COVERING FOR CATEGORY
AND TRIGONOMETRIC THIN SETS

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Abstract. In this work we consider several combinatorial principles satisfied for cardinals smaller than cov(\(\mathcal{M}\)), the covering number of the ideal of first category sets on real line. Using these principles we prove that there exist \(\aleph_0\)-sets (similarly \(N\)-sets, \(A\)-sets) which cannot be covered by fewer than cov(\(\mathcal{M}\)) \(pD\)-sets (\(A\)-sets, \(N\)-sets, respectively). This improves the results of our previous paper (1997).

The study of special sets of reals related to the convergence of trigonometric series, generally called `trigonometric thin sets', was one of the classical topics of harmonic analysis (see e.g. [4]). Recently, many investigations were made concerning the set-theoretic properties of such sets. A serious interest turns on the cardinal characteristics of structures related to trigonometric thin sets (see e.g. [3], [5], [6], [8], [10], [11]).

This paper has two parts. In the first part we present two new combinatorial characterizations of the cardinal cov(\(\mathcal{M}\)), the covering number of the ideal of first category sets (Corollary 5). We use these characterizations in the second part to prove some theorems about trigonometric thin sets (Theorems 12, 13, and 16).

1. Combinatorics

We use the standard set-theoretic notation. Hence, \(\omega\) is the set of all natural numbers, \(|\omega|^{<\omega}\) and \(|\omega|^\omega\) denote the sets of all finite and infinite subsets of \(\omega\), \(A^B\) is the set of all functions from \(B\) to \(A\), and \(|A|\) is the cardinality of a set \(A\). Quantifiers \(\exists^\infty\) and \(\forall^\infty\) stand for `there are infinitely many' and `for all but finitely many', respectively.

Let \(\mathcal{M}\) denote the ideal of first category (meager) sets on real line. Let us recall that cov(\(\mathcal{M}\)), the covering number of \(\mathcal{M}\), is the minimum size of a family \(\mathcal{F} \subseteq \mathcal{M}\) such that \(\bigcup \mathcal{F} = \mathbb{R}\). We will use the following combinatorial characterizations of this cardinal, due to Bartoszyński ([1], [2]).

Lemma 1. Let \(\kappa\) be a cardinal. The following conditions are equivalent:

\((1)\) \(\kappa < \text{cov}(\mathcal{M})\),
(2) for every family $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, there exists a function $g \in \omega^\omega$ such that 
$\left( \forall f \in \mathcal{F}\right) (\exists n) g(n) = f(n), \ \ (3)$ for every family $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, and for every family $X \subseteq [\omega]^\omega$, $|X| \leq \kappa$, there exists $g \in \omega^\omega$ such that 
$\left( \forall f \in \mathcal{F}\right) (\forall X \in X)(\exists n \in X) g(n) = f(n), \ \ (4)$ for every family $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| \leq \kappa$, there exists a function $S \in ([\omega]^{<\omega})^\omega$, such that $\left( \forall n \right) |S(n)| = n + 1$, and $\left( \forall f \in \mathcal{F}\right) (\exists n) f(n) \in S(n).

We will add two more conditions to this list. First, we need to prove some lemmas.

**Lemma 2.** For every family $\mathcal{F} \subseteq [\omega]^\omega$, $|\mathcal{F}| < \text{cov}(\mathcal{M})$, and for every $h \in \omega^\omega$, there exists $g \in \omega^\omega$ such that 
$\left( \forall n \right) g(n + 1) \geq h(g(n))$, and $\left( \forall X \in \mathcal{F}\right) |\text{rng}(g) \cap X| = \omega$.

*Proof.* Let $\mathcal{F} = \{X_\alpha : \alpha < \kappa\}$, $\kappa < \text{cov}(\mathcal{M})$. Without loss of generality, we assume that $h$ is non-decreasing and $h(n) > n$ for all $n$.

Put $k_0 = 0$, $k_{n+1} = h(k_n)$ for $n \in \omega$. For $\alpha < \kappa$ and $n \in \omega$, let $f_\alpha(n) \in \omega$ be such that $\left( \{j : j < f_\alpha(n) \land [k_j, k_{j+1}) \land X_\alpha \neq \emptyset\} \right) = 3n + 1$.

Put $f_\alpha^*(n) = [0, k_{f_\alpha(n)}) \land X_\alpha$. Since $f_\alpha^*$-s can be coded as elements of $\omega^\omega$, by Lemma 1 there exists $g^*$ such that $\left( \forall \alpha < \kappa \right) (\exists n) g^*(n) = f_\alpha^*(n)$. Moreover, we may assume that $\left( \forall n \right) (\exists \alpha < \kappa) g^*(n) = f_\alpha^*(n)$. Thus for each $n$,

$\left( \{j : j < f_\alpha(n) \land [k_j, k_{j+1}) \land g^*(n) \neq \emptyset\} \right) = 3n + 1,$

and we can take $n_\alpha \in g^*(n)$ such that for every $i < n$ and $j \in \omega$, if $m_i \in [k_j, k_{j+1})$, then $m_i \notin [k_{j-1}, k_{j+2})$.

Let $g \in \omega^\omega$ be an increasing function such that $\text{rng}(g) = \{m_\alpha : n \in \omega\}$. For every $n$, if $g(n) \in [k_{j+1}, k_{j+2})$, then $[k_{j+1}, k_{j+2}) \land \text{rng}(g) = \emptyset$, and thus $h(n + 1) \geq k_{j+2} = h(k_{j+1}) \geq h(g(n))$, since $k_{j+1} > g(n)$. Moreover, if $g^*(n) = f_\alpha^*(n)$, then $n_\alpha \in X_\alpha$. Hence $\text{rng}(g) \cap X_\alpha$ is infinite for every $\alpha < \kappa$. \hfill $\Box$

**Lemma 3.** Let $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \text{cov}(\mathcal{M})$, and let every $f \in \mathcal{F}$ be increasing. Then for every $h \in \omega^\omega$ there exists $g \in \omega^\omega$ such that $\left( \forall n \right) g(n + 1) \geq h(g(n))$, and $\left( \forall f \in \mathcal{F}\right) (\exists n) (\exists k \leq n) f(k) = f(n)$.

*Proof.* By Lemma 1 there exists $g$ such that $\left( \forall f \in \mathcal{F}\right) (\exists n) g^*(n) = f(n)$. Moreover, we may assume that $\left( \forall n \in \mathcal{F}\right) (\exists f \in \mathcal{F}) g^*(n) = f(n)$, and hence $g^*(n) \geq n$ for all $n$. For $f \in \mathcal{F}$, put $X_f = \{n : g^*(n) = f(n)\}$ and $h^*(n) = \max\{n + 1, h(g^*(n))\}$. By Lemma 2 there exists $g^* \in \omega^\omega$ such that $\left( \forall n \right) g^*(n + 1) \geq h^*(g^*(n))$, and $\left( \forall f \in \mathcal{F}\right) |\text{rng}(g^*) \cap X_f| = \omega$. Clearly $g^*$ is increasing.

For $n \in \omega$, let $g(n) = g^*(g(n))$. Then for every $n$, $g(n + 1) = g(g^*(n)) \geq g^*(n + 1) \geq h^*(g^*(n)) \geq h(g(g^*(n))) = h(g(n))$. Moreover, if $n \in \text{rng}(g^*) \cap X_f$, then $n = g^*(k)$ for some $k \leq n$, and $g(k) = g^*(g^*(k)) = g^*(n) = f(n)$. \hfill $\Box$

**Lemma 4.** Let $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \text{cov}(\mathcal{M})$. Then there exists a function $g \in \omega^\omega$ such that $\left( \forall n \in \mathcal{F}\right) (\exists f \in \mathcal{F}) g^*(n + 1) = f(g(n))$, and $\left( \forall f \in \mathcal{F}\right) (\exists n) (\exists k \leq n) f(k) = f(g(n))$.

*Proof.* First, we show the existence of $g' \in \omega^\omega$ such that $\left( \forall n \in \mathcal{F}\right) (\exists f \in \mathcal{F}) g'(n + 1) \geq f(g(n))$, and $\left( \forall f \in \mathcal{F}\right) (\exists n) (\exists k \leq n) f(k) = f(g(n))$. Using Lemma 1 we can find $h(n) \in \omega^\omega$ such that $h(n) \geq n$ for all $n$, and the set $X_f = \{n : f(n) \leq h(n)\}$ is infinite for every $f \in \mathcal{F}$. By Lemma 2 there exists $g' \in \omega^\omega$ such that for all $n$, $g'(n + 1) \geq h(g'(n))$, and for all $f \in \mathcal{F}$, $|\text{rng}(g') \cap X_f| = \omega$. Moreover, we may assume that $\left( \forall n \in \mathcal{F}\right) g'(n) \in X_f$. Since $g'$ is increasing, we have $\left( \forall f \in \mathcal{F}\right) \overset{\text{nat}}{\ldots}$

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$F(\exists n) g'(n) \in X_f$. Clearly, if $g'(n) \in X_f$, then $f(g'(n)) \leq h(g'(n)) \leq g'(n + 1)$. Thus $g'$ is as we have expected.

For $f \in F$, $n \in \omega$, denote $f^*(n) = \max\{f(0), \ldots, f(n)\}$. As we have just proved, there exists $g^* \in \omega^\omega$ such that for all $f \in F$, the set $Y_f = \{n : g^*(n+1) \geq f^*(g^*(n))\}$ is infinite, and $\bigcup_{f \in F} Y_f = \omega$.

For given $f \in F$ and $n \in \omega$, put $H_f(n) = f \upharpoonright \{0, \ldots, g^*(n)\}$. Since $H_f(n)$-s can be coded as elements of $\omega$, by Lemma 4 there exists a function $H$ such that $(\forall f \in F)(\exists n \in Y_f) H(n) = H_f(n)$. Without loss of generality we may assume also that $(\exists n)(\forall f \in F)(n \in Y_f \land H(n) = H_f(n))$.

Put $g(0) = 0, g(n + 1) = H(n)(g(n))$. For every $n$, we have $H(n) = H_f(n)$ for some $f \in F$ such that $n \in Y_f$, and hence if $k \in \text{dom}(H(n))$, then $k \leq g^*(n)$, and $H(n)(k) = H_f(n)(k) = f(k) \leq f^*(g^*(n)) \leq g^*(n + 1)$. Thus $H(n)(k) \in \text{dom}(H(n+1))$, and the definition of $g$ makes sense. Moreover, if $H(n) = H_f(n)$, then $g(n + 1) = H_f(n)(g(n)) = f(g(n))$. Hence $g$ has the required properties. \hfill \square

**Corollary 5.** Let $\kappa$ be a cardinal. The following conditions are equivalent to (1)-(4) from Lemma 1:

5. for every family $F \subseteq \omega^\omega$, $|F| \leq \kappa$, such that every $f \in F$ is increasing, and for every $h \in \omega^\omega$, there exists $g \in \omega^\omega$ such that $(\forall n) g(n+1) \geq h(g(n))$, and $(\forall f \in F)(\exists n \leq \kappa) g(k) = f(n),

6. for every family $F \subseteq [\omega]^\omega$, $|F| \leq \kappa$, there exists $g \in \omega^\omega$ such that $(\forall n)(\exists f \in F) g(n+1) = f(g(n)), and $(\forall f \in F)(\exists n \leq \kappa) g(n+1) = f(g(n)).

**Proof.** As we have shown in the previous two lemmas, (1) implies both (5) and (6). We will prove that (5) implies (4), and (6) implies (2).

Assume that (5) holds. Let $F \subseteq \omega^\omega$, $|F| \leq \kappa$, and let $h(n) = n + 1$ for all $n \in \omega$. Clearly if $g \in \omega^\omega$ satisfies the conditions of (5), then $g$ is increasing, and the function $S$ such that $S(n) = \{g(0), \ldots, g(n)\}, n \in \omega$, satisfies the condition of (4). Thus (5) implies (4).

Let $F \subseteq \omega^\omega$, $|F| \leq \kappa$, and let (6) hold true. Fix a bijection $p \in (\omega \times \omega)^\omega$. For $f \in F$, let $f^* \in \omega^\omega$ be such that for all $k, l \in \omega$, $f^*(p(k, l)) = p(k + 1, l)(f(k))$. By (6), there exists $g^* \in \omega^\omega$ such that $(\forall n)(\exists f \in F) g^*(n+1) = f^*(g^*(n))$, and $(\forall f \in F)(\exists n) g^*(n+1) = f^*(g^*(n))$. For $n \in \omega$, let $k_n, l_n$ be such that $g^*(n) = p(k_n, l_n)$. Now, if $g^*(n + 1) = f^*(g^*(n))$ for some $f \in F$ and $n \in \omega$, then $p(k_{n+1}, l_{n+1}) = f^*(p(k_n, l_n)) = p(k_n + 1, f(k_n))$. It follows that for all $n$, $k_n = n + k_0$ and $f(k_n) = l_{n+1}$. If we put $g(n) = 0$ for $n < k_0$, and $g(n + k_0) = l_{n+1}$ for all $n$, then $g$ will satisfy the condition of (2). Thus (6) implies (2). \hfill \square

An open problem is whether the condition from Lemma 2 is equivalent to (1)-(6). It can be formulated as follows.

**Problem 6.** Let $\kappa$ be the minimum cardinality of a family $F \subseteq [\omega]^\omega$ for which there exists $h \in \omega^\omega$ such that for all $g \in \omega^\omega$, if $(\forall n) g(n + 1) \geq h(g(n))$, then $(\exists X \in F) |\text{rng}(g) \cap X| < \omega$. Is it consistent that $\kappa > \text{cov}(M)$?

## 2. Trigonometric thin sets

From many types of trigonometric thin sets, we will consider the following four: $N$-sets, $N_0$-sets, $A$-sets, and $pD$-sets. A set $X$ of real numbers is called an $N$-set if there exists a trigonometric series absolutely converging on $X$ while not absolutely converging everywhere, or equivalently, if there exists a sequence of positive reals
exists an interval $J$. Let $a < b$. Thus an interval will always be compact and have non-empty interior. The length of an interval $I$, denoted by $|I|$, is called an $N_0$-set (or an $A$-set, a p$D$-set) if there exists an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ such that for all $x \in X$, $\sum_{k=0}^{\infty} |\sin \pi n_k x| < \infty$ (or $\lim_{k \to \infty} |\sin \pi n_k x| = 0$, $|\sin \pi n_k x| \leq 2^{-k}$ for almost all $k$, respectively). Let us note that in the definitions above, we can use the distance of $x$ to the nearest integer, i.e. the function $|x| = \min \{x - k : k \in \mathbb{Z}\}$, instead of the function $|\sin \pi x|$. More about these types of sets can be found, e.g., in [3, 7].

We denote by $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$, and $\mathcal{pD}$ the families of all N-sets, $N_0$-sets, A-sets, and p$D$-sets, respectively. Directly from the definitions we can see that $\mathcal{N}_0 \subseteq \mathcal{N}$, $\mathcal{N}_0 \subseteq \mathcal{A}$, and $\mathcal{pD} \subseteq \mathcal{N}_0$. In [9], S. Kahane showed that there exist $N_0$-sets which cannot be covered by countably many p$D$-sets, N-sets which cannot be covered by countably many A-sets, and A-sets which cannot be covered by countably many N-sets. In [11] we have proved that ‘countably many’ can be replaced by ‘fewer than add$($M$)$’, where add$($M$)$ is the additivity of the ideal $M$ of all first category sets. Now, we will improve these results to ‘countably many’ (Theorems 12, 13 and 14). Let us note that it is consistent with ZFC that add$($M$) < \text{cov}(\mathcal{M})$; the consistency is established by the Cohen model (see e.g. [2]).

By an interval we will mean a set of the form $\{x \in \mathbb{R} : a \leq x \leq b\}$, for some $a < b$. Thus an interval will always be compact and have non-empty interior. The length of an interval $I$, denoted by $|I|$, is the value $b-a$.

**Lemma 7.** Let $n \geq 1$, $0 < \varepsilon \leq 1$. For every interval $I$ such that $\lambda(I) = 1/n$ there exists an interval $J \subseteq I$ such that $|nx| \leq \varepsilon$ for all $x \in J$, and $\lambda(J) = \varepsilon/n$.

**Proof.** Since $\lambda(I) = 1/n$, there exists $x_0 \in I$ such that $|nx_0| = 0$. Let $J$ be any sub-interval of $I$ such that $x_0 \in J$ and $\lambda(J) = \varepsilon/n$. \hfill $\square$

**Lemma 8.** Let $n \geq 1$, $0 < \varepsilon < 1/2$. For every interval $I$ such that $\lambda(I) > 2\varepsilon/n$ there exists an interval $J \subseteq I$ such that $|nx| \geq \varepsilon$ for all $x \in J$, and

$$
\lambda(J) = \min \left\{ \frac{\lambda(I)}{2} - \frac{\varepsilon}{n}, \frac{1 - 2\varepsilon}{n} \right\}.
$$

**Proof.** Assume that $I = [a-b, a+b]$, thus $b = \lambda(I)/2 > \varepsilon/n$. If $|nx_0| = 0$ for some $x_0 \in [a-b-\varepsilon/n, a]$, then let us take

$$
J = \left[ x_0 + \frac{\varepsilon}{n}, x_0 + \min \left\{ b, \frac{1 - \varepsilon}{n} \right\} \right] .
$$

Clearly $J \subseteq I$, $\lambda(J) = \min \{ b-\varepsilon/n, (1-2\varepsilon)/n \}$, and if $x \in J$, then $\varepsilon/n \leq |x-x_0| \leq (1-\varepsilon)/n$, thus $|nx| \geq \varepsilon$. Similarly, if $|nx_0| = 0$ for some $x_0 \in [a, a+b+\varepsilon/n]$, then we can take

$$
J = \left[ x_0 - \min \left\{ b, \frac{1 - \varepsilon}{n} \right\}, x_0 - \frac{\varepsilon}{n} \right] .
$$

Finally, if $|nx| > 0$ for all $x \in [a-b-\varepsilon/n, a+b+\varepsilon/n]$, then clearly $|nx| \geq \varepsilon$ for all $x \in I$. We can take arbitrary interval $J \subseteq I$ with $\lambda(J) = \min \{ b - \varepsilon/n, (1-2\varepsilon)/n \}$. \hfill $\square$

**Corollary 9.** Let $1 \leq n \leq n'$. For every interval $I$ such that $\lambda(I) = 1/n$, there exists an interval $J \subseteq I$ such that $\lambda(J) = 1/n'$, and if $x \in J$, then $|nx| \leq n/n'$.

**Proof.** Use Lemma 7 with $\varepsilon = n/n'$. \hfill $\square$
Corollary 10. Let $1 \leq n \leq 2m \leq n'$, $0 < \varepsilon \leq 1/4$, and $n/n' \leq \varepsilon$. For every interval $I$ such that $\lambda(I) = 1/n$, there exists an interval $J \subseteq I$ such that $\lambda(J) = 1/n'$, and if $x \in J$, then $\|nx\| \leq 4\varepsilon$, and $\|mx\| \geq \varepsilon/2$.

Proof. By Lemma 7 there exists an interval $I' \subseteq I$ such that $\|nx\| \leq 4\varepsilon$ for all $x \in I'$, and $\lambda(I') = 4\varepsilon/n$. By Lemma 8 there exists an interval $J' \subseteq I'$ such that $\|nx\| \geq \varepsilon/2$ for all $x \in J'$, and

$$\lambda(J') = \min \left\{ \frac{2\varepsilon}{n} - \frac{\varepsilon}{2m}, \frac{1 - \varepsilon}{m} \right\} \geq \min \left\{ \frac{\varepsilon}{n}, \frac{1}{2m} \right\} \geq \frac{1}{n'}.$$

Let $J$ be any interval such that $J \subseteq J'$ and $\lambda(J) = 1/n'$.

Lemma 11. Let $\{n_k\}_{k \in \omega}$, $\{m_i\}_{i \in \omega}$ be increasing sequences of natural numbers, $\{\varepsilon_i\}_{i \in \omega}$ be a sequence of reals, and $k \in \omega^n$ be such that for all $i$,

1. $n_k(i) \leq 2m_i < n_{k(i)+1}$,
2. $k(i+1) > k(i)$,
3. $n_k(i)/n_{k(i)+1} \leq \varepsilon_i \leq 1/4$.

Then there exists a real $x$ such that

1. for all $k$, $\|n_kx\| \leq \begin{cases} \frac{4\varepsilon_i}{n_k/n_{k+1}} & \text{if } k = k(i) \text{ for some } i, \\ n_k/n_{k+1} & \text{if } k \notin \{k(i) : i \in \omega\}, \end{cases}$
2. for all $i$, $\|m_ix\| \geq \varepsilon_i/2$.

Proof. We take $x \in \bigcap_{k \in \omega} I_k$ where $\{I_k\}_{k \in \omega}$ is a nested sequence of intervals defined as follows. Let $I_0$ be such that $\lambda(I_0) = 1/n_0$. For every $k$, assume that $\lambda(I_k) = 1/n_k$.

If $k \notin \{k(i) : i \in \omega\}$, then by Corollary 9 there exists an interval $I_{k+1} \subseteq I_k$ such that $\lambda(I_{k+1}) = 1/n_{k+1}$ and $\|n_kx\| \leq n_k/n_{k+1}$ for all $x \in I_{k+1}$. If $k = k(i)$ for some $i$, then by Corollary 10 there exists $I_{k+1} \subseteq I_k$ such that $\lambda(I_{k+1}) = 1/n_{k+1}$, and if $x \in I_{k+1}$, then $\|n_kx\| \leq 4\varepsilon_i$, and $\|m_ix\| \geq \varepsilon_i/2$.

Theorem 12. Let $\{n_k\}_{k \in \omega}$ be an increasing sequence of natural numbers such that $\sum_{k \in \omega} n_k/n_{k+1} < \infty$. Then the $N_0$-set $\{x : \sum_{k \in \omega} \|n_kx\| < \infty\}$ cannot be covered by a union of fewer than $\cov(M)$ pD-sets.

Proof. Let $\{X_\alpha\}_{\alpha < \kappa}$ be a sequence of pD-sets, $\kappa < \cov(M)$. We will find a real $x$ such that $\sum_{k \in \omega} \|n_kx\| < \infty$ and $x \notin \bigcup_{\alpha < \kappa} X_\alpha$.

Fix a monotone sequence of positive reals $\{\varepsilon_j\}_{j \in \omega}$ such that $\sum \varepsilon_j < \infty$, and $\varepsilon_j \leq 1/4$ for all $j$. Find an increasing function $k^* \in \omega^\omega$ such that if $j \geq k^*(i)$, then $n_j/n_{j+1} \leq \varepsilon_i$, and put $h(i) = n_k(i+1)$. For $\alpha < \kappa$, let $\{m^\alpha_j\}_{j \in \omega}$ be an increasing sequence of natural numbers such that $\forall x \in X_\alpha(x) \|m^\alpha_jx\| < \varepsilon_j/2$ and $m^\alpha_j \geq n_{k^*(i)}/2$. By Lemma 4 there exists $g \in \omega^\omega$ such that $(\forall i) g(i+1) \geq h(g(i))$, and $(\forall \alpha < \kappa)(\exists \varepsilon) (\exists i \leq j) g(i) = 2m^\alpha_i$. Without a loss of generality we may assume that $(\forall i)(\exists \alpha < \kappa)(\exists j \geq i) g(i) = 2m^\alpha_i$, thus every $g(i)$ is even. Put $m_i = g(i)/2$.

Let $k \in \omega^\omega$ be such that for all $i$, $n_k(i) \leq 2m_i < n_{k(i)+1}$.

Clearly $m_0 \geq n_{k(0)+1}/2$, and thus $k(0) \geq k^*(0)$. We have $g(i+1) \geq n_{k'} > g(i)$, where $k' = k^*(g(i)+1)$, hence $k(i+1) \geq k' > k(i)$. Since $g$ is increasing, we have $k' \geq k^*(i+1)$. Thus for all $i \in \omega$, $k(i) \geq k^*(i)$, and hence $n_{k(i)}/n_{k(i)+1} \leq \varepsilon_i$.

Now, for a real $x$ obtained from Lemma 11 we have $\sum_{k \in \omega} \|n_kx\| \leq \sum_{i \in \omega} 4\varepsilon_i + \sum_{k \in \omega} n_k/n_{k+1} < \infty$, and for every $\alpha < \kappa$ there exist infinitely many $j$-s such that for some $i \leq j$, $\|m^\alpha_jx\| = \|m_jx\| \geq \varepsilon_i/2 \geq \varepsilon_j/2$, thus $x \notin \bigcup_{\alpha < \kappa} X_\alpha$. □
Theorem 13. Let \( \{n_k\}_{k \in \omega} \) be an increasing sequence of natural numbers and let \( \{g_k\}_{k \in \omega} \) be a sequence of non-negative reals such that \( \lim_{k \to \infty} g_k = 0 \), \( \sum_{k \in \omega} g_k = \infty \), and \( \sum_{k \in \omega} g_k n_k/n_{k+1} < \infty \). Then the N-set \( \{x : \sum_{k \in \omega} g_k \|n_kx\| < \infty \} \) cannot be covered by a union of fewer than \( \text{cov}(M) \) A-sets.

Proof. Put \( K = \{k : n_k/n_{k+1} \leq 1/4\} \). Since \( \sum_{k \in K} g_k < \infty \), and since \( \sum_{k \in \omega} g_k = \infty \), the set \( K \) is infinite. For \( j \in \omega \), let \( k_j \) denote its \( j \)-th element. Clearly \( \lim_{j \to \infty} g_{k_j} = 0 \), \( \sum_{j \in \omega} g_{k_j} = \infty \), and for every \( x \), \( \sum_{k \in \omega} g_k \|n_kx\| < \infty \) iff \( \sum_{j \in \omega} g_{k_j} \|n_{k_j}x\| < \infty \). Moreover, \( n_{k_j}/n_{k_{j+1}} < n_{k_j}/n_{k_{j+1}} \leq 1/4 \) for all \( j \in \omega \). Hence by taking \( \{n_{k_j}\}_{j \in \omega} \) instead of \( \{n_k\}_{k \in \omega} \) we can ensure that \( n_k/n_{k+1} \leq 1/4 \) for all \( k \in \omega \).

Let \( k^* \in \omega^\omega \) be an increasing function such that for all \( i \), if \( j \leq k^*(i) \), then \( g_j \leq 2^{-i} \). Let \( \{X_{\alpha}\}_{\alpha < \kappa} \) be a sequence of A-sets, \( \kappa < \text{cov}(M) \). For every \( \alpha < \kappa \), let \( \{m_j^\alpha\}_{j \in \omega} \) be an increasing sequence of natural numbers such that \( m_0^\alpha \geq n_{k^*(0)} \), and for all \( x \in X_{\alpha} \), \( \lim_{j \to \infty} \|m_j^\alpha x\| = 0 \).

Similarly as in Theorem 12 there exists an increasing sequence of natural numbers \( \{m_i\}_{i \in \omega} \) and an increasing function \( k \in \omega^\omega \) such that for all \( i \), \( n_{k(i)} \leq 2m_i < n_{k(i)+1} \) and \( k(i) \geq k^*(i) \) and thus \( g_{k(i)} \leq 2^{-i} \). By putting \( \varepsilon_i = 1/4 \) in Lemma 13 we can find a real \( x \) such that \( \|n_kx\| \leq n_k/n_{k+1} \) for all \( k \notin \{k(i) : i \in \omega \} \), and \( \|m_i x\| \geq \alpha /8 \) for all \( i \). Clearly \( \sum_{k \in \omega} g_k \|n_kx\| \leq \sum_{i \in \omega} 2^{-i} + \sum_{k \in \omega} g_k n_k/n_{k+1} < \infty \), and for every \( \alpha < \kappa \) there exist infinitely many \( j \)-s such that for some \( i \), \( \|m_j^\alpha x\| = \|m_i x\| \geq \alpha/8 \), thus \( x \notin \bigcup_{\alpha < \kappa} X_{\alpha} \).

The following lemma is a reformulation of Lemma 2.9 in 7.

Lemma 14. Let \( \{g_n\}_{n \in \omega} \) be a sequence of non-negative reals, let \( \alpha \leq \beta \leq 0 < \theta \leq 1/7 \), and let \( I \) be an interval such that \( \lambda(I) > 3\theta/\alpha \). Then there exists an interval \( J \subseteq I \) such that \( \lambda(J) \geq \theta/\beta \), and if \( x \in J \), then

\[
\sum_{\alpha \leq n \leq \beta} g_n \|nx\| \geq \frac{\theta}{8} \sum_{\alpha \leq n \leq \beta} g_n.
\]

Proof. At first let us consider the case \( \beta \leq 3\alpha \). There exists \( x_0 \) such that

\[
\left[ x_0 - \frac{\theta}{2\beta}, x_0 + \frac{3\theta}{2\alpha} + \frac{\theta}{2\beta} \right] \subseteq I.
\]

Let \( K = \{n \in \omega : \alpha \leq n \leq \beta \land \|nx_0\| \geq \theta\} \); \( L = \{n \in \omega : \alpha \leq n \leq \beta \land \|nx_0\| < \theta\} \).

If \( \sum_{n \in K} g_n \geq \frac{1}{4} \sum_{n \in K \cup L} g_n \), put \( J = \left[ x_0 - \frac{\theta}{2\beta}, x_0 + \frac{\theta}{2\beta} \right] \). Then for \( x \in J \) and \( n \in K \) we obtain \( \|nx\| \geq \|nx_0\| - n|x - x_0| \geq \theta - \theta/2 = \theta/2 \), and thus

\[
\sum_{n \in K} g_n \|nx\| \geq \frac{\theta}{4} \sum_{n \in K \cup L} g_n.
\]

Otherwise, we have \( \sum_{n \in L} g_n \geq \frac{1}{4} \sum_{n \in K \cup L} g_n \). Put \( J = \left[ x_0 + \frac{3\theta}{2\alpha}, x_0 + \frac{3\theta}{2\alpha} + \frac{\theta}{2\beta} \right] \). For \( x \in J \) and \( n \in L \) we obtain \( \theta/2 \leq n|x - x_0| \leq 11\theta/2 \leq 1 - \theta/2 \), and thus \( \|nx\| \geq \|nx_0 - n|x - x_0|\| - \|nx_0\| \geq 3\theta/2 - \theta = \theta/2 \). Hence (1) holds true again.

In the case that \( \beta > 3\alpha \), define a sequence \( \alpha_0 < \alpha_1 < \cdots < \alpha_k \) as follows: put \( \alpha_0 = \alpha \), \( \alpha_{i+1} = 3\alpha_i \) for \( 0 \leq i < k \), and \( \alpha_k = \beta \leq 3\alpha_{k-1} \). Let \( K' = \{n \in \omega : \alpha_i \leq n \leq \alpha_{i+1} \text{ for some } i \text{ even}\} \); \( L' = \{n \in \omega : \alpha_i \leq n \leq \alpha_{i+1} \text{ for some } i \text{ odd}\} \).
If $\sum_{n \in K'} \varrho_n \geq \frac{1}{2} \sum_{n \in K' \cup L'} \varrho_n$, find a sequence of intervals $I \supseteq I_0 \supseteq I_2 \supseteq \ldots$ such that for every $i$, $0 \leq i \leq k$, $i$ even,

\[(2) \quad \lambda(I_i) \geq \frac{\theta}{\alpha_{i+1}} \geq \frac{3\theta}{\alpha_{i+2}}, \quad \text{and}
\]

\[(3) \quad \sum_{a_i \leq n \leq a_{i+1}} \varrho_n \|nx\| \geq \frac{\theta}{4} \sum_{a_i \leq n \leq a_{i+1}} \varrho_n \text{ for all } x \in I_i.
\]

Similarly, if $\sum_{n \in L'} \varrho_n \geq \frac{1}{2} \sum_{n \in K' \cup L'} \varrho_n$, find a sequence of intervals $I \supseteq I_1 \supseteq I_3 \supseteq \ldots$ such that (2) and (3) hold true for every $i$, $0 \leq i \leq k$, $i$ odd. In both cases, let $J$ be the last interval in the sequence, i.e. $I_k$ if $k$ is even, $I_{k-1}$ if $k$ is odd. \hfill \square

Lemma 15. Let $\{n_k\}_{k \in \omega}$ be an increasing sequence of natural numbers and let $\{\varrho_n\}_{n \in \omega}$ be a sequence of non-negative reals such that

\[\lim_{k \to \infty} \frac{n_k}{n_{k+1}} = 0, \quad \text{and} \quad \sum_{n \in \omega} \varrho_n = \infty.
\]

Let $k'$, $\varepsilon \leq 1$ be such that $n_k/n_{k+1} \leq \varepsilon$ for all $k \geq k'$. Then for every interval $I$ such that $\lambda(I) = 1/n_k$, there exist $k'' > k'$ and an interval $J \subseteq I$ such that $\lambda(J) = 1/n_{k''}$, and for all $x \in J$, $\|nx\| \leq \varepsilon$ whenever $k' \leq k < k''$, and

\[\sum_{n_{k'} \leq n < n_{k''}} \varrho_n \|nx\| \geq 1.
\]

Proof. For given reals $\alpha, \beta$, we denote

\[S(\alpha, \beta) = \sum_{\alpha \leq n \leq \beta} \varrho_n.
\]

We put $S(\alpha, \beta) = 0$ if there is no $n \in \omega$ such that $\alpha \leq n \leq \beta$.

Let $\gamma = 1/7$. We will consider two cases.

(a) There exists $\delta \geq \gamma$ such that $\sum_{k \in \omega} S(\gamma n_k, \delta n_k) = \infty$. Then we can find a sequence $\{\theta_k\}_{k \in \omega}$ such that $\lim_{k \to \infty} \theta_k = 0$, $\sum_{k \in \omega} \theta_k S(\gamma n_k, \delta n_k) = \infty$, and for all $k$, $\theta_k \geq \delta n_k/n_{k+1}$. Put $\alpha_k = \gamma n_k$, $\beta_k = \delta n_k$, for $k \in \omega$.

(b) For all $\delta \geq \gamma$, we have $\sum_{k \in \omega} S(\gamma n_k, \delta n_k) < \infty$. Then for all $\delta \geq \gamma$, we must have $\sum_{k \in \omega} S(\delta n_k, \gamma n_{k+1}) = \infty$. Thus there exists a sequence $\{\delta_k\}_{k \in \omega}$ such that $\lim_{k \to \infty} \delta_k = \infty$, $\sum_{k \in \omega} S(\delta_k n_k, \gamma n_{k+1}) = \infty$, and for all $k$, $\gamma < \delta_k \leq \gamma n_{k+1}/n_k$. Put $\alpha_k = \delta_k n_k$, $\beta_k = \gamma n_{k+1}$, $\theta_k = \gamma$, for $k \in \omega$.

As can be easily checked, in both cases there exists $k_0$ such that for all $k \geq k_0$,

\[\alpha_k \leq \beta_k < \alpha_{k+1}, \quad 0 < \theta_k \leq 1/7, \quad \varepsilon/n_k \geq 3\theta_k/\alpha_k, \quad \text{and} \quad \theta_k/\beta_k \geq 1/n_k + 1.
\]

Moreover, we have $\sum_{k \in \omega} \theta_k S(\alpha_k, \beta_k) = \infty$.

Find $i \geq \max\{k_0, k'\}$ such that $\alpha_i \geq n_{k'}$. There exists $j \geq i$ such that

\[\sum_{i \leq k \leq j} \theta_k S(\alpha_k, \beta_k) \geq 8.
\]

Put $k'' = j + 1$. Clearly $\beta_j \leq \theta_j n_{k''} < n_{k''}$.

Let us define a nested sequence of intervals $\{I_k\}_{k' < k < k''}$ as follows. Put $I_{k'} = I$. We will always have $\lambda(I_k) = 1/n_k$. By Lemma 17 there exists an interval $J_k \subseteq I_k$ such that $\lambda(J_k) = \varepsilon/n_k$, and if $x \in J_k$, then $\|nx\| \leq \varepsilon$. If $k < i$, let $I_{k+1}$ be any interval such that $I_{k+1} \subseteq J_k$ and $\lambda(I_{k+1}) = 1/n_{k+1}$. For $i \leq k \leq j$, we have
Theorem 16. Let \( \{n_k\}_{k \in \omega} \) be an increasing sequence of natural numbers such that \( \lim_{k \to \infty} n_k/n_{k+1} = 0 \). Then the \( A \)-set \( \{ x : \lim_{k \to \infty} \|n_k x\| = 0 \} \) cannot be covered by a union of fewer than \( \text{cov}(\mathcal{M}) \) \( N \)-sets.

Proof. Let \( \{I_i\}_{i \in \omega} \) be an enumeration of all intervals of the form \( [m/n_k, (m+1)/n_k] \), with integer \( m \). Let \( \{X_\alpha\}_{\alpha < \kappa} \) be a sequence of \( N \)-sets, \( \kappa < \text{cov}(\mathcal{M}) \). We will find a real \( x \) such that \( \lim_{k \to \infty} \|n_k x\| = 0 \) and \( x \notin \bigcup_{\alpha < \kappa} X_\alpha \).

For every \( \alpha < \kappa \), let \( \{g^n_\alpha\}_{n \in \omega} \) be a sequence of non-negative reals such that \( \sum_{n \in \omega} g^n_\alpha \|nx\| = \infty \), for all \( x \in X_\alpha \). We will define a function \( f_\alpha \in \omega^\omega \) as follows.

For given \( i \in \omega \), let \( k' \) be such that \( \lambda(I_i) = 1/n_{k'} \). Put \( \varepsilon = \max\{n_k/n_{k+1} : k \geq k'\} \). By Lemma 15 there exist \( k'' > k' \) and an interval \( J \subseteq I_i \) such that \( \lambda(J) = 1/n_{k''} \), and for all \( x \in J \), \( \|n_k x\| \leq \varepsilon \) whenever \( k' \leq k < k'' \), and

\[
\sum_{n_k \leq n < n_{k''}} g^n_\alpha \|nx\| \geq 1.
\]

Let \( k^* \geq k'' \) be such that \( n_{k^*}/n_{k+1} \leq \varepsilon/2 \). There exists an interval \( J^* \subseteq J \) such that for all \( x \in J^* \), \( \|n_k x\| \leq n/k_{k+1} \) whenever \( k'' \leq k < k^* \), \( \|n_k x\| \leq \varepsilon \), and \( \lambda(J^*) = \varepsilon/n_{k^*} \geq 2/n_{k+1} \). Hence we can find \( i_j \subseteq J^* \) such that \( \lambda(I_j) = 1/n_{k+1} \). We put \( f(i) = j \).

By Lemma 13 there exists \( g \in \omega^\omega \) such that \( (\forall j)(\exists \alpha < \kappa) g(j+1) = f_\alpha(g(j)) \) and \( (\forall \alpha < \kappa)(\exists j) g(j+1) = f_\alpha(g(j)) \). For \( j \in \omega \), let \( k(j) \) be such that \( \lambda(I_j) = 1/n_{k(j)} \). Clearly \( I_{g(j)+1} \subseteq I_{g(j)} \) and \( k(j+1) > k(j) \) for all \( j \), and if we put \( \varepsilon_j = \max\{n_k/n_{k+1} : k \geq k(j)\} \), then \( \lim_{j \to \infty} \varepsilon_j = 0 \). Let us take \( x \in \bigcap_{j \in \omega} I_{g(j)} \). We have \( \|n_k x\| \leq \varepsilon_j \) whenever \( k \geq k(j) \), and thus \( \lim_{k \to \infty} \|n_k x\| = 0 \). Moreover, if \( g(j) = f_\alpha(g(j)) \), then

\[
\sum_{n_k \leq n < n_{k(j)+1}} g^n_\alpha \|nx\| \geq 1,
\]

and hence for all \( \alpha < \kappa \), \( \sum_{n \in \omega} g^n_\alpha \|nx\| = \infty \), and thus \( x \notin X_\alpha \).

We finish with an open problem.

Problem 17. Is it possible to replace the cardinal \( \text{cov}(\mathcal{M}) \) in Theorems 12, 13 and 16 by some consistently greater cardinal?
References


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