WEAK COMPACTNESS OF CERTAIN SETS OF MEASURES

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Abstract. For a compact Hausdorff space $X$ and a Montel Hausdorff locally convex space $E$, let $F = (C(X, E), u)$, $u$ being the uniform topology. We determine the necessary and sufficient conditions for an equicontinuous $H \subseteq F$ to be $\sigma(F', F'')$-compact. Special results are obtained when $X$ is an $F$-space or a $\sigma$-Stonian space.

1. Introduction and notations

For locally convex spaces, the notations and results from [14] are used and for general topology we refer to [6]. All vector spaces are taken over $K$, the field of real or complex numbers (we call $K$ the scalar field). For measures and vector measures, we will follow the notations and results from [2], [7], [8], [9], [13], [16].

For a locally convex space $G$, $G'$ will be its dual. For a completely regular Hausdorff space $Y$, $\hat{Y}$ will denote the Stone-Cech compactification of $Y$. $N$ will denote the set of natural numbers. For two vector spaces in duality, $\langle \cdot, \cdot \rangle$ will denote the bilinear form on their product.

In this paper, $X$ is a compact Hausdorff space and $E$ a Montel Hausdorff locally convex space. Recall that a locally convex space is called a Montel space if it is barreled and every bounded set is relatively compact ([10], pp. 369-378). As stated by Köthe ([10], p. 372), some of the most important spaces in analysis are Montel (the space $H(D)$, of analytic functions on the open unit disc, with the topology of uniform convergence on the compact subsets of $D$ is a Fréchet-Montel space ([10], p. 373)). We denote by $C(X, E)$ the space of all $E$-valued continuous functions on $X$. If $E = K$, then we denote $C(X, E)$ by $C(X)$. Let $\{\| \cdot \|_p : p \in P\}$ be the collections of all continuous semi-norms on $E$. For a $p \in P$, $E_p$ denotes the normed space arising from $E$ by the semi-norm $\| \cdot \|_p$. $E'_p$ can be considered a subspace of $E'$: $E'_p = \{ f \in E' : f(p^{-1}[0, 1])$ is bounded}. We take the uniform topology on $C(X, E)$ and denote the locally convex space $(C(X, E), u)$ by $F$. The dual of $F$ is denoted by $M(X, E')$. If $E$ is a normed space, then with norm topology on $E'$, $M(X, E')$ consists of regular Borel $E'$-valued measures, of finite variations, on $X$. If $E$ is locally convex, then $(C(X, E), u)$ can be considered a subspace of $\prod_{p \in P}(C(X, E_p), u)$, with induced topology. So for every $\mu \in M(X, E')$, there

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is a $p \in P$ such that $\mu \in (C(X,E_p), u)'$. Thus every $\mu \in F'$ gives a $p \in P$ such that $\mu$ is an $E_p'$-valued regular Borel measure, of finite variation $|\mu|_p$, on $X$ (8). If $E = K$, then $M(X,E')$ is denoted by $M(X)$ and its positive elements are denoted by $M^+(X)$. If $H \subset F'$ is equicontinuous, then there is a $p \in P$ such that $|\mu|_p(X) \leq 1$, $\forall \mu \in H$, and all $\mu \in H$ are $E_p'$-valued. We denote by $\gamma$ the canonical mapping $E \to E_p$, by $B$ all closed, absolutely convex bounded subsets of $E$ (since $E$ is Montel, these subsets are compact), and by $B_0$ the collection $\{\gamma(B) : B \in B\}$; the elements of $B_0$ are absolutely convex and compact in $E_p$.

In the celebrated paper of Grothendieck [5], some remarkable characterizations of weakly compact subsets of $M(X)$ are obtained and then are applied to the special case when $X$ is a Stonian space. These results are extended to the case of $C(X,E)$ by many authors ([4], [1], [17]). In this paper we give necessary and sufficient conditions for an equicontinuous subset of $F'$ to be $\sigma(F',F'')$-compact (i.e. weakly compact in $F'_\beta$—note, in the notations of [14], that $F'_\beta$ is the space $F'$ with topology of uniform convergence on the $\sigma(F,F')$-bounded subsets of $F$ and $(F'_\beta)' = F''$) and then apply the results to the special cases when $X$ is a $\sigma$-Stonian or an $F$-space. Recall that $X$ is Stonian if the closure of every open set is open, is $\sigma$-Stonian if the closure of every open $F_\sigma$-set is open, and is $F$-space if disjoint open $F_\sigma$-sets have disjoint closures.

We will also need the following lemmas:

**Lemma 1.** Let $\lambda_n : 2^N \to K$ be a sequence of countably additive functions (on all subsets of $N$). If $\lim\lambda_n(A)$ exists for all subsets $A \subset N$, then the convergence is uniform on $2^N$.

**Proof.** This is a particular case of classical Phillips’ Lemma ([2], p. 33).

**Lemma 2.** Let $T$ be a Hausdorff topological space having a $\sigma$-compact dense subset, $C(T)$ all scalar-valued continuous functions on $T$ with the topology of pointwise convergence, and $A \subset C(T)$ such that every sequence in $A$ has a cluster point in $C(T)$. Let $g$ be in the closure of $A$ in $K^T$. Then there exists a sequence, in $A$, which converges to $g$ pointwise.

**Proof.** This is proved in [12].

2. **Main results**

**Theorem 3.** Let $X$ be a compact set, $E$ a Montel locally convex space and $F = (C(X,E), u)$ (here $u$ denotes the uniform topology). Suppose $H \subset F'$ is equi-
continuous. Then $H$ is relatively weakly compact in $F'_\beta$, if and only if, $\forall x \in E, \{\mu_x : \mu \in H\}$ is relatively weakly compact in $(M(X), \|\|)$.

**Proof.** Suppose $H$ is relatively weakly compact in $F'_\beta$. Fix an $x \in E$. Since the mapping $\phi : (C(X), \|\|) \to (C(X,E), u), f \to f \otimes x$ is continuous, the adjoint $\phi' : F'_\beta \to (M(X), \|\|)$ is continuous and so the result follows.

Now we come to the converse. Since $H$ is equi-continuous, $H_0$, the closed convex hull of $H$ in $F'_\sigma$, is compact. From this it is easy to verify that $H_0$ is complete in $F'_\beta$. So to prove the weak-compactness of $H$ in $F'_\beta$, it is enough to prove that every sequence in $H$ has a cluster point ([14], Theorem 11.2). Take a sequence $\{\mu_n\} \subset H$ and let $\hat{H}$ be the closure of $H$ in $F'_\sigma$. Let $\mu_0 \in \hat{H}$ be a cluster point $\{\mu_n\}$ in $F'_\sigma$. There exists a $p \in P$ such that $|\mu_n|_p(X) \leq 1$, $0 \leq n < \infty$. We fix a
lifting \( \rho \) for the finite measure \( \lambda = \sum_{n=0}^{\infty} \lambda_n \) \( |\mu_n|_p \) (10, 13). We take the lifting topology \( T_0 \) on \( X \) (this topology is denoted by \( T_1 \) in [13]) and denote the associated completely regular Hausdorff space by \((X_0, T_0)\). \( \hat{X}_0 \) will denote the Stone-Cech compactification of \( X_0 \).

We state some notations and results from [9]:

\( M_0 \) is the linear span of \( \mu_n \), \( 0 \leq n < \infty \), in \( F' \).

\[ V = \{ x \in E : ||x||_p \leq 1 \} \]

\[ L_1 = \{ \phi : (X_0, T_0) \to E'_p, \phi \text{ continuous and } \phi(X_0) \subset \alpha V^0, \alpha > 0 \} ; \text{ here } V^0 \text{ is the polar of } V \text{ in the duality } (E, E') \text{. Since } E \text{ is Montel, } \phi \text{ is also continuous in } E'_p \text{.} \]

Also every \( \phi \in L_1 \) is \( E'_p \)-valued.

For a \( B \in \mathcal{B} \), \( \phi \in L_1 \), \( |\phi_B(t)| = \sup \{|\langle x, \phi(t) \rangle | : x \in B \} \). \( |\phi_B \) is continuous on \( X_0 \). Since \( \phi \) is \( E'_p \)-valued, if we put \( B_0 = \gamma(B) \), then we also have \( |\phi|_B = \sup \{|\langle x, \phi(t) \rangle | : x \in B_0 \} \).

\[ \{ f \in C(X, E) : f(X) \subset B \} \text{, as } B \text{ varies in } \mathcal{B} \text{, form a fundamental system of bounded sets in } F' \text{.} \]

For a \( \mu \in M_0 \), there exists a \( q \in L_1 \) such that \( \mu = q \lambda \), in the sense that \( \mu_x = q_x \lambda \) (here \( q_x(t) = \langle x, q(t) \rangle \); also \( \sup \{|\mu(f)| : f(X) \subset B \} = \int |q|_B d\lambda, \forall B \in \mathcal{B} \)). Further \( |\mu|_p(X) = \int |q|_X d\lambda \) where \( |q|_X = \sup \{|x \circ q| : x \in V \} \). From this it follows that \( \mu \to q \) is a linear isomorphism if the topology on \( M_0 \) is the one induced by \( F'_p \) and the topology on \( L_1 \) is generated by \( \mathcal{B} \) (each \( B \in \mathcal{B} \) gives a semi-norm on \( L_1 \), \( q \to \int |q|_B d\lambda \)).

\[ L_{\infty} = \{ f : X_0 \to E_p, f(X_0) \subset B \), \( B \in \mathcal{B}, f \text{ continuous with norm topology on } E_p \} \]

\[ L_{\infty} = (L_1, B)', \text{ algebraically.} \]

Now we continue with the proof. Take an \( f \in L_{\infty} \). This means there is a \( B \in \mathcal{B} \) such that if \( B_0 = \gamma(B) \), then \( f(X_0) \subset B_0 \). By continuity, extend \( f \) to \( \hat{f} : \hat{X}_0 \to B_0 \).

Fix \( c > 0 \) and take an \( \hat{f}_0 \in C(\hat{X}_0) \otimes E_p \) such that \( \| \hat{f} - \hat{f}_0 \|_p < c \). Put \( \hat{f}_0 |_{X_0} = f_0 \).

Take a net \( \{ \mu_{\alpha} \} \subset M_0 : \mu_{\alpha} \to \mu_0 \) in \( F'_p \). Also take a net \( \{ q_{\alpha} \} \subset L_1 \) such that \( \mu_{\alpha} = q_{\alpha} \lambda \); further take \( q_0 \in L_1 \), satisfying \( \mu_0 = q_0 \lambda \). Since \( \int |q_0|_X d\lambda \leq 1 \) and \( \int |q_0|_X d\lambda \leq 1 \), we have \( \int |f - f_0, q_0 - q_0|_X d\lambda \leq c \). Also since \( \{ (\mu_{\alpha})_X \} \) is relatively weakly compact in \((M(X), ||.||)\) for every \( x \in E \), \( \int (f_0, q_0 - q_0) d\lambda \to 0 \). The result now follows.

**Corollary 4.** Let \( X \) be a compact Hausdorff space, \( E \) a Montel-Fréchet locally convex space and \( F = (C(X, E), u) \). Suppose \( H \subset F' \) is bounded in \( F'_p \). Then \( H \) is relatively weakly compact in \( F'_p \), if and only if, \( \forall x \in E \), \( \{ \mu_x : \mu \in H \} \) is relatively weakly compact in \((M(X), ||.||)\).

**Proof.** In this case \( F \) is barreled and so \( H \) will be equicontinuous. The result follows from Theorem 3.

The following theorem is proved in [17]; we give below a different proof.

**Theorem 5 (13, Theorem 3.3, Theorem 3.4).** Let \( X \) be a compact Hausdorff space. Assume \( X \) is either an \( F \)-space or a \( \sigma \)-Stonean space. Suppose \( H \) is a compact subset of \((M(X), \sigma(M(X), C(X))) \). The \( H \) is weakly compact if there is a \( \lambda \in M^+(X) \) such that \( \mu \ll \lambda \) for every \( \mu \in H \).

**Proof.** Assume that there is a \( \lambda \in M^+(X) \) such that \( \mu \ll \lambda \) for every \( \mu \in H \). Suppose \( H \) is not weakly compact. By (17, Theorem 2(ii), p. 4846) there is a sequence \( \{ \mu_n \} \subset H \), a disjoint sequence \( \{ V_n \} \) of open subsets of \( X \), and a \( c > 0 \)
such that $|\mu_n(V_n)| > c$, $\forall n$. By regularity, we can assume that each $V_n$ is $F_\sigma$. Take a sequence $\{f_n\} \subset C(X)$, $0 \leq f_n \leq \chi_{V_n}$ such that $|\mu_n(f_n)| > c$, $\forall n$. Now the supp$(\mu)$ is relative weakly compact, $\forall \mu \in H$. If $X$ is an $F$-space, for notational convenience, we take $X = \text{supp}(\lambda)$ (note that every continuous scalar-valued function on a closed subset of $X$ can be extended to a continuous function on the whole of $X$); this means $X$ is Stonian [15, Theorem 2.2]. In both the cases $X$ is $\sigma$-Stonian and so the real-valued functions in $C(X)$ are an order $\sigma$-complete vector lattice. For any $M \subset N$, let

$$f_{|M} = \sup_{i \in C(M)} \{ \sum_{i \in M, i \leq n} f_i : n \in N \}$$

(for $M = \emptyset$, $f_{|M} = 0$). It is easy to verify that if $M_1 \cap M_2 = \emptyset$, then $f_{|M_1 \cap f_{|M_2} = 0}$. Define $\lambda_0 : 2^X \rightarrow R^+$, $\lambda_0(M) = \lambda(f_{|M})$. $\lambda_0$ is finitely additive and exhaustive (3) and $\lambda_0(\{n\}) = \lambda(f_n)$, $\forall n$. By (3, Proposition 1, p. 728), there is an infinite subset of $N$, which again we denote by $N$, such that $\lambda_0$ is countably additive on $2^N$. Bringing this information on $\lambda_0$, we get $\lambda(f_{|M} - \sum_{i \in M} f_i) = 0$, $\forall M \subset N$, and $\forall \mu \in H$, $|\mu|(f_{|M} - \sum_{i \in M} f_i) = 0$, $\forall M \subset N$. Define $\nu_n : 2^N \rightarrow K$, $\nu_n(M) = \mu_n(\sum_{i \in M} f_i) = \mu(f_{|M})$. These measures are countably additive. Let $Y = \{f_{|M} : M \subset N\}$ be given the topology induced by $K^H$ with product topology. This means the countable set $\{f_{|M} : M$ a finite subset of $N\}$ is dense in $Y$. Applying Lemma 2 to $H \subset C(Y)$ (note that $H$ is playing the role of $A$ in Lemma 2), we get a subsequence of $\{\mu_n\}$, which again we denote by $\{\mu_n\}$, and a $\mu_0 \in H$ such that $\mu_n \rightarrow \mu_0$, pointwise on $Y$. This implies that $\lim \mu_n(M)$ exists for any $M \subset N$. By Lemma 1, $\mu_n(f_n) \rightarrow 0$ which is a contradiction.

The converse is trivial.

Theorem 6. Let $X$ be a compact Hausdorff space. Assume that $X$ is also either an $F$-space or a $\sigma$-Stonian space. Let $E$ be a Montel locally convex space and $F = (C(X), E, u)$ (here $u$ denotes the uniform topology). Suppose $H \subset F'$ is equicontinuous and $F'_\sigma$-closed. Then $H$ is relatively weakly compact in $F'_\beta$, if and only if, for every $x \in E$, there exists a $\lambda_x \in M^+(X)$ such that $\mu_x << \lambda_x$, for every $\mu \in H$.

Proof. Suppose the condition of the theorem is satisfied. Fix an $x \in E$. By Theorem 3, we have only to prove that $\{\mu_x : \mu \in H\}$ is relatively weakly compact in $M(X)$. The mapping $F'_\sigma \rightarrow (M(X), \sigma(M(X), C(X)))$, $\mu \rightarrow \mu_x$ is continuous and so $\{\mu_x : \mu \in H\}$ is $\sigma(M(X), C(X))$-compact. The result follows from Theorem 5.

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