PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 10, Pages 3021–3023
S 0002-9939(03)06954-5
Article electronically published on February 20, 2003

F-SPLIT GALOIS REPRESENTATIONS ARE POTENTIALLY ABELIAN

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(Communicated by David E. Rohrlich)

Abstract. In this note we relate the property of a semisimple ℓ-adic Galois representation being "F-split" to its having abelian image.

If E is an elliptic curve defined over a number field K, then it gives rise to a *strictly compatible system* (ρ_ℓ) of 2-dimensional ℓ-adic representations (see I-11 of [Se]) of the absolute Galois group G_K of K. This arises from the action of G_K on the ℓ-adic Tate module $\mathrm{Ta}_\ell(E)$ of E.

This strictly compatible system has very different properties when E has CM and when E does not have CM. For instance in the latter case the image of ρ_ℓ is all of $GL_2(\mathbf{Z}_\ell)$ for almost all ℓ while in the former case (ρ_ℓ) is potentially abelian, i.e., $\rho_\ell|_{G_L}$ is abelian for all ℓ for a fixed finite extension L of K. These cases are also markedly different from the point of view of the the distribution of the eigenvalues of the Frobenii (see I-25 of [Se]).

Yet another notable difference is in the behaviour of the compositum of the splitting fields of the *Frobenius polynomials* attached to E, i.e., characteristic polynomials of $\rho_\ell|_{G_L}(\mathrm{Frob}_r)$ (r prime to ℓ) for primes r of L at which $E_{/L}$ has good reduction: these characteristic polynomials depend only on r and not on ℓ, and here again L is a fixed number field which contains K. When E does not have CM, this compositum is always an infinite extension of $\mathbf{Q}$ (see exercise on page IV-13 of [Se]). If E has CM, this compositum is a finite extension whenever all the endomorphisms (i.e., CM) of E are defined over L.

This difference is the most pertinent to this short note. We study here in a more abstract setting the relationship between a compatible system of ℓ-adic Galois representations being potentially abelian and the nature of the field generated by the splitting fields of the Frobenius polynomials.

Definition 1. Let E, F be number fields and fix embeddings of E, F in each completion of $\overline{\mathbf{Q}}$.

(1) Consider (ρ_λ) a strictly compatible system of E-rational, continuous, semisimple, n-dimensional, λ-adic representations $\rho_\lambda : G_K \to GL_n(E_\lambda)$ for a number field K and λ running through the places of E (see I-11 and I-13 of [Se]). We say that (ρ_λ) is F-split if for almost all places r of L the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_r)$ (which is defined when r and λ are of coprime residue characteristics, and then is independent of λ) *splits* over

Received by the editors May 13, 2002.
2000 *Mathematics Subject Classification.* Primary 11R32.

F. We say that (ρ_λ) is potentially F-split if $(\rho_\lambda|_{G_L})$ is F-split for a finite extension L of K.

(2) We say that a continuous, semisimple representation $\rho_\lambda : G_K \to GL_n(E_\lambda)$ is F-split if the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_r)$ of L for all places r of L unramified in ρ_λ splits over F. We say that ρ_λ is potentially F-split if $\rho_\lambda|_{G_L}$ is F-split for a finite extension L of K. (Note: By [KhRa], the set of places r of L unramified in ρ_λ has density 1.)

Definition 2. We say that either a strictly compatible system (ρ_λ) of E-rational, continuous, semisimple, n-dimensional λ-adic representations of G_K or an E-rational, continuous, semisimple representation $\rho_\lambda : G_K \to GL_n(E_\lambda)$ is abelian (resp., potentially abelian) if each ρ_λ has abelian image (resp., if for some finite extension L of K each $\rho_\lambda|_{G_L}$ has abelian image).

Theorem 1. *A strictly compatible system (ρ_λ) of E-rational, continuous, semisimple, n-dimensional λ-adic representations of G_K is potentially F-split for some number field F if and only if it is potentially abelian.*

Proof. We first prove the "only if" statement of the theorem. By restriction of scalars, it is enough to prove that a $\mathbf{Q}$-rational, continuous, semisimple, n-dimensional ℓ-adic system of representations of G_K that is potentially F-split for some number field F is potentially abelian. (We reduce to this case for ease of comparison with [LP].) Consider a prime ℓ_0 that splits completely in F. Let L be a finite extension of K such that the compatible system $(\rho_\ell|_{G_L})$ is F-split and such that the Zariski closure of the image $\rho_{\ell_0}(G_L)$ is connected (such an L exists!). From the assumption of F-splitness and the Cebotarev density theorem we deduce that the subgroup $\rho_{\ell_0}(G_L)$ of $GL_n(\mathbf{Q}_{\ell_0})$ contains no non-split torus. Since ρ_ℓ is semisimple this implies that the ℓ_0-adic representation has toral image (and in particular is abelian). This together with the $\mathbf{Q}$-rationality of ρ_{ℓ_0}, and a consequence of a result of Waldschmidt in transcendental number theory (see Theorem 2 of [H]), implies that ρ_{ℓ_0} arises as the direct sum of 1-dimensional representations arising from algebraic Hecke characters χ_i of L, $i = 1, \cdots, n$. The χ_i's give rise (see II of [H]) to a strictly compatible system of (continuous, semisimple, n-dimensional) λ-adic representations. Comparing this with $(\rho_\ell|_{G_L})$, we deduce that $(\rho_\ell|_{G_L})$ itself "arises" from the direct sum of the algebraic Hecke characters χ_i proving the proposition. (Note that without using [H] it follows from Proposition 6.14 of [LP] that there is a finite extension L of K such that for a density 1 set of primes ℓ, $\rho_\ell|_{G_L}$ has abelian image, and thus $(\rho_\ell|_{G_L})$ is an "abelian system" with ℓ running through a density 1 set of primes. But we do not get the stronger assertion that $(\rho_\ell|_{G_L})$ itself is an abelian system.)

We now prove the other direction of the statement of the theorem. Consider a strictly compatible system (ρ_λ) that is potentially abelian and consider an extension L of K such that $(\rho_\lambda|_{G_L})$ is abelian. Choose any place λ of E. By appealing to [H] again, and using that ρ_λ is rational over E, semisimple and abelian, we deduce that $\rho_\lambda|_{G_L}$ arises as the direct sum of 1-dimensional representations arising from algebraic Hecke characters χ_i of L. Then by standard proprties of algebraic Hecke characters (see II of [H]), we conclude that $\rho_\lambda|_{G_L}$ is F-split for some number field F, which in its turn implies that $(\rho_\lambda|_{G_L})$ is an F-split strictly compatible system.

Remark. While it is true that an E-rational abelian, semisimple representation ρ_λ (or compatible system (ρ_λ)) is always F-split for some number field F (as follows

from the proof), it is not true that an F-split representation ρ_λ (or compatible system (ρ_λ)) is abelian (consider the "constant", compatible systems arising from Artin representations).

The case of a single, F-split, λ-adic representation of G_K we cannot settle, even in the case when the image is in $GL_2(\mathbf{Q}_\ell)$ (if the completions of F contain all the quadratic extensions of $\mathbf{Q}_\ell$ we do not know how to proceed). We end with a question.

Question 1. Is a continuous, semisimple, F-split λ-adic representation $\rho : G_K \to GL_n(E_\lambda)$ potentially abelian?

Acknowledgements

I thank Richard Pink, Dipendra Prasad and J-P. Serre for helpful correspondence: each came up with different proofs of the theorem of this note (independently of each other and of the author, when responding to messages posing mainly the question above). The terminology "F-split" was suggested by Richard Pink, and J-P. Serre pointed out the example in his book. I also thank David Rohrlich for helpful comments that improved the exposition.

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