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THE MINIMUM NUMBER OF ACUTE DIHEDRAL ANGLES OF A SIMPLEX

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Abstract. For any n-dimensional simplex $\Omega \subset R^n$, we confirm a conjecture of Klamkin and Pook (1988) that there are always at least n acute dihedral angles in Ω.

Let Ω be an n-simplex in R^n with vertices $A_1, A_2, \cdots, A_{n+1}$ (i.e., $\Omega = \langle A_1, A_2, \cdots, A_{n+1}\rangle$), let $\Omega_i = \langle A_1, \cdots, A_{i-1}, A_{i+1}, \cdots, A_{n+1}\rangle$ denote its facet which lies in a hyperplane π_i, and let e_i be the unit outer normal vector of π_i ($i = 1, 2, \cdots, n+1$). Denote by θ_{ij} the supplement of the angle between e_i and e_j; we call θ_{ij} the dihedral angle between Ω_i and Ω_j.

Obviously, for any n-simplex, there exist $\binom{n+1}{2}$ dihedral angles. In [1], Klamkin shows that there are always at least three acute dihedral angles in any tetrahedron. Further, he posed the question: can one generalize this result to an n-simplex? Pook [1] conjectures that n is the required minimum number of acute dihedral angles in any n-simplex.

In this paper, we confirm this supposition.

Theorem 1. *There exist at least n acute dihedral angles in any n-simplex, and there exists an n-simplex which has only n acute dihedral angles.*

It is clear that Theorem 1 can be replaced equivalently by the following statement.

Theorem 1′. *There exist at most $\frac{1}{2}n(n-1)$ obtuse dihedral angles in any n-simplex, and there exists an n-simplex which has only $\frac{1}{2}n(n-1)$ obtuse dihedral angles.*

To prove Theorem 1′, we need the following three lemmas.

Lemma 1. *Let $\{x_1, x_2, \cdots, x_n\}$ be a given linearly independent set of vectors from R^n. Then there exists an n-simplex Ω which takes $\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \cdots, \frac{x_n}{\|x_n\|}$ as n unit outer normal vectors of facets.*

Proof. Since $x_1, x_2, \cdots, x_n$ are linearly independent, there exist n linearly independent vectors $v_1, v_2, \cdots, v_n$ such that

$$\langle x_i, v_j\rangle = \delta_{ij}\ \|x_i\|^2\ (i, j = 1, 2, \cdots, n),$$

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where δ_{ij} is the Kronecker delta symbol and $\langle , \rangle$ denotes the ordinary inner product of R^n.

Without loss of generality, we may assume that $v_i = \overrightarrow{A_{n+1}A_i}$ $(i = 1, 2, \cdots, n)$. We now consider the n-simplex $\Omega = \langle A_1, A_2, \cdots, A_{n+1} \rangle$. Since $x_i \perp v_j$ $(i \neq j)$, it follows that

$$x_i \perp \pi_i \ (i = 1, 2, \cdots, n),$$

where π_i is the hyperplane spanned by the facet Ω_i. This completes the proof of Lemma 1. □

Lemma 2 (Yang and Zhang [3])**.** *Let Ω and Ω' be two n-simplices with dihedral angles θ_{ij} and θ'_{ij} $(1 \leq i < j \leq n+1)$, respectively. If*

$$\theta_{ij} \leq \theta'_{ij} \ (1 \leq i < j \leq n+1),$$

then Ω and Ω' are similar, namely,

$$\theta_{ij} = \theta'_{ij} \ (1 \leq i < j \leq n+1). \quad \square$$

Let $\{u_1, \cdots, u_n\}$ and $\{s_1, \cdots, s_n\}$ be two bases of R^n. They are called dual if

$$\langle u_i, s_j \rangle = \delta_{ij} \ (1 \leq i \leq j \leq n).$$

It is well known that there exists a unique dual basis for a given basis of R^n.

Lemma 3. *Let $U = \{u_1, \cdots, u_n\}$ be a basis of R^n and $\{s_1, \cdots, s_n\}$ be the dual basis of U. If $\langle u_i, u_j \rangle < 0$ $(i \neq j)$, then $\langle s_i, s_j \rangle > 0$ for all i, j.*

Proof. We first consider $n = 2$. Suppose that $\langle u_1, u_2 \rangle < 0$ and $\{s_1, s_2\}$ is the dual basis. Then $s_1 = \langle s_1, s_1 \rangle u_1 + \langle s_1, s_2 \rangle u_2$. Hence

$$0 = \langle s_1, u_2 \rangle = \langle s_1, s_1 \rangle \langle u_1, u_2 \rangle + \langle s_1, s_2 \rangle \ \|u_2\|^2.$$

Thus $\langle s_1, s_2 \rangle > 0$.

Turning to the case of general n, let

$$H_1 = \{x \in R^n \ | \langle u_1, x \rangle = 0\}.$$

Then the vectors $u_2, \cdots, u_n$ are on the opposite side of H_1 from u_1. For $j \geq 2$, let $u_j = y_j + z_j, \ y_j \perp H_1, \ z_j \in H_1$; then

$$y_j = \frac{\langle u_1, u_j \rangle}{\|u_1\|^2} u_1, \ z_j = u_j - \frac{\langle u_1, u_j \rangle}{\|u_1\|^2} u_1.$$

If $i \neq j$ and $i, j \geq 2$, then

$$\begin{aligned} \langle z_i, z_j \rangle &= \langle u_i - \frac{\langle u_1, u_i \rangle}{\|u_1\|^2} u_1, \ u_j - \frac{\langle u_1, u_j \rangle}{\|u_1\|^2} u_1 \rangle \\ &= \langle u_i, u_j \rangle - 2 \frac{\langle u_1, u_i \rangle \langle u_1, u_j \rangle}{\|u_1\|^2} + \frac{\langle u_1, u_i \rangle \langle u_1, u_j \rangle}{\|u_1\|^2} \\ &= \langle u_i, u_j \rangle - \frac{\langle u_1, u_i \rangle \langle u_1, u_j \rangle}{\|u_1\|^2} < 0. \end{aligned}$$

Note that $s_k \in H_1$ for $k \geq 2$, and

$$\langle z_j, s_k \rangle = \langle u_j - \frac{\langle u_1, u_i \rangle}{\|u_1\|^2} u_1, s_k \rangle = \langle u_j, s_k \rangle = \delta_{jk}$$

for $k, j \geq 2$.

Since $\{z_2, \cdots, z_n\}$ and $\{s_2, \cdots, s_n\}$ are biorthognal sets of $n-1$ vectors in the $(n-1)$-dimensional subspace H_1, they must be dual bases in H_1, and $\langle z_i, z_j \rangle < 0$

when $i \neq j$. If dim $H_1 > 2$, then we can continue projecting to lower-dimensional subspaces until $\{u_1, \cdots, u_n\}$ is mapped to a basis $G = \langle g_{n-1}, g_n \rangle$ in a two-dimensional subspace with $\langle g_{n-1}, g_n \rangle < 0$ and with $\{s_{n-1}, s_n\}$ the dual basis of G. But then in dimension 2, $\langle s_{n-1}, s_n \rangle > 0$, completing the proof. □

Proof of Theorem 1′. We will first construct an n-simplex that has exactly $\frac{1}{2}n(n-1)$ obtuse dihedral angles. Denote by $\overline{\Omega}$ an n-simplex all of whose dihedral angles are acute (obviously, such an $\overline{\Omega}$ exists, e.g., regular simplex).

Let $u_1, u_2, \cdots, u_n$ be unit outer normal vectors of n facets of $\overline{\Omega}$. Then

$$\langle u_i, u_j \rangle < 0, \ i \neq j,$$

and $U = \{u_1, \cdots, u_n\}$ is a basis of R^n. Further let $\{s_1, \cdots, s_n\}$ be the dual basis of U. By Lemma 1, there exists an n-simplex Ω which takes $e_1, \cdots, e_n$ as unit outer normal vectors of n facets, where $e_i = \frac{s_i}{\|s_i\|}$. By Lemma 3, we have

$$\langle e_i, e_j \rangle = \frac{1}{\|s_i\| \ \|s_j\|} \ \langle s_i, s_j \rangle > 0, \ 1 \leq i, j \leq n. \tag{1.1}$$

Let e_{n+1} be the unit outer normal vector of the $(n+1)$-th facet of Ω, and f_i the $(n-1)$-dimensional volume of the facet Ω_i of Ω. By Minowski's realization theorem (see [2, p. 390]), we have

$$\sum_{i=1}^{n+1} f_i e_i = 0.$$

Therefore, for $j \in \{1, 2, \cdots, n\}$,

$$\begin{aligned} \langle e_{n+1}, e_j \rangle &= \langle -\sum_{i=1}^{n} e_i \ \frac{f_i}{f_{n+1}}, \ e_j \rangle \\ &= -\sum_{i=1}^{n} \langle e_i, e_j \rangle \ \frac{f_i}{f_{n+1}} < 0. \end{aligned} \tag{1.2}$$

By (1.1) and (1.2), we know that Ω has $\frac{1}{2}n(n-1)$ obtuse dihedral angles and n acute dihedral angles.

Assume that there exists an n-simplex whose obtuse dihedral angles are more than $\frac{1}{2}n(n-1)$. Without loss of generality, we may assume that

$$\alpha_1, \cdots, \alpha_{\frac{1}{2}n(n-1)}, \alpha_{\frac{1}{2}n(n-1)+1}, \cdots, \alpha_{\frac{1}{2}n(n-1)+j}$$

are its obtuse dihedral angles, while $\alpha_{\frac{1}{2}n(n-1)+j+1}, \cdots, \alpha_{\frac{1}{2}n(n+1)}$ are acute. Thus, we can construct a rectangular n-simplex with acute dihedral angles $\alpha_{\frac{1}{2}n(n-1)+j+1}$, $\cdots, \alpha_{\frac{1}{2}n(n+1)}$. In fact, let $\{\overline{e}_1, \cdots, \overline{e}_n\}$ be a standard base of R^n, and let

$$\overline{e}_{n+1} = -\sqrt{\frac{1-\lambda}{j}}\overline{e}_1 - \cdots - \sqrt{\frac{1-\lambda}{j}}\overline{e}_j - \cos\alpha_{\frac{1}{2}n(n-1)+j+1}\overline{e}_{j+1} - \cdots - \cos\alpha_{\frac{1}{2}n(n+1)}\overline{e}_n,$$

where

$$\lambda = \cos^2\alpha_{\frac{1}{2}n(n-1)+j+1} + \cdots + \cos^2\alpha_{\frac{1}{2}n(n+1)}.$$

Then the rectangular simplex that takes $\overline{e}_1, \cdots, \overline{e}_n, \overline{e}_{n+1}$ as $n+1$ unit normal vectors of facets has n acute dihedral angles

$$\arccos\sqrt{\frac{1-\lambda}{j}}, \cdots, \arccos\sqrt{\frac{1-\lambda}{j}}, \alpha_{\frac{1}{2}n(n-1)+j+1}, \cdots, \alpha_{\frac{1}{2}n(n+1)}.$$

By Lemma 2, this is contradictory. This completes the proof. □

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