HIGHER DIMENSIONAL APOSYNDETIC DECOMPOSITIONS

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Abstract. Let $X$ be a homogeneous, decomposable continuum that is not aposyndetic. The Aposyndetic Decomposition Theorem yields a cell-like decomposition of $X$ into homogeneous continua with quotient space $Y$ being an aposyndetic, homogeneous continuum.

Assume the dimension of $X$ is greater than one. About 20 years ago the author asked the following questions:

Can this aposyndetic decomposition raise dimension? Can it lower dimension? We answer these questions by proving the following theorem.

Theorem. The dimension of the quotient space $Y$ is one.

1. Introduction

The Aposyndetic Decomposition Theorem [J1] of F. Burton Jones is essential to the study of homogeneous continua. It goes like this.

Theorem 1. If $X$ is a homogeneous, decomposable continuum that is not aposyndetic, then $X$ admits a continuous decomposition into mutually homeomorphic, indecomposable, homogeneous continua such that the quotient space $Y$ is an aposyndetic, homogeneous continuum.

The author has strengthened this result by showing that (1) the elements of this decomposition must be cell-like continua [R1], and (2) the elements of this decomposition have the same dimension as $X$ [R6].

In 1983 the author wrote a survey paper [R4] exposing the state of the art in the study of homogeneous continua. He raised a number of questions, about half of which have been answered. One of the unanswered ones is the following [R4, Question 11, p. 224]:

Question 2. Can this aposyndetic decomposition raise dimension? lower dimension?

In this paper we answer this question by proving the following theorem:

Theorem. If $X$ is a homogeneous, decomposable continuum that is not aposyndetic, then the dimension of the quotient space $Y$ of the aposyndetic decomposition of $X$ is one.
2. Results

A continuum is a compact, connected, nonvoid metric space. A continuum is indecomposable if it is not the union of two of its proper subcontinua. A continuum is hereditarily indecomposable if each of its subcontinua is indecomposable. R. H. Bing \[B\] has constructed hereditarily indecomposable continua of dimension \( n \) for \( 1 \leq n \leq \infty \).

Let \( x \) and \( y \) be points of the continuum \( X \). If \( X \) contains an open set \( G \) and a continuum \( H \) such that \( x \in G \subset H \subset X - \{y\} \), then \( X \) is aposydentic at \( x \) with respect to \( y \). If \( X \) is aposydentic at each of its points with respect to every other point, then \( X \) is aposydentic.

A continuum \( X \) is cell-like if each map of \( X \) into a polyhedron is homotopic to a constant map. A continuum is tree-like if it is cell-like and one-dimensional.

A continuum \( X \) is homogeneous if for each pair of points \( p \) and \( q \) belonging to \( X \), there exists a homeomorphism \( h : X \to X \) such that \( h(p) = q \). The author \[R5\] has shown that each homogeneous, hereditarily indecomposable, nondegenerate continuum is tree-like, and hence one-dimensional.

A subcontinuum \( Z \) of the continuum \( X \) is terminal if each subcontinuum \( W \) of \( X \) that intersects \( Z \) satisfies either \( W \subset Z \) or \( Z \subset W \). For example, if \( X \) is the topologist’s sin \( 1/x \) curve and \( Z \) is the “limit bar,” then \( Z \) is a terminal subcontinuum of \( X \).

A decomposition of \( X \) into continua is terminal if each element of the decomposition is a terminal subcontinuum of \( X \). Jones \[J2\] has shown that his aposydentic decomposition is terminal.

If \( f : X \to Y \) is a map and \( y \) is a point of \( Y \), then the set \( f^{-1}(y) \) is a fiber of \( f \). If each fiber of the map \( f \) is a cell-like continuum, then \( f \) is a cell-like map.

We use reduced Čech cohomology with integral coefficients. A space is acyclic if each of its cohomology groups is trivial. Note that a cell-like continuum is acyclic.

A nondegenerate continuum \( Y \) has cohomological dimension one if \( H^q(Y, B) = 0 \) for every closed subset \( B \) of \( Y \) and for every \( q > 1 \). It is known that a continuum is one-dimensional if and only if it has cohomological dimension one [W, p. 109].

Theorem 3. Let \( X \) be a decomposable, homogeneous continuum that is not aposydentic, and let \( f : X \to Y \) be the quotient map of the aposydentic decomposition of \( X \). Then \( \dim Y = 1 \).

Proof. Since \( X \) is decomposable, \( Y \) is a nondegenerate continuum. Hence \( \dim Y \geq 1 \). If \( \dim X = 1 \), then \( Y \), being the cell-like image of \( X \), is also one-dimensional [W, p. 113].

Suppose \( \dim X > 1 \). Each fiber \( f^{-1}(y) \) is a homogeneous continuum with the property that \( \dim f^{-1}(y) = \dim X \) [R6]. Homogeneous, hereditarily indecomposable continua have dimension less than or equal to one [R5], and fibers of \( f \) are homogeneous continua, so no fiber of \( f \) is hereditarily indecomposable. Since the decomposition is terminal, each hereditarily indecomposable subcontinuum of \( X \) is contained in a fiber of \( f \).

M. Levin [L] and J. Krasinkiewicz [Kra] have shown that there exists a map \( p : X \to I \) of \( X \) onto the unit interval \( I \) such that each component of each fiber \( p^{-1}(t) \) is hereditarily indecomposable. Let \( g : X \to Z \) and \( h : Z \to I \) be the monotone-light factorization [N, p. 279] of \( p \). Each fiber \( g^{-1}(z) \) is a hereditarily indecomposable continuum (possibly a point), so each fiber of \( g \) is contained in
a fiber of $f$. It follows that there exists a monotone map $k: Z \to Y$ satisfying $f = k \circ g$.

Each fiber of $h$ is totally disconnected, so $Z$ is one-dimensional [HW, Theorem VI7, p. 91]. It suffices to show that the cohomological dimension of $Y$ is one. Let $B$ be a closed subset of $Y$. Consider $H^q(Y, B)$ for $q > 1$. Let $A = f^{-1}(B)$ and $C = k^{-1}(B)$. Since each fiber of $f$ is cell-like, each fiber has trivial cohomology. The Vietoris-Begle Theorem [S, p. 344] implies that $f^*: H^q(Y, B) \to H^q(X, A)$ is an isomorphism. Since $f = k \circ g$, we have $f^* = g^* \circ k^*$. Since $Z$ is one-dimensional, $H^q(Z, C) = 0$. Hence $H^q(Y, B) = 0$ as well. This completes the proof.

As a consequence of this theorem, we obtain some information on the cohomology groups of homogeneous, decomposable continua that are not aposyndetic. First we need a definition.

A continuum $X$ is unicoherent if each pair of subcontinua of $X$ whose union is $X$ has a connected intersection. A continuum is hereditarily unicoherent if each of its subcontinua is unicoherent.

**Corollary 4.** If $X$ is a homogeneous, decomposable continuum that is not aposyndetic, then $H^1(X) \neq 0$, and $H^q(X) = 0$ for $q > 1$.

**Proof.** Since $f^*: H^q(Y) \to H^q(X)$ is an isomorphism, it suffices to show these claims for the cohomology groups of $Y$. The second claim is true because $Y$ is one-dimensional. To prove the first claim, recall [R2, Theorem 1, p. 450] that an acyclic one-dimensional continuum is hereditarily unicoherent. Jones [J2] has shown that a hereditarily unicoherent, homogeneous continuum is indecomposable.

There is a theorem for indecomposable, homogeneous continua that corresponds to the Aposyndetic Decomposition Theorem. It is called the Terminal Decomposition Theorem [R3], and it goes like this.

**Theorem 5.** If $X$ is a homogeneous, indecomposable continuum that contains a nondegenerate, proper, terminal subcontinuum and if $H^1(X) \neq 0$, then $X$ admits a continuous decomposition into mutually homeomorphic, indecomposable, homogeneous continua such that the quotient space $Y$ is a homogeneous, indecomposable continuum that contains no proper, nondegenerate, terminal subcontinuum.

The condition of nontrivial first cohomology of $X$ is necessary to insure the existence of maximal terminal, proper subcontinua of $X$; these are the elements of the decomposition. As before, the elements of the decomposition are cell-like continua of the same dimension as $X$. Hence, by the same proof as Theorem 3, we have the following theorem.

**Theorem 6.** If $X$ satisfies the hypotheses of Theorem 5, then the quotient space $Y$ of the terminal decomposition is one-dimensional.

**Question 7.** If $X$ satisfies the hypotheses of Theorem 5, is the quotient space $Y$ of the terminal decomposition of $X$ a solenoid?
According to a theorem of Pavel Krupski [Kru, Theorem 3.1, p. 167], the answer is yes, provided that the complement of any subcontinuum of \( Y \) has finitely many components.

References


