PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 10, Pages 3241–3249
S 0002-9939(03)06869-2
Article electronically published on February 12, 2003

COVERING FOR CATEGORY AND TRIGONOMETRIC THIN SETS

PETER ELIAŠ

(Communicated by Carl G. Jockusch, Jr.)

Abstract. In this work we consider several combinatorial principles satisfied for cardinals smaller than $\mathrm{cov}(\mathcal{M})$, the covering number of the ideal of first category sets on real line. Using these principles we prove that there exist N_0-sets (similarly N-sets, A-sets) which cannot be covered by fewer than $\mathrm{cov}(\mathcal{M})$ pD-sets (A-sets, N-sets, respectively). This improves the results of our previous paper (1997).

The study of special sets of reals related to the convergence of trigonometric series, generally called 'trigonometric thin sets', was one of the classical topics of harmonic analysis (see e.g. [4]). Recently, many investigations were made concerning the set-theoretic properties of such sets. A serious interest turns on the cardinal characteristics of structures related to trigonometric thin sets (see e.g. [3], [5], [6], [8], [10], [11]).

This paper has two parts. In the first part we present two new combinatorial characterizations of the cardinal $\mathrm{cov}(\mathcal{M})$, the covering number of the ideal of first category sets (Corollary 5). We use these characterizations in the second part to prove some theorems about trigonometric thin sets (Theorems 12, 13, and 16).

1. Combinatorics

We use the standard set-theoretic notation. Hence, ω is the set of all natural numbers, $[\omega]^{<\omega}$ and $[\omega]^{\omega}$ denote the sets of all finite and infinite subsets of ω, A^B is the set of all functions from B to A, and $|A|$ is the cardinality of a set A. Quantifiers $\exists^\infty$ and $\forall^\infty$ stand for 'there are infinitely many' and 'for all but finitely many', respectively.

Let $\mathcal{M}$ denote the ideal of first category (meager) sets on real line. Let us recall that $\mathrm{cov}(\mathcal{M})$, the covering number of $\mathcal{M}$, is the minimum size of a family $\mathcal{F} \subseteq \mathcal{M}$ such that $\bigcup \mathcal{F} = \mathbb{R}$. We will use the following combinatorial characterizations of this cardinal, due to Bartoszyński ([1], [2]).

Lemma 1. *Let κ be a cardinal. The following conditions are equivalent:*

(1) $\kappa < \mathrm{cov}(\mathcal{M})$,

Received by the editors October 23, 2001 and, in revised form, May 17, 2002.
2000 *Mathematics Subject Classification.* Primary 03E75; Secondary 03E17, 40A30, 43A46.
Key words and phrases. Covering, first category sets, meager sets, trigonometric thin sets, A-sets, N-sets, N_0-sets, pD-sets.
The work on this paper was supported by grant 1/7555/20 of Slovak Grant Agency VEGA.

(2) *for every family* $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, *there exists a function* $g \in \omega^\omega$ *such that* $(\forall f \in \mathcal{F})(\exists^\infty n)\, g(n) = f(n)$,
(3) *for every family* $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, *and for every family* $\mathcal{X} \subseteq [\omega]^\omega$, $|\mathcal{X}| \leq \kappa$, *there exists* $g \in \omega^\omega$ *such that* $(\forall f \in \mathcal{F})(\forall X \in \mathcal{X})(\exists^\infty n \in X)\, g(n) = f(n)$,
(4) *for every family* $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, *there exists a function* $S \in ([\omega]^{<\omega})^\omega$, *such that* $(\forall n)\ |S(n)| = n+1$, *and* $(\forall f \in \mathcal{F})(\exists^\infty n)\, f(n) \in S(n)$.

We will add two more conditions to this list. First, we need to prove some lemmas.

Lemma 2. *For every family* $\mathcal{F} \subseteq [\omega]^\omega$, $|\mathcal{F}| < \mathrm{cov}(\mathcal{M})$, *and for every* $h \in \omega^\omega$, *there exists* $g \in \omega^\omega$ *such that* $(\forall n)\, g(n+1) \geq h(g(n))$, *and* $(\forall X \in \mathcal{F})\ |\mathrm{rng}(g) \cap X| = \omega$.

Proof. Let $\mathcal{F} = \{X_\alpha : \alpha < \kappa\}$, $\kappa < \mathrm{cov}(\mathcal{M})$. Without loss of generality, we assume that h is non-decreasing and $h(n) > n$ for all n.

Put $k_0 = 0$, $k_{n+1} = h(k_n)$ for $n \in \omega$. For $\alpha < \kappa$ and $n \in \omega$, let $f_\alpha(n) \in \omega$ be such that

$$\big|\{j : j < f_\alpha(n) \wedge [k_j, k_{j+1}) \cap X_\alpha \neq \emptyset\}\big| = 3n+1.$$

Put $f^*_\alpha(n) = \big[0, k_{f_\alpha(n)}\big) \cap X_\alpha$. Since f^*_α-s can be coded as elements of ω^ω, by Lemma 1 there exists g^* such that $(\forall \alpha < \kappa)(\exists^\infty n)\, g^*(n) = f^*_\alpha(n)$. Moreover, we may assume that $(\forall n)(\exists \alpha < \kappa)\, g^*(n) = f^*_\alpha(n)$. Thus for each n,

$$\big|\{j : [k_j, k_{j+1}) \cap g^*(n) \neq \emptyset\}\big| = 3n+1,$$

and we can take $m_n \in g^*(n)$ such that for every $i < n$ and $j \in \omega$, if $m_i \in [k_j, k_{j+1})$, then $m_n \notin [k_{j-1}, k_{j+2})$.

Let $g \in \omega^\omega$ be an increasing function such that $\mathrm{rng}(g) = \{m_n : n \in \omega\}$. For every n, if $g(n) \in [k_j, k_{j+1})$, then $[k_{j+1}, k_{j+2}) \cap \mathrm{rng}(g) = \emptyset$, and thus $g(n+1) \geq k_{j+2} = h(k_{j+1}) \geq h(g(n))$, since $k_{j+1} > g(n)$. Moreover, if $g^*(n) = f^*_\alpha(n)$, then $m_n \in X_\alpha$. Hence $\mathrm{rng}(g) \cap X_\alpha$ is infinite for every $\alpha < \kappa$. □

Lemma 3. *Let* $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \mathrm{cov}(\mathcal{M})$, *and let every* $f \in \mathcal{F}$ *be increasing. Then for every* $h \in \omega^\omega$ *there exists* $g \in \omega^\omega$ *such that* $(\forall n)\, g(n+1) \geq h(g(n))$, *and* $(\forall f \in \mathcal{F})(\exists^\infty n)(\exists k \leq n)\, g(k) = f(n)$.

Proof. By Lemma 1, there exists g' such that $(\forall f \in \mathcal{F})(\exists^\infty n)\, g'(n) = f(n)$. Moreover, we may assume that $(\forall n)(\exists f \in \mathcal{F})\, g'(n) = f(n)$, and hence $g'(n) \geq n$ for all n. For $f \in \mathcal{F}$, put $X_f = \{n : g'(n) = f(n)\}$ and $h^*(n) = \max\{n+1, h(g'(n))\}$. By Lemma 2 there exists $g^* \in \omega^\omega$ such that $(\forall n)\, g^*(n+1) \geq h^*(g^*(n))$, and $(\forall f \in \mathcal{F})\ |\mathrm{rng}(g^*) \cap X_f| = \omega$. Clearly g^* is increasing.

For $n \in \omega$, let $g(n) = g'(g^*(n))$. Then for every n, $g(n+1) = g'(g^*(n+1)) \geq g^*(n+1) \geq h^*(g^*(n)) \geq h(g'(g^*(n))) = h(g(n))$. Moreover, if $n \in \mathrm{rng}(g^*) \cap X_f$, then $n = g^*(k)$ for some $k \leq n$, and $g(k) = g'(g^*(k)) = g'(n) = f(n)$. □

Lemma 4. *Let* $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \mathrm{cov}(\mathcal{M})$. *Then there exists a function* $g \in \omega^\omega$ *such that* $(\forall n)(\exists f \in \mathcal{F})\, g(n+1) = f(g(n))$, *and* $(\forall f \in \mathcal{F})(\exists^\infty n)\, g(n+1) = f(g(n))$.

Proof. First, we show the existence of $g' \in \omega^\omega$ such that $(\forall n)(\exists f \in \mathcal{F})\, g'(n+1) \geq f(g'(n))$, and $(\forall f \in \mathcal{F})(\exists^\infty n)\, g'(n+1) \geq f(g'(n))$. Using Lemma 1, we can find $h \in \omega^\omega$ such that $h(n) > n$ for all n, and the set $X_f = \{n : f(n) \leq h(n)\}$ is infinite for every $f \in \mathcal{F}$. By Lemma 2, there exists $g' \in \omega^\omega$ such that for all n, $g'(n+1) \geq h(g'(n))$, and for all $f \in \mathcal{F}$, $|\mathrm{rng}(g') \cap X_f| = \omega$. Moreover, we may assume that $(\forall n)(\exists f \in \mathcal{F})\, g'(n) \in X_f$. Since g' is increasing, we have $(\forall f \in$

$\mathcal{F})(\exists^\infty n)\, g'(n) \in X_f$. Clearly, if $g'(n) \in X_f$, then $f(g'(n)) \leq h(g'(n)) \leq g'(n+1)$. Thus g' is as we have expected.

For $f \in \mathcal{F}$, $n \in \omega$, denote $f^*(n) = \max\{f(0), \dots, f(n)\}$. As we have just proved, there exists $g^* \in \omega^\omega$ such that for all $f \in \mathcal{F}$, the set $Y_f = \{n : g^*(n+1) \geq f^*(g^*(n))\}$ is infinite, and $\bigcup_{f\in\mathcal{F}} Y_f = \omega$.

For given $f \in \mathcal{F}$ and $n \in \omega$, put $H_f(n) = f \restriction \{0, \dots, g^*(n)\}$. Since $H_f(n)$-s can be coded as elements of ω, by Lemma 1 there exists a function H such that $(\forall f \in \mathcal{F})(\exists^\infty n \in Y_f)\, H(n) = H_f(n)$. Without loss of generality we may assume also that $(\forall n)(\exists f \in \mathcal{F})(n \in Y_f \wedge H(n) = H_f(n))$.

Put $g(0) = 0$, $g(n+1) = H(n)(g(n))$. For every n, we have $H(n) = H_f(n)$ for some $f \in \mathcal{F}$ such that $n \in Y_f$, and hence if $k \in \mathrm{dom}(H(n))$, then $k \leq g^*(n)$, and $H(n)(k) = H_f(n)(k) = f(k) \leq f^*(g^*(n)) \leq g^*(n+1)$. Thus $H(n)(k) \in \mathrm{dom}(H(n+1))$, and the definition of g makes sense. Moreover, if $H(n) = H_f(n)$, then $g(n+1) = H_f(n)(g(n)) = f(g(n))$. Hence g has the required properties. □

Corollary 5. *Let κ be a cardinal. The following conditions are equivalent to* (1)–(4) *from Lemma 1:*

(5) *for every family $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, such that every $f \in \mathcal{F}$ is increasing, and for every $h \in \omega^\omega$, there exists $g \in \omega^\omega$ such that $(\forall n)\, g(n+1) \geq h(g(n))$, and $(\forall f \in \mathcal{F})(\exists^\infty n)(\exists k \leq n)\, g(k) = f(n)$,*

(6) *for every family $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, there exists $g \in \omega^\omega$ such that $(\forall n)(\exists f \in \mathcal{F})\, g(n+1) = f(g(n))$, and $(\forall f \in \mathcal{F})(\exists^\infty n)\, g(n+1) = f(g(n))$.*

Proof. As we have shown in the previous two lemmas, (1) implies both (5) and (6). We will prove that (5) implies (4), and (6) implies (2).

Assume that (5) holds. Let $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, and let $h(n) = n+1$ for all $n \in \omega$. Clearly if $g \in \omega^\omega$ satisfies the conditions of (5), then g is increasing, and the function S such that $S(n) = \{g(0), \dots, g(n)\}$, $n \in \omega$, satisfies the condition of (4). Thus (5) implies (4).

Let $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, and let (6) hold true. Fix a bijection $p \in (\omega \times \omega)^\omega$. For $f \in \mathcal{F}$, let $f^* \in \omega^\omega$ be such that for all $k, l \in \omega$, $f^*(p(k,l)) = p(k+1, f(k))$. By (6), there exists $g^* \in \omega^\omega$ such that $(\forall n)(\exists f \in \mathcal{F})\, g^*(n+1) = f^*(g^*(n))$, and $(\forall f \in \mathcal{F})(\exists^\infty n)\, g^*(n+1) = f^*(g^*(n))$. For $n \in \omega$, let k_n, l_n be such that $g^*(n) = p(k_n, l_n)$. Now, if $g^*(n+1) = f^*(g^*(n))$ for some $f \in \mathcal{F}$ and $n \in \omega$, then $p(k_{n+1}, l_{n+1}) = f^*(p(k_n, l_n)) = p(k_n + 1, f(k_n))$. It follows that for all n, $k_n = n + k_0$ and $f(k_n) = l_{n+1}$. If we put $g(n) = 0$ for $n < k_0$, and $g(n + k_0) = l_{n+1}$ for all $n \in \omega$, then g will satisfy the condition of (2). Thus (6) implies (2). □

An open problem is whether the condition from Lemma 2 is equivalent to (1)–(6). It can be formulated as follows.

Problem 6. Let κ be the minimum cardinality of a family $\mathcal{F} \subseteq [\omega]^\omega$ for which there exists $h \in \omega^\omega$ such that for all $g \in \omega^\omega$, if $(\forall n)\, g(n+1) \geq h(g(n))$, then $(\exists X \in \mathcal{F})\ |\mathrm{rng}(g) \cap X| < \omega$. Is it consistent that $\kappa > \mathrm{cov}(\mathcal{M})$?

2. Trigonometric thin sets

From many types of trigonometric thin sets, we will consider the following four: N-sets, N_0-sets, A-sets, and pD-sets. A set X of real numbers is called an *N-set* if there exists a trigonometric series absolutely converging on X while not absolutely converging everywhere, or equivalently, if there exists a sequence of positive reals

$\{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n = \infty$, and for all $x \in X$, $\sum_{n=1}^{\infty} a_n \left|\sin \pi n x\right| < \infty$. It is called an *$N_0$-set* (or an *A-set*, a *pD-set*) if there exists an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ such that for all $x \in X$, $\sum_{k=0}^{\infty} \left|\sin \pi n_k x\right| < \infty$ (or $\lim_{k\to\infty} \left|\sin \pi n_k x\right| = 0$, $\left|\sin \pi n_k x\right| \leq 2^{-k}$ for almost all k, respectively). Let us note that in the definitions above, we can use the distance of x to the nearest integer, i.e. the function $\|x\| = \min\{x - k : k \in \mathbb{Z}\}$, instead of the function $\left|\sin \pi x\right|$. More about these types of sets can be found, e.g., in [6], [7].

We denote by $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$, and $p\mathcal{D}$ the families of all N-sets, N_0-sets, A-sets, and pD-sets, respectively. Directly from the definitions we can see that $\mathcal{N}_0 \subseteq \mathcal{N}$, $\mathcal{N}_0 \subseteq \mathcal{A}$, and $p\mathcal{D} \subseteq \mathcal{N}_0$. In [9], S. Kahane showed that there exist N_0-sets which cannot be covered by countably many pD-sets, N-sets which cannot be covered by countably many A-sets, and A-sets which cannot be covered by countably many N-sets. In [7] we have proved that 'countably many' can be replaced by 'fewer than $\mathrm{add}(\mathcal{M})$', where $\mathrm{add}(\mathcal{M})$ is the additivity of the ideal $\mathcal{M}$ of all first category sets. Now, we will improve these results to 'fewer than $\mathrm{cov}(\mathcal{M})$' (Theorems 12, 13, and 16). Let us note that it is consistent with ZFC that $\mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$; the consistency is established by the Cohen model (see e.g. [2]).

By *an interval* we will mean a set of the form $\{x \in \mathbb{R} : a \leq x \leq b\}$, for some $a < b$. Thus an interval will always be compact and have non-empty interior. The length of an interval I, denoted by $\lambda(I)$, is the value $b - a$.

Lemma 7. *Let $n \geq 1$, $0 < \varepsilon \leq 1$. For every interval I such that $\lambda(I) = 1/n$ there exists an interval $J \subseteq I$ such that $\|nx\| \leq \varepsilon$ for all $x \in J$, and $\lambda(J) = \varepsilon/n$.*

Proof. Since $\lambda(I) = 1/n$, there exists $x_0 \in I$ such that $\|nx_0\| = 0$. Let J be any sub-interval of I such that $x_0 \in J$ and $\lambda(J) = \varepsilon/n$. □

Lemma 8. *Let $n \geq 1$, $0 < \varepsilon < 1/2$. For every interval I such that $\lambda(I) > 2\varepsilon/n$ there exists an interval $J \subseteq I$ such that $\|nx\| \geq \varepsilon$ for all $x \in J$, and*

$$\lambda(J) = \min\left\{\frac{\lambda(I)}{2} - \frac{\varepsilon}{n}, \frac{1-2\varepsilon}{n}\right\}.$$

Proof. Assume that $I = [a-b, a+b]$, thus $b = \lambda(I)/2 > \varepsilon/n$. If $\|nx_0\| = 0$ for some $x_0 \in [a-b-\varepsilon/n, a]$, then let us take

$$J = \left[x_0 + \frac{\varepsilon}{n}, x_0 + \min\left\{b, \frac{1-\varepsilon}{n}\right\}\right].$$

Clearly $J \subseteq I$, $\lambda(J) = \min\{b - \varepsilon/n, (1-2\varepsilon)/n\}$, and if $x \in J$, then $\varepsilon/n \leq |x - x_0| \leq (1-\varepsilon)/n$, thus $\|nx\| \geq \varepsilon$. Similarly, if $\|nx_0\| = 0$ for some $x_0 \in [a, a+b+\varepsilon/n]$, then we can take

$$J = \left[x_0 - \min\left\{b, \frac{1-\varepsilon}{n}\right\}, x_0 - \frac{\varepsilon}{n}\right].$$

Finally, if $\|nx\| > 0$ for all $x \in [a-b-\varepsilon/n, a+b+\varepsilon/n]$, then clearly $\|nx\| \geq \varepsilon$ for all $x \in I$. We can take arbitrary interval $J \subseteq I$ with $\lambda(J) = \min\{b - \varepsilon/n, (1-2\varepsilon)/n\}$. □

Corollary 9. *Let $1 \leq n \leq n'$. For every interval I such that $\lambda(I) = 1/n$, there exists an interval $J \subseteq I$ such that $\lambda(J) = 1/n'$, and if $x \in J$, then $\|nx\| \leq n/n'$.*

Proof. Use Lemma 7 with $\varepsilon = n/n'$. □

Corollary 10. *Let $1 \le n \le 2m \le n'$, $0 < \varepsilon \le 1/4$, and $n/n' \le \varepsilon$. For every interval I such that $\lambda(I) = 1/n$, there exists an interval $J \subseteq I$ such that $\lambda(J) = 1/n'$, and if $x \in J$, then $\|nx\| \le 4\varepsilon$, and $\|mx\| \ge \varepsilon/2$.*

Proof. By Lemma 7 there exists an interval $I' \subseteq I$ such that $\|nx\| \le 4\varepsilon$ for all $x \in I'$, and $\lambda(I') = 4\varepsilon/n$. By Lemma 8 there exists an interval $J' \subseteq I'$ such that $\|mx\| \ge \varepsilon/2$ for all $x \in J'$, and

$$\lambda(J') = \min\left\{\frac{2\varepsilon}{n} - \frac{\varepsilon}{2m}, \frac{1-\varepsilon}{m}\right\} \ge \min\left\{\frac{\varepsilon}{n}, \frac{1}{2m}\right\} \ge \frac{1}{n'}.$$

Let J be any interval such that $J \subseteq J'$ and $\lambda(J) = 1/n'$. □

Lemma 11. *Let $\{n_k\}_{k\in\omega}$, $\{m_i\}_{i\in\omega}$ be increasing sequences of natural numbers, $\{\varepsilon_i\}_{i\in\omega}$ be a sequence of reals, and $k \in \omega^\omega$ be such that for all i,*

(1) $n_{k(i)} \le 2m_i < n_{k(i)+1}$,
(2) $k(i+1) > k(i)$,
(3) $n_{k(i)}/n_{k(i)+1} \le \varepsilon_i \le 1/4$.

Then there exists a real x such that

(4) *for all k,* $\|n_k x\| \le \begin{cases} 4\varepsilon_i & \text{if } k = k(i) \text{ for some } i, \\ n_k/n_{k+1} & \text{if } k \notin \{k(i) : i \in \omega\}, \end{cases}$
(5) *for all i,* $\|m_i x\| \ge \varepsilon_i/2$.

Proof. We take $x \in \bigcap_{k\in\omega} I_k$ where $\{I_k\}_{k\in\omega}$ is a nested sequence of intervals defined as follows. Let I_0 be such that $\lambda(I_0) = 1/n_0$. For every k, assume that $\lambda(I_k) = 1/n_k$. If $k \notin \{k(i) : i \in \omega\}$, then by Corollary 9 there exists an interval $I_{k+1} \subseteq I_k$ such that $\lambda(I_{k+1}) = 1/n_{k+1}$ and $\|n_k x\| \le n_k/n_{k+1}$ for all $x \in I_{k+1}$. If $k = k(i)$ for some i, then by Corollary 10 there exists $I_{k+1} \subseteq I_k$ such that $\lambda(I_{k+1}) = 1/n_{k+1}$, and if $x \in I_{k+1}$, then $\|n_k x\| \le 4\varepsilon_i$, and $\|m_i x\| \ge \varepsilon_i/2$. □

Theorem 12. *Let $\{n_k\}_{k\in\omega}$ be an increasing sequence of natural numbers such that $\sum_{k\in\omega} n_k/n_{k+1} < \infty$. Then the N_0-set $\left\{x : \sum_{k\in\omega} \|n_k x\| < \infty\right\}$ cannot be covered by a union of fewer than $\operatorname{cov}(\mathcal{M})$ pD-sets.*

Proof. Let $\{X_\alpha\}_{\alpha<\kappa}$ be a sequence of pD-sets, $\kappa < \operatorname{cov}(\mathcal{M})$. We will find a real x such that $\sum_{k\in\omega} \|n_k x\| < \infty$ and $x \notin \bigcup_{\alpha<\kappa} X_\alpha$.

Fix a monotone sequence of positive reals $\{\varepsilon_j\}_{j\in\omega}$ such that $\sum \varepsilon_j < \infty$, and $\varepsilon_j \le 1/4$ for all j. Find an increasing function $k^* \in \omega^\omega$ such that if $j \ge k^*(i)$, then $n_j/n_{j+1} \le \varepsilon_i$, and put $h(i) = n_{k^*(i+1)}$. For $\alpha < \kappa$, let $\left\{m_j^\alpha\right\}_{j\in\omega}$ be an increasing sequence of natural numbers such that $(\forall x \in X_\alpha)(\forall^\infty j)\ \left\|m_j^\alpha x\right\| < \varepsilon_j/2$ and $m_0^\alpha \ge n_{k^*(0)}/2$. By Lemma 3 there exists $g \in \omega^\omega$ such that $(\forall i)\, g(i+1) \ge h(g(i))$, and $(\forall \alpha < \kappa)(\exists^\infty j)(\exists i \le j)\, g(i) = 2m_j^\alpha$. Without a loss of generality we may assume that $(\forall i)(\exists \alpha < \kappa)(\exists j \ge i)\, g(i) = 2m_j^\alpha$, thus every $g(i)$ is even. Put $m_i = g(i)/2$.

Let $k \in \omega^\omega$ be such that for all i, $n_{k(i)} \le 2m_i < n_{k(i)+1}$. Clearly $m_0 \ge n_{k^*(0)}/2$, and thus $k(0) \ge k^*(0)$. We have $g(i+1) \ge n_{k'} > g(i)$, where $k' = k^*(g(i)+1)$, hence $k(i+1) \ge k' > k(i)$. Since g is increasing, we have $k' \ge k^*(i+1)$. Thus for all $i \in \omega$, $k(i) \ge k^*(i)$, and hence $n_{k(i)}/n_{k(i)+1} \le \varepsilon_i$.

Now, for a real x obtained from Lemma 11 we have $\sum_{k\in\omega} \|n_k x\| \le \sum_{i\in\omega} 4\varepsilon_i + \sum_{k\in\omega} n_k/n_{k+1} < \infty$, and for every $\alpha < \kappa$ there exist infinitely many j-s such that for some $i \le j$, $\left\|m_j^\alpha x\right\| = \|m_i x\| \ge \varepsilon_i/2 \ge \varepsilon_j/2$, thus $x \notin \bigcup_{\alpha<\kappa} X_\alpha$. □

Theorem 13. *Let $\{n_k\}_{k\in\omega}$ be an increasing sequence of natural numbers and let $\{\varrho_k\}_{k\in\omega}$ be a sequence of non-negative reals such that $\lim_{k\to\infty}\varrho_k = 0$, $\sum_{k\in\omega}\varrho_k = \infty$, and $\sum_{k\in\omega}\varrho_k n_k/n_{k+1} < \infty$. Then the N-set $\left\{x : \sum_{k\in\omega}\varrho_k \|n_k x\| < \infty\right\}$ cannot be covered by a union of fewer than $\operatorname{cov}(\mathcal{M})$ A-sets.*

Proof. Put $K = \{k : n_k/n_{k+1} \le 1/4\}$. Since $\sum_{k\in\omega}\varrho_k n_k/n_{k+1} < \infty$, we have $\sum_{k\notin K}\varrho_k < \infty$, and since $\sum_{k\in\omega}\varrho_k = \infty$, the set K is infinite. For $j \in \omega$, let k_j denote its j-th element. Clearly $\lim_{j\to\infty}\varrho_{k_j} = 0$, $\sum_{j\in\omega}\varrho_{k_j} = \infty$, and for every x, $\sum_{k\in\omega}\varrho_k\|n_k x\| < \infty$ iff $\sum_{j\in\omega}\varrho_{k_j}\left\|n_{k_j}x\right\| < \infty$. Moreover, $n_{k_j}/n_{k_{j+1}} \le n_{k_j}/n_{k_j+1} \le 1/4$ for all $j \in \omega$. Hence by taking $\{n_{k_j}\}_{j\in\omega}$ instead of $\{n_k\}_{k\in\omega}$ we can ensure that $n_k/n_{k+1} \le 1/4$ for all $k \in \omega$.

Let $k^* \in \omega^\omega$ be an increasing function such that for all i, if $j \ge k^*(i)$, then $\varrho_j \le 2^{-i}$. Let $\{X_\alpha\}_{\alpha<\kappa}$ be a sequence of A-sets, $\kappa < \operatorname{cov}(\mathcal{M})$. For every $\alpha < \kappa$, let $\left\{m_j^\alpha\right\}_{j\in\omega}$ be an increasing sequence of natural numbers such that $m_0^\alpha \ge n_{k^*(0)}$, and for all $x \in X_\alpha$, $\lim_{j\to\infty}\left\|m_j^\alpha x\right\| = 0$.

Similarly as in Theorem 12, there exists an increasing sequence of natural numbers $\{m_i\}_{i\in\omega}$ and an increasing function $k \in \omega^\omega$ such that for all i, $n_{k(i)} \le 2m_i < n_{k(i)+1}$ and $k(i) \ge k^*(i)$ and thus $\varrho_{k(i)} \le 2^{-i}$. By putting $\varepsilon_i = 1/4$ in Lemma 11 we can find a real x such that $\|n_k x\| \le n_k/n_{k+1}$ for all $k \notin \{k(i) : i \in \omega\}$, and $\|m_i x\| \ge 1/8$ for all i. Clearly $\sum_{k\in\omega}\varrho_k\|n_k x\| \le \sum_{i\in\omega}2^{-i} + \sum_{k\in\omega}\varrho_k n_k/n_{k+1} < \infty$, and for every $\alpha < \kappa$ there exist infinitely many j-s such that for some i, $\left\|m_j^\alpha x\right\| = \|m_i x\| \ge 1/8$, thus $x \notin \bigcup_{\alpha<\kappa} X_\alpha$. □

The following lemma is a reformulation of Lemma 2.9 in [7].

Lemma 14. *Let $\{\varrho_n\}_{n\in\omega}$ be a sequence of non-negative reals, let $\alpha \le \beta$, $0 < \theta \le 1/7$, and let I be an interval such that $\lambda(I) \ge 3\theta/\alpha$. Then there exists an interval $J \subseteq I$ such that $\lambda(J) \ge \theta/\beta$, and if $x \in J$, then*

$$\sum_{\alpha\le n\le\beta}\varrho_n\|nx\| \ge \frac{\theta}{8}\sum_{\alpha\le n\le\beta}\varrho_n.$$

Proof. At first let us consider the case $\beta \le 3\alpha$. There exists x_0 such that

$$\left[x_0 - \frac{\theta}{2\beta}, x_0 + \frac{3\theta}{2\alpha} + \frac{\theta}{2\beta}\right] \subseteq I.$$

Let $K = \{n \in \omega : \alpha \le n \le \beta \wedge \|nx_0\| \ge \theta\}$, $L = \{n \in \omega : \alpha \le n \le \beta \wedge \|nx_0\| < \theta\}$.

If $\sum_{n\in K}\varrho_n \ge \frac{1}{2}\sum_{n\in K\cup L}\varrho_n$, put $J = \left[x_0 - \frac{\theta}{2\beta}, x_0 + \frac{\theta}{2\beta}\right]$. Then for $x \in J$ and $n \in K$ we obtain $\|nx\| \ge \|nx_0\| - n\,|x - x_0| \ge \theta - \theta/2 = \theta/2$, and thus

$$\sum_{n\in K\cup L}\varrho_n\|nx\| \ge \frac{\theta}{4}\sum_{n\in K\cup L}\varrho_n. \tag{1}$$

Otherwise, we have $\sum_{n\in L}\varrho_n \ge \frac{1}{2}\sum_{n\in K\cup L}\varrho_n$. Put $J = \left[x_0 + \frac{3\theta}{2\alpha}, x_0 + \frac{3\theta}{2\alpha} + \frac{\theta}{\beta}\right]$. For $x \in J$ and $n \in L$ we obtain $3\theta/2 \le n\,|x - x_0| \le 11\theta/2 \le 1 - 3\theta/2$, and thus $\|nx\| \ge \|n(x - x_0)\| - \|nx_0\| \ge 3\theta/2 - \theta = \theta/2$. Hence (1) holds true again.

In the case that $\beta > 3\alpha$, define a sequence $\alpha_0 < \alpha_1 < \cdots < \alpha_k$ as follows: put $\alpha_0 = \alpha$, $\alpha_{i+1} = 3\alpha_i$ for $0 \le i < k$, and $\alpha_k = \beta \le 3\alpha_{k-1}$. Let $K' = \{n \in \omega : \alpha_i \le n \le \alpha_{i+1}$ for some i even$\}$, $L' = \{n \in \omega : \alpha_i \le n \le \alpha_{i+1}$ for some i odd$\}$.

If $\sum_{n\in K'} \varrho_n \geq \frac{1}{2}\sum_{n\in K'\cup L'} \varrho_n$, find a sequence of intervals $I \supseteq I_0 \supseteq I_2 \supseteq \dots$ such that for every i, $0 \leq i \leq k$, i even,

$$\lambda(I_i) \geq \frac{\theta}{\alpha_{i+1}} \geq \frac{3\theta}{\alpha_{i+2}}, \text{ and} \tag{2}$$

$$\sum_{\alpha_i \leq n \leq \alpha_{i+1}} \varrho_n \|nx\| \geq \frac{\theta}{4} \sum_{\alpha_i \leq n \leq \alpha_{i+1}} \varrho_n \text{ for all } x \in I_i. \tag{3}$$

Similarly, if $\sum_{n\in L'} \varrho_n \geq \frac{1}{2}\sum_{n\in K'\cup L'} \varrho_n$, find a sequence of intervals $I \supseteq I_1 \supseteq I_3 \supseteq \dots$ such that (2) and (3) hold true for every i, $0 \leq i \leq k$, i odd. In both cases, let J be the last interval in the sequence, i.e. I_k if k is even, I_{k-1} if k is odd. □

Lemma 15. *Let $\{n_k\}_{k\in\omega}$ be an increasing sequence of natural numbers and let $\{\varrho_n\}_{n\in\omega}$ be a sequence of non-negative reals such that*

$$\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = 0, \text{ and } \sum_{n\in\omega} \varrho_n = \infty.$$

Let k', $\varepsilon \leq 1$ be such that $n_k/n_{k+1} \leq \varepsilon$ for all $k \geq k'$. Then for every interval I such that $\lambda(I) = 1/n_{k'}$ there exist $k'' > k'$ and an interval $J \subseteq I$ such that $\lambda(J) = 1/n_{k''}$, and for all $x \in J$, $\|n_k x\| \leq \varepsilon$ whenever $k' \leq k < k''$, and

$$\sum_{n_{k'} \leq n < n_{k''}} \varrho_n \|nx\| \geq 1.$$

Proof. For given reals α, β, we denote

$$S(\alpha,\beta) = \sum_{\alpha\leq n\leq\beta} \varrho_n.$$

We put $S(\alpha,\beta) = 0$ if there is no $n \in \omega$ such that $\alpha \leq n \leq \beta$.

Let $\gamma = 1/7$. We will consider two cases.

(a) There exists $\delta \geq \gamma$ such that $\sum_{k\in\omega} S(\gamma n_k, \delta n_k) = \infty$. Then we can find a sequence $\{\theta_k\}_{k\in\omega}$ such that $\lim_{k\to\infty}\theta_k = 0$, $\sum_{k\in\omega}\theta_k S(\gamma n_k, \delta n_k) = \infty$, and for all k, $\theta_k \geq \delta n_k/n_{k+1}$. Put $\alpha_k = \gamma n_k$, $\beta_k = \delta n_k$, for $k \in \omega$.

(b) For all $\delta \geq \gamma$, we have $\sum_{k\in\omega} S(\gamma n_k, \delta n_k) < \infty$. Then for all $\delta \geq \gamma$, we must have $\sum_{k\in\omega} S(\delta n_k, \gamma n_{k+1}) = \infty$. Thus there exists a sequence $\{\delta_k\}_{k\in\omega}$ such that $\lim_{k\to\infty}\delta_k = \infty$, $\sum_{k\in\omega} S(\delta_k n_k, \gamma n_{k+1}) = \infty$, and for all k, $\gamma < \delta_k \leq \gamma n_{k+1}/n_k$. Put $\alpha_k = \delta_k n_k$, $\beta_k = \gamma n_{k+1}$, $\theta_k = \gamma$, for $k \in \omega$.

As can be easily checked, in both cases there exists k_0 such that for all $k \geq k_0$,

$$\alpha_k \leq \beta_k < \alpha_{k+1},\ 0 < \theta_k \leq 1/7,\ \varepsilon/n_k \geq 3\theta_k/\alpha_k, \text{ and } \theta_k/\beta_k \geq 1/n_{k+1}.$$

Moreover, we have $\sum_{k\in\omega}\theta_k S(\alpha_k,\beta_k) = \infty$.

Find $i \geq \max\{k_0, k'\}$ such that $\alpha_i \geq n_{k'}$. There exists $j \geq i$ such that

$$\sum_{i\leq k\leq j} \theta_k S(\alpha_k,\beta_k) \geq 8.$$

Put $k'' = j+1$. Clearly $\beta_j \leq \theta_j n_{k''} < n_{k''}$.

Let us define a nested sequence of intervals $\{I_k\}_{k'\leq k<k''}$ as follows. Put $I_{k'} = I$. We will always have $\lambda(I_k) = 1/n_k$. By Lemma 7, there exists an interval $J_k \subseteq I_k$ such that $\lambda(J_k) = \varepsilon/n_k$, and if $x \in J_k$, then $\|n_k x\| \leq \varepsilon$. If $k < i$, let I_{k+1} be any interval such that $I_{k+1} \subseteq J_k$ and $\lambda(I_{k+1}) = 1/n_{k+1}$. For $i \leq k \leq j$, we have

$\lambda(J_k) \geq 3\theta_k/\alpha_k$, and hence by Lemma 14 there exists an interval $J'_k \subseteq J_k$ such that $\lambda(J'_k) \geq \theta_k/\beta_k$, and if $x \in J'_k$, then

$$\sum_{\alpha_k \leq n \leq \beta_k} \varrho_n \|nx\| \geq \theta_k S(\alpha_k, \beta_k)/8.$$

Let I_{k+1} be any interval such that $I_{k+1} \subseteq J'_k$ and $\lambda(I_{k+1}) = 1/n_{k+1}$.

Put $J = I_{k''}$. For $x \in J$, we have $\|n_k x\| \leq \varepsilon$ whenever $k' \leq k < k''$, and

$$\sum_{n_{k'} \leq n < n_{k''}} \varrho_n \|nx\| \geq \sum_{i \leq k \leq j} \sum_{\alpha_k \leq n \leq \beta_k} \varrho_n \|nx\| \geq \sum_{i \leq k \leq j} \theta_k S(\alpha_k, \beta_k)/8 \geq 1.$$

□

Using this lemma we can prove the last theorem.

Theorem 16. *Let $\{n_k\}_{k\in\omega}$ be an increasing sequence of natural numbers such that $\lim_{k\to\infty} n_k/n_{k+1} = 0$. Then the A-set $\{x : \lim_{k\to\infty} \|n_k x\| = 0\}$ cannot be covered by a union of fewer than $\mathrm{cov}(\mathcal{M})$ N-sets.*

Proof. Let $\{I_i\}_{i\in\omega}$ be an enumeration of all intervals of the form $[m/n_k, (m+1)/n_k]$, with integer m. Let $\{X_\alpha\}_{\alpha<\kappa}$ be a sequence of N-sets, $\kappa < \mathrm{cov}(\mathcal{M})$. We will find a real x such that $\lim_{k\to\infty} \|n_k x\| = 0$ and $x \notin \bigcup_{\alpha<\kappa} X_\alpha$.

For every $\alpha < \kappa$, let $\{\varrho_n^\alpha\}_{n\in\omega}$ be a sequence of non-negative reals such that $\sum_{n\in\omega} \varrho_n^\alpha = \infty$, and for all $x \in X_\alpha$, $\sum_{n\in\omega} \varrho_n^\alpha \|nx\| < \infty$. We will define a function $f_\alpha \in \omega^\omega$ as follows.

For given $i \in \omega$, let k' be such that $\lambda(I_i) = 1/n_{k'}$. Put $\varepsilon = \max\{n_k/n_{k+1} : k \geq k'\}$. By Lemma 15, there exist $k'' > k'$ and an interval $J \subseteq I_i$ such that $\lambda(J) = 1/n_{k''}$, and for all $x \in J$, $\|n_k x\| \leq \varepsilon$ whenever $k' \leq k < k''$, and

$$\sum_{n_{k'} \leq n < n_{k''}} \varrho_n^\alpha \|nx\| \geq 1.$$

Let $k^* \geq k''$ be such that $n_{k^*}/n_{k^*+1} \leq \varepsilon/2$. There exists an interval $J^* \subseteq J$ such that for all $x \in J^*$, $\|n_k x\| \leq n_k/n_{k+1}$ whenever $k'' \leq k < k^*$, $\|n_{k^*} x\| \leq \varepsilon$, and $\lambda(J^*) = \varepsilon/n_{k^*} \geq 2/n_{k^*+1}$. Hence we can find $I_j \subseteq J^*$ such that $\lambda(I_j) = 1/n_{k^*+1}$. We put $f(i) = j$.

By Lemma 4, there exists $g \in \omega^\omega$ such that $(\forall j)(\exists \alpha < \kappa)\, g(j+1) = f_\alpha(g(j))$ and $(\forall \alpha < \kappa)(\exists^\infty j)\, g(j+1) = f_\alpha(g(j))$. For $j \in \omega$, let $k(j)$ be such that $\lambda(I_{g(j)}) = 1/n_{k(j)}$. Clearly $I_{g(j+1)} \subseteq I_{g(j)}$ and $k(j+1) > k(j)$ for all j, and if we put $\varepsilon_j = \max\{n_k/n_{k+1} : k \geq k(j)\}$, then $\lim_{j\to\infty} \varepsilon_j = 0$. Let us take $x \in \bigcap_{j\in\omega} I_{g(j)}$. We have $\|n_k x\| \leq \varepsilon_j$ whenever $k \geq k(j)$, and thus $\lim_{k\to\infty} \|n_k x\| = 0$. Moreover, if $g(j+1) = f_\alpha(g(j))$, then

$$\sum_{n_{k(j)} \leq n < n_{k(j+1)}} \varrho_n^\alpha \|nx\| \geq 1,$$

and hence for all $\alpha < \kappa$, $\sum_{n\in\omega} \varrho_n^\alpha \|nx\| = \infty$, and thus $x \notin X_\alpha$. □

We finish with an open problem.

Problem 17. Is it possible to replace the cardinal $\mathrm{cov}(\mathcal{M})$ in Theorems 12, 13, and 16 by some consistently greater cardinal?

References

[1] Bartoszyński, T., *Combinatorial aspects of measure and category*, Fund. Math. **127** (1987), 225–239. MR **88m:**04001

[2] Bartoszyński, T., Judah, H., *Set theory. On the structure of the real line*, A K Peters, Ltd., Wellesley, MA, 1995. MR **96k:**03002

[3] Bartoszyński, T., Scheepers, M., *Remarks on sets related to trigonometric series*, Topology Appl. **64** (1995), 133–140. MR **96h:**42004

[4] Bary, N. K., *A treatise on trigonometric series*, The Macmillan Co., New York, 1964. MR **30:**1347

[5] Bukovská, Z., Bukovský, L., *Adding small sets to an N-set*, Proc. Amer. Math. Soc. **123** (1995), 3867–3873. MR **96b:**04002

[6] Bukovský, L., Kholshchevnikova, N. N., Repický, M., *Thin sets of harmonic analysis and infinite combinatorics*, Real Anal. Exchange **20** (1994/95), 454–509. MR **97b:**43004

[7] Eliaš, P., *A classification of trigonometrical thin sets and their interrelations*, Proc. Amer. Math. Soc. **125** (1997), 1111-1121. MR **97f:**43006

[8] Kada, M., *Slalom, prediction, and permitted trigonometric thin sets*, preprint.

[9] Kahane, S., *Antistable classes of thin sets in harmonic analysis*, Illinois J. Math. **37** (1993), 186–223. MR **94g:**43005

[10] Laflamme, C., *Combinatorial aspects of F_σ filters with an application to $\mathcal{N}$-sets*, Proc. Amer. Math. Soc. **125** (1997), 3019–3025. MR **97m:**04003

[11] Repický, M., *Towers and permitted trigonometric thin sets*, Real Anal. Exchange **21** (1995/96), 648–655. MR **97f:**42013

Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 04001 Košice, Slovakia

E-mail address: elias@kosice.upjs.sk