_1[u, v] - T_2[u, v], u, v \in D[T_1] \).

*Proof.* The form \( \tilde{t}[u, v] := T_1[u, v] - T_2[u, v], u, v \in D[\tilde{t}] = D[T_1] \) is nonnegative and closed in the Hilbert space \( D[T_2] \). Let \( u \in D[T_1] \). Then there exists a sequence \( \{f_n\} \subset D[T_1] \) such that \( \lim_{n \to \infty} f_n = u \) and \( \lim_{m, n \to \infty} (T_1(f_n - f_m), f_n - f_m) = 0 \). Then \( \lim_{m, n \to \infty} T_2(f_n - f_m) = 0 \); therefore, the sequence \( \{f_n\} \) converges to \( u \) in the space \( D[T_2] \) and \( \lim_{n \to \infty} \{T_1(f_n - u) - T_2(f_n - u)\} = 0 \). Thus, the form \( \tilde{t} \) is the closure of the form \( (T_1(f), g) - T_2(f, g) \) in the Hilbert space \( D[T_2] \).

3.

Let \( S \) be a closed densely defined symmetric and nonnegative operator in a Hilbert space \( H \) and let \( S^* \) be its adjoint. As is well known [14], the extension \( S_F \) of \( S \) obtained by K.Friedrichs [10] is defined as a nonnegative self-adjoint extension associated with the form \( S[\cdot, \cdot] \). Clearly, \( D(S_F) = D[S] \cap D(S^*) \), \( S_F = S^*|D(S_F) \).

If \( \mathfrak{N}_z = \text{Ker}(S^* - zI) \) are the defect subspaces, then \( D[S] \cap \mathfrak{N}_z = \{0\}, z \in \rho(S_F) \) and

\[
\mathcal{R}(S_F^{1/2}) = \{h \in H : \sup \left[ \frac{(h, f)^2}{(Sf, f)} , f \in D(S) \right] < \infty \}.
\]

In addition, \( ||S_F^{-1/2}h||^2 = \sup \left[ \frac{(h, f)^2}{(Sf, f)} , f \in D(S) \right] \).

As was established by M.G.Krein [15], [16], a nonnegative symmetric operator \( S \) has a minimal nonnegative self-adjoint extension which coincides with the extension obtained by J.von Neumann [18] (in the case of a positive lower bound of \( S \)). This
extension we call the Krein-von Neumann extension $S_N$. The operator $S_N$ can be
defined as follows \[2, 7\]:
\[ S_N = ((S^{-1})_F)^{-1}, \]
where $S^{-1}$ denotes in this context
the inverse l.r. to the $Gr(S)$. Thus, for every nonnegative self-adjoint extension $\tilde{S}$
of $S$, the inequality $S_N \leq \tilde{S} \leq S_F$ holds. In addition,
\[ S[f, u] = (f, S^*u), \quad f \in D[S], \quad u \in D[\tilde{S}] \cap D[S^*], \]
\[ D[\tilde{S}] = D[S] + \mathfrak{N}_z \cap D[\tilde{S}]. \]
We will use the following relations (see \[2\]):
\[ D[S_N] = \{ u \in H : \sup \left[ |(u, Sf)|^2/\langle Sf, f \rangle, \quad f \in D(S) \right] < \infty \} \]
and $S_N[u] = \sup \left[ |(u, Sf)|^2/\langle Sf, f \rangle, \quad f \in D(S) \right], \quad u \in D[S_N]$. From \[4\] and \[1\] we
see the equivalence
\[ u \in D(S^*) \cap D[S_N] \iff S^*u \in \mathcal{R}(S_F^{1/2}) \]
and the equality
\[ S_N[u] = S_F^{-1}[S^*u], \quad u \in D(S^*) \cap D[S_N]. \]
In particular,
\[ \mathfrak{N}_z \cap D[S_N] = \mathfrak{N}_z \cap \mathcal{R}(S_F^{1/2}) \]
and for $\varphi_z \in \mathfrak{N}_z \cap D[S_N]$ we have
\[ S_N[\varphi_z] = |z|^2 S_F^{-1}[\varphi_z]. \]
From \[2\], \[3\] and \[5\] and the polarization identity we obtain for all $f, g \in D[S], \quad \varphi_z, \psi_z \in \mathfrak{N}_z \cap D[S_N]$ and $z \in \rho(S_F)$
\[ S_N[f + \varphi_z, g + \psi_z] = \left( S_F^{1/2} f + z S_F^{-1/2} \varphi_z, S_F^{1/2} g + z S_F^{-1/2} \psi_z \right). \]
The next theorem gives a description of all closed forms associated with nonnegative
self-adjoint extensions of $S$.

**Theorem 1** \[5, 9\]. Let $\tilde{S}$ be a nonnegative self-adjoint extension of $S$. Then
the form $(\tilde{S}u, v) - S_N[u, v], \quad u, v \in D(\tilde{S}),$ is nonnegative and closable
in the Hilbert space $D[S_N]$. Moreover, the formulas
\[ \tilde{S}[u, v] = S_N[u, v] + \tau[u, v], \quad u, v \in D[\tilde{S}] = D[\tau] \]
give a one-to-one correspondence between all closed forms $\tilde{S}[\cdot, \cdot]$ associated with
nonnegative self-adjoint extensions $\tilde{S}$ of $S$ and all nonnegative forms $\tau[\cdot, \cdot]$ which
are closed in the Hilbert space $D[S_N]$ and such that $\tau[f] = 0$ for all $f \in D[S].$

Recall that two self-adjoint extensions $\tilde{S}_1$ and $\tilde{S}_2$ of a symmetric operator $S$ are
disjoint (relatively prime) if $D(\tilde{S}_1) \cap D(\tilde{S}_2) = D(S)$ and transversal if in addition
$D(\tilde{S}_1) + D(\tilde{S}_2) = D(S^*)$. The necessary and sufficient condition for transversality
of the Friedrichs and Krein-von Neumann extensions is the following:
$\mathfrak{N}_z \subset \mathcal{R}[S_N]$ for
some (and then for all) $z \in \rho(S_F)$. This condition is equivalent to $\mathfrak{N}_z \subset \mathcal{R}[S_F]$ for
some (and then for all) $z \in \rho(S_F)$. M.G.Krein established criterion for uniqueness
of nonnegative self-adjoint extensions of a nonnegative densely defined operator $S$.
Below we present an equivalent condition for uniqueness.
Consider the domain \( D(S^*) \) of the adjoint operator \( S^* \) to a closed densely defined nonnegative symmetric operator \( S \) as a Hilbert space \( H_+ \) with the inner product \( (f, g)_+ = (f, g) + (S^* f, S^* g) \). The (+)-orthogonal decomposition holds: \( H_+ = D(S) \oplus \mathcal{N}_i \oplus \mathcal{N}_{-i} \). Denote

\[
\mathcal{M}_F = D(S_F) \ominus D(S), \quad \mathcal{M}_F = H_+ \ominus D(S_F)
\]

((+)-orthogonal complements of \( D(S) \) in \( D(S_F) \) and of \( D(S_F) \) in \( H_+ \)). Note that

\[
\mathcal{M}_F = S_F \mathcal{N}_F, \quad \mathcal{N}_i = (S_F \pm iI) \mathcal{N}_F, \quad H_+ = D(S) \oplus \mathcal{N}_F \oplus \mathcal{M}_F.
\]

**Proposition 3.** A necessary and sufficient condition for the uniqueness of a nonnegative self-adjoint extension of \( S \) is the equality \( \mathcal{R}(S_F^{1/2}) \cap \mathcal{M}_F = \{0\} \).

Suppose that

\[
(10) \quad \mathcal{N}_0 = \mathcal{R}(S_F^{1/2}) \cap \mathcal{M}_F \neq \{0\}.
\]

According to Proposition 3 an operator \( S \) has nonunique nonnegative self-adjoint extensions. In the following we give a parametrization of all of them. First of all we describe the closed form, associated with the Kren-von Neumann extension \( S_N \) of \( S \).

**Proposition 4.** The following equalities hold:

\[
\begin{cases}
D[S_N] = D[S] + S_F \mathcal{N}_0, \\
S_N[f + S_F e] = \|S_F^{1/2} f - \hat{S}_F^{-1/2} e\|^2, \quad f \in D[S], \quad e \in \mathcal{N}_0.
\end{cases}
\]

**Proof.** From \( S_F \geq 0 \) and the equality \( S^* S_F e = -e \) for all \( e \in \mathcal{M}_F \) it follows that \( D[S] \cap S_F \mathcal{N}_F = \{0\} \). From \( \mathcal{N}_i = (S_F + iI) \mathcal{M}_F \), \( 3 \) and \( 7 \), we have \( D[S_N] = D[S] + \mathcal{R}(S_F^{1/2}) \cap \mathcal{N}_i \). Hence, we get that \( D[S_N] = D[S] + S_F \mathcal{N}_0 \). From \( 9 \) it follows for all \( f \in D[S] \) and all \( e \in \mathcal{N}_0 \) that

\[
S_N[f + S_F e] = S_N[f - ie + (S_F + iI)e] = \|S_F^{1/2} (f - ie) + i\hat{S}_F^{-1/2} (S_F + iI)e\|^2 = \|S_F^{1/2} f - \hat{S}_F^{-1/2} e\|^2. \quad \Box
\]

**Proposition 5.** Suppose that condition \( 10 \) is fulfilled and define a nonnegative sesquilinear form

\[
(12) \quad w_0[e, g] = (S_F^{1/2} e, S_F^{1/2} g) + (\hat{S}_F^{-1/2} e, \hat{S}_F^{-1/2} g), \quad e, g \in \mathcal{N}_0.
\]

Then the form \( w_0 \) is closed in the Hilbert space \( H_+ \) and

\[
(12) \quad w_0[e] \geq 2\|e\|^2 \text{ for all } e \in \mathcal{N}_0.
\]

Denote by \( \mathcal{H}_0^{w_0} \) the Hilbert space with the inner product

\[
(e, h)_{w_0} = w_0[e, h] + (e, h)_+, \quad e, h \in \mathcal{N}_0.
\]

Then the operator \( S_F \) is an isomorphism of the Hilbert space \( \mathcal{H}_0^{w_0} \) and the subspace \( S_F \mathcal{N}_0 \) of the Hilbert space \( D[S_N] \).
Proof. From (12) it follows that
\[ w_0[e] = \| S_F^{1/2} e - \hat{S}_F^{-1/2} e \|^2 + 2| e \|^2 = \| S_F^{1/2} e \|^2 \pm i \hat{S}_F^{-1/2} e. \]
Therefore, \( w_0[e] \geq 2| e \|^2 \) for all \( e \in \mathfrak{N}_0 \). Let \( \lim_{n \to \infty} e_n = e \) in \( H_+ \), and let \( \lim_{n,m \to \infty} w_0[e_n - e_m] = 0 \). Then \( \lim_{n \to \infty} e_n = e \) in \( H \) and \( \{ S_F^{1/2} e_n \}, \{ \hat{S}_F^{-1/2} e_n \} \) are Cauchy sequences. Since the operators \( S_F^{1/2} \) and \( \hat{S}_F^{-1/2} \) are closed, we get that
\[ S_F^{1/2} e = \lim_{n \to \infty} S_F^{1/2} e_n, e \in \mathcal{R}(S_F^{1/2}) \quad \text{and} \quad \hat{S}_F^{-1/2} e = \lim_{n \to \infty} S_F^{-1/2} e_n. \]
Thus \( e \in \mathfrak{N}_0 \) and \( \lim_{n \to \infty} w_0[e - e_n] = 0 \). This implies that the form \( w_0 \) is closed in \( H_+ \). From the inequality \( \| \Lambda^{1-\theta} u \| \leq \| \Lambda u \|^{1-\theta} \| u \|^\theta \) for all \( u \in \mathcal{D}(\Lambda) \), which is true for an arbitrary nonnegative self-adjoint operator \( \Lambda \) and an arbitrary \( \theta \in [0,1] \), we obtain, for all \( e \in \mathcal{D}(S_F) \cap \mathcal{R}(S_F^{1/2}) \),
\[ |e| \leq \| S_F e \|^{1/3} / \| S_F^{1/2} e \|^{2/3}, \quad | S_F^{1/2} e \| \leq \| S_F e \|^{2/3} / \| S_F^{1/2} e \|^{1/3}. \]
Therefore, statement 2. follows from (10) and (11). \( \square \)

Let \( \mathbf{W}_0 \) be a \((+)-\)nonnegative self-adjoint linear relation in \( \mathfrak{H}_F \) associated with \( w_0 \). In view of \( w_0[f] > 0 \) for all \( f \neq 0 \in \mathfrak{N}_0 \), the inverse l.r. \( \mathbf{W}_0^{-1} \) is densely defined in \( \mathfrak{H}_F \) and therefore is the graph of a \((+)-\)self-adjoint nonnegative operator. We denote this operator by \( \mathbf{W}_0^{-1} \). Clearly, \( \text{Ker} \mathbf{W}_0^{-1} = \mathbf{W}(0) = \mathfrak{H}_F \cap \mathfrak{N}_0 \) (the \((+)-\)orthogonal complement).

The next theorem gives a description of all nonnegative self-adjoint extensions of \( S \) and their associated closed forms in terms of \( \mathbf{W}_0^{-1} \) and some auxiliary operators in \( \mathfrak{H}_F \).

**Theorem 2.** The formulas
\begin{equation}
(13) \quad \begin{cases}
\mathcal{D}(\tilde{S}) = \mathcal{D}(S) \oplus (I + S_F \tilde{U}) \mathcal{D}(\tilde{U}), \\
\tilde{S}(f_0 + e + S_F \tilde{U} e) = S_F (f_0 + e) - \tilde{U} e, \quad f_0 \in \mathcal{D}(S), \quad e \in \mathcal{D}(\tilde{U}),
\end{cases}
\end{equation}
\begin{equation}
(14) \quad \begin{cases}
\mathcal{D}(\tilde{S}) = \mathcal{D}(S) + S_F \mathcal{R}(\tilde{U}^{1/2}), \\
\tilde{S}[f + S_F h] = \| S_F^{1/2} f - S_F^{-1/2} h \|^2 + \tilde{U}^{-1}[h] - w_0[h], \\
f \in \mathcal{D}(S), \quad h \in \mathcal{R}(\tilde{U}^{1/2})
\end{cases}
\end{equation}
give a one-to-one correspondence between all nonnegative self-adjoint extensions \( \tilde{S} \) of \( S \), their associated closed forms and all \((+)-\)nonnegative self-adjoint operators \( \tilde{U} \) in \( \mathfrak{H}_F \) satisfying the condition
\begin{equation}
(15) \quad \tilde{U} \leq \mathbf{W}_0^{-1}.
\end{equation}
An extension \( \tilde{S} \) coincides with \( S_N \) if \( \tilde{U} = \mathbf{W}_0^{-1} \). The extensions \( S_F \) and \( S_N \) are disjoint if and only if \( \mathfrak{N}_0 \) is dense in \( \mathfrak{H}_F \), and are transversal if and only if \( \mathfrak{N}_0 = \mathfrak{H}_F \).

Proof. For every \( h \in \mathfrak{N}_F \) we have \( (S^* S_F h, S_F h) = -(h, S_F h) \leq 0 \). It follows that, if \( \tilde{S} = \tilde{S}^* \) is a nonnegative extension of \( S \), then \( \mathcal{D}(\tilde{S}) \cap S_F \mathfrak{H}_F = \{0\} \). Therefore, we obtain \( \mathcal{D}(\tilde{S}) = \mathcal{D}(S) \oplus (I + S_F \tilde{U}) \mathcal{D}(\tilde{U}), \) where \( \tilde{U} \) is a closed linear operator in the subspace \( \mathfrak{H}_F \) with the domain \( \mathcal{D}(\tilde{U}) \). Let us show that \( \tilde{U} \) is a \((+)-\)self-adjoint
operator in $\mathfrak{H}_F$ and satisfies condition (13). Consider an arbitrary vector $f \in \mathcal{D}(\tilde{S})$ of the form $f = h + S_F \tilde{U}h$, $h \in \mathcal{D}(\tilde{U})$. Then $Sf = S^*f = S_F h - \tilde{U}h$ and

\begin{align*}
(Sf, f) &= (S_F h - \tilde{U}h, h + S_F \tilde{U}h) \\
&= (S_F h, h) - (\tilde{U}h, S_F \tilde{U}h) + (h, \tilde{U}h)_+ - 2 \text{Re}(\tilde{U}h, h).
\end{align*}

Since $\text{Im}(Sf, f) = 0$, we get that $\text{Im}(h, \tilde{U}h)_+ = 0$. This means that $\tilde{U}$ is a (+)-symmetric operator in $\mathfrak{H}_F$. Since $\tilde{S}$ is a nonnegative self-adjoint extension of $S$, then for every vector $f \in \mathcal{D}(\tilde{S})$ the inequality $(\tilde{S}f, f) \geq S_N[f]$ holds. Again let $f = h + S_F \tilde{U}h$, $h \in \mathcal{D}(\tilde{U})$. Then (5) implies $\tilde{U}h \in \mathcal{R}(S_F^{1/2})$ for all $h \in \mathcal{D}(\tilde{U})$ and from (6) we obtain

$$S_N[f] = (S_F h, h) + \|S_F^{1/2} \tilde{U}h\|^2 - 2 \text{Re}(\tilde{U}h, h).$$

Thus, the inequality $(\tilde{S}f, f) \geq S_N[f]$ yields

$$(S_F h, h) - (\tilde{U}h, S_F \tilde{U}h) + (h, \tilde{U}h)_+ - 2 \text{Re}(\tilde{U}h, h)$$

$$\geq (S_F h, h) + \|S_F^{1/2} \tilde{U}h\|^2 - 2 \text{Re}(\tilde{U}h, h).$$

Finally we have $\|S_F^{1/2} \tilde{U}h\|^2 + (\tilde{U}h, S_F \tilde{U}h) \leq (h, \tilde{U}h)_+$. It follows that $w_0[\tilde{U}h] \leq (\tilde{U}h, h)_+$ for all $h \in \mathcal{D}(\tilde{U})$.

In particular, $\tilde{U}$ is a (+)-nonnegative operator in $\mathfrak{H}_F$. It follows that $\tilde{U}$ has a (+)-self-adjoint extension $\tilde{\tilde{U}}$ in $\mathfrak{H}_F$. One can check that the operator $\tilde{S}$ given by $\mathcal{D}(\tilde{S}) = \mathcal{D}(S) \oplus (I + S_F \tilde{U})\mathcal{D}(\tilde{U})$, $\tilde{S} = S^*|\mathcal{D}(\tilde{S})$ is a symmetric extension of $\tilde{S}$ in $H$. Since $\tilde{S}$ is self-adjoint, we get $\tilde{\tilde{U}} = \tilde{\tilde{U}}$. Thus, $\tilde{\tilde{U}}$ is a (+)-self-adjoint and nonnegative operator in $\mathfrak{H}_F$. According to Proposition 1 the inequality $w_0[\tilde{\tilde{U}}h] \leq (\tilde{\tilde{U}}h, h)_+$, $h \in \mathcal{D}(\tilde{U})$, is equivalent to $\tilde{\tilde{U}} \leq W_0^{-1}$.

Conversely, let $\tilde{\tilde{U}}$ be a (+)-self-adjoint nonnegative operator in $\mathfrak{H}_F$ satisfying the condition $\tilde{\tilde{U}} \leq W_0^{-1}$. Then also $w_0[\tilde{\tilde{U}}h] \leq (\tilde{\tilde{U}}h, h)_+$ for all $h \in \mathcal{D}(\tilde{U})$. It follows that

$$(S^* f, f) \geq S_N[f]$$

for all $f = h + S_F h$, $h \in \mathcal{D}(\tilde{U})$, and from (4) we obtain $|(S\varphi, f)|^2 \leq (S\varphi, \varphi)(S^* f, f)$. Hence,

$$2 \text{Re}(S\varphi, f) \geq -(S\varphi, \varphi) - (S^* f, f).$$

Further for $g = \varphi + f$, where $\varphi \in \mathcal{D}(S)$, we have

$$2 \text{Re}(S\varphi, f) \geq - (S\varphi, \varphi) - (S^* f, f).$$

Thus, the operator $\tilde{S} = S^*|\mathcal{D}(S) \oplus (I + S_F \tilde{U})\mathcal{D}(\tilde{U})$ is a self-adjoint nonnegative extension of $S$. Define a nonnegative self-adjoint extension

$$\tilde{S}_0 = S^*|\mathcal{D}(S) \oplus (I + S_F W_0^{-1})\mathfrak{H}_0).$$

Let us prove that $\tilde{S}_0$ coincides with the Krein-von Neumann extension $S_N$. We will show the equality $(\tilde{S}_0 u, v) = S_N[u, v]$ for all $u \in \mathcal{D}(\tilde{S}_0)$ and all $v \in \mathcal{D}(S_N)$. Let $u = f + (I + S_F W_0^{-1})e$, $v = g + S_F h$, where $f \in \mathcal{D}(S)$, $e \in \mathcal{D}(W_0^{-1})$, $g \in \mathcal{D}(S)$, $h \in \mathfrak{H}_0$. 

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Theorem 3. Let $P_i^+$ be the orthogonal projection onto $\mathfrak{N}_i$ in $H_+$ and $D(S_F) = D(S) + (I + V_F)\mathfrak{N}_i$. Then the operator

$$\tilde{V} P_i^+ h = -V_F P_i^+ (\tilde{U} + iI)(\tilde{U} - iI)^{-1} h, \ h \in \mathfrak{N}_F,$$

defines $D(\tilde{S})$ by the von Neumann formula $D(\tilde{S}) = D(S) + (I + \tilde{V})\mathfrak{N}_i$, where $\tilde{U}$ is a (+)-self-adjoint operator in $\mathfrak{N}_F$ satisfying the condition $0 \leq \tilde{U} \leq W_0^{-1}$.

Proof. Let $\tilde{U}$ be a (+)-selfadjoint operator in the subspace $\mathfrak{N}_F$ which satisfies the condition $0 \leq \tilde{U} \leq W_0^{-1}$ and let $\tilde{Z} = (\tilde{U} + iI) (\tilde{U} - iI)^{-1}$ be the Cayley transform.
of $\tilde{U}$. Then $\tilde{Z}$ is a $(+)$-unitary operator in $\mathfrak{N}_F$ and $\tilde{U}$ can be defined as follows: 
\[ e = i(I - \tilde{Z})h, \quad \tilde{U}e = (I + \tilde{Z})h, \quad h \in \mathfrak{N}_F. \]
Since $\mathfrak{N}_F = (I + V_F)\mathfrak{N}_i$ and $V_F$ is a $(+)$-isometry from $\mathfrak{N}_i$ onto $\mathfrak{N}_{-i}$, we get that if $h = (I + V_F)g$, $g \in \mathfrak{N}_i$, then 
\[ \tilde{Z}h = (I + V_F)\tilde{K}g, \]
where $\tilde{K}$ is a $(+)$-unitary operator in $\mathfrak{N}_i$. This implies that 
\[ P_i^+h = g, \quad P_i^+\tilde{Z}h = \tilde{K}g, \]
and 
\[ e = i(I + V_F)(g - \tilde{K}g), \quad \tilde{U}e = (I + V_F)(g + \tilde{K}g), \]
\[ (I + S_F\tilde{U})e = i(I + V_F)(g - \tilde{K}g) + i(I - V_F)(g + \tilde{K}g) \]
\[ = 2ig - 2iV_F\tilde{K}g. \]

Hence, for a self-adjoint extension $\tilde{S}$ given by (13) we get 
\[ \mathcal{D}(\tilde{S}) = \mathcal{D}(S) + (I - V_F\tilde{K})\mathfrak{N}_i. \]

5.

Let $y \in \mathbb{R}^3$. Consider the operator $S$ defined as follows:
\[ \mathcal{D}(S) = \{ \varphi(x) \in H^2_2(\mathbb{R}^3) : \varphi(y) = 0 \}, \quad S\varphi = -\Delta \varphi, \]
where $x \in \mathbb{R}^3$, $H^2_2(\mathbb{R}^3)$ is the Sobolev space and $\Delta$ denotes the Laplacian. As is well known the operator $S$ is a nonnegative symmetric operator in $L^2(\mathbb{R}^3, dx)$ with defect numbers 1, 1 and its Friedrichs extension $S_F$ is given by 
\[ \mathcal{D}(S_F) = H^2_2(\mathbb{R}^3), \quad S_F = -\Delta. \]

Let $\mathcal{F} : L^2(\mathbb{R}^3, dx) \to L^2(\mathbb{R}^3, dp)$,
\[ \mathcal{F}f = \hat{f}(p) = s - \lim_{R \to \infty} (2\pi)^{-3/2} \int_{|x| \leq R} f(x) \exp(-ipx) dx, \quad p = (p_1, p_2, p_3), \]
be the Fourier transform. In the $p$-representation we obtain the nonnegative symmetric operator $\hat{A}$ and its Friedrichs extension $A_F$:
\[ \mathcal{D}(\hat{A}) = \{ h(p) \in L^2(\mathbb{R}^3, dp), \int_{\mathbb{R}^3} h(p) \exp(ipy) dp = 0 \}, \]
\[ \mathcal{D}(A_F) = H^2_2(\mathbb{R}^3) := L^2(\mathbb{R}^3, (|p|^4 + 1)dp), \]
\[ \hat{A}h = |p|^2 h(p), \quad h(p) \in \mathcal{D}(A), \]
\[ A_F f = |p|^2 f(p), \quad f(p) \in \mathcal{D}(A_F). \]

Let $e(p) = \exp(-ipy) \left(1 + |p|^4\right)^{-1}$. Clearly, $\mathfrak{N}_F = \text{span}\{ e(p) \}$, $\mathfrak{M}_F = A_F\mathfrak{N}_F = \text{span}\{ |p|^2 e(p) \}$. The adjoint operator $\hat{A}^*$ is given by the following relations:
\[ \mathcal{D}(\hat{A}^*) = \mathcal{D}(A^*) + \mathfrak{N}_F + \mathfrak{M}_F = H^2_2(\mathbb{R}^3) + \mathfrak{M}_F, \]
\[ \hat{A}^* (f(p) + \lambda |p|^2 e(p)) = |p|^2 f(p) - \lambda e(p), \quad f(p) \in H^2_2(\mathbb{R}^3), \lambda \in \mathbb{C}. \]
Let $H_+ = \mathcal{D}(\hat{A}^*).$ Since

$$\mathcal{D}(A_F^{1/2}) = H_1(\mathbb{R}^3) := L^2(\mathbb{R}^3, (|p|^2 + 1)dp), \ A_F^{1/2} f = |p|f(p),$$

we obtain that

$$A_F^{-1/2} e(p) = \frac{\exp(-ipy)}{|p|(1 + |p|^4)} \in H_1(\mathbb{R}^3).$$

By Proposition 3 we have $A_F \neq A_N.$ By the direct calculation we get

$$(e(p), e(p))_+ = \int_{\mathbb{R}^3} \frac{dp}{1 + |p|^4} = \sqrt{2\pi^2},$$

$$\left( A_F^{1/2} e(p), A_F^{1/2} e(p) \right) + \left( A_F^{-1/2} e(p), A_F^{-1/2} e(p) \right) = \int_{\mathbb{R}^3} \frac{dp}{|p|^2(1 + |p|^4)} = \sqrt{2\pi^2}.$$

From Theorem 2 we get the following descriptions of non-negative self-adjoint extensions of $\hat{A}$:

$$\mathcal{D}(A_N) = \left\{ f_0(p) + \lambda \frac{(1 + |p|^2) \exp(-ipy)}{1 + |p|^4}, \ f_0(p) \in \mathcal{D}(A), \ \lambda \in \mathbb{C} \right\},$$

$$A_N \left( f_0(p) + \lambda \frac{(1 + |p|^2) \exp(-ipy)}{1 + |p|^4} \right) = |p|^2 f_0(p) + \lambda \frac{(|p|^2 - 1) \exp(-ipy)}{1 + |p|^4},$$

$$\mathcal{D}(\tilde{A}_u) = \left\{ f_0(p) + \lambda \frac{(1 + u |p|^2) \exp(-ipy)}{1 + |p|^4}, \ f_0(p) \in \mathcal{D}(A), \ \lambda \in \mathbb{C} \right\},$$

$$\tilde{A}_u \left( f_0(p) + \lambda \frac{(1 + u |p|^2) \exp(-ipy)}{1 + |p|^4} \right) = |p|^2 f_0(p) + \lambda \frac{(|p|^2 - u) \exp(-ipy)}{1 + |p|^4},$$

where $0 \leq u \leq 1.$ The inverse Fourier transform $\mathcal{F}^{-1}$ is given by the equality

$$\mathcal{F}^{-1} \hat{f} = f(x) = s - \lim_{R \to \infty} (2\pi)^{-3/2} \int_{|p| \leq R} \hat{f}(p) \exp(ipx)dp.$$

We have $S = \mathcal{F}^{-1} \hat{A} \mathcal{F}, \ S_F = \mathcal{F}^{-1} A_F \mathcal{F}, \ S_N = \mathcal{F}^{-1} A_N \mathcal{F}.$ A calculation gives

$$\mathcal{F}^{-1} e(p) = \sqrt{\frac{\pi}{2}} \frac{\exp\left(-\frac{|x-y|}{\sqrt{2}}\right)}{|x-y|} \sin \frac{|x-y|}{\sqrt{2}},$$

$$\mathcal{F}^{-1} A_F e(p) = \sqrt{\frac{\pi}{2}} \frac{\exp\left(-\frac{|x-y|}{\sqrt{2}}\right)}{|x-y|} \cos \frac{|x-y|}{\sqrt{2}}.$$
Hence,
\[ \mathcal{D}(S_N) = \left\{ f_0(x) + \lambda \exp\left( -\frac{|x-y|}{|x-y|} \right) \left( \sin \frac{|x-y|}{\sqrt{2}} + \cos \frac{|x-y|}{\sqrt{2}} \right), f_0(y) = 0, \lambda \in \mathbb{C} \right\}, \]
\[ S_N \left( f_0(x) + \lambda \exp\left( -\frac{|x-y|}{|x-y|} \right) \left( \sin \frac{|x-y|}{\sqrt{2}} + \cos \frac{|x-y|}{\sqrt{2}} \right) \right) = -\Delta f_0(x) + \lambda \exp\left( -\frac{|x-y|}{|x-y|} \right) \left( \cos \frac{|x-y|}{\sqrt{2}} - \sin \frac{|x-y|}{\sqrt{2}} \right), \]
\[ \mathcal{D}(\tilde{S}_u) = \left\{ f_0(x) + \lambda \exp\left( -\frac{|x-y|}{|x-y|} \right) \left( \sin \frac{|x-y|}{\sqrt{2}} + u \cos \frac{|x-y|}{\sqrt{2}} \right), f_0(y) = 0, \lambda \in \mathbb{C} \right\}, \]
\[ \tilde{S}_u \left( f_0(x) + \lambda \exp\left( -\frac{|x-y|}{|x-y|} \right) \left( \sin \frac{|x-y|}{\sqrt{2}} + u \cos \frac{|x-y|}{\sqrt{2}} \right) \right) = -\Delta f_0(x) + \lambda \exp\left( -\frac{|x-y|}{|x-y|} \right) \left( \cos \frac{|x-y|}{\sqrt{2}} - u \sin \frac{|x-y|}{\sqrt{2}} \right), 0 \leq u \leq 1. \]

ACKNOWLEDGMENTS

The authors thank Fritz Gesztesy and Konstantin Makarov for valuable discussions and the referee for some remarks. Special thanks go to Joseph Ball for supplying us with numerous helpful suggestions, both mathematical and linguistic.

REFERENCES


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