F-SPLIT GALOIS REPRESENTATIONS ARE POTENTIALLY ABELIAN

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Abstract. In this note we relate the property of a semisimple $\ell$-adic Galois representation being “$F$-split” to its having abelian image.

If $E$ is an elliptic curve defined over a number field $K$, then it gives rise to a strictly compatible system $(\rho_{\ell})$ of 2-dimensional $\ell$-adic representations (see I-11 of [Se]) of the absolute Galois group $G_K$ of $K$. This arises from the action of $G_K$ on the $\ell$-adic Tate module $T_{\ell}(E)$ of $E$.

This strictly compatible system has very different properties when $E$ has CM and when $E$ does not have CM. For instance in the latter case the image of $\rho_{\ell}$ is all of $GL_2(\mathbb{Z}_\ell)$ for almost all $\ell$ while in the former case $(\rho_{\ell})$ is potentially abelian, i.e., $\rho_{\ell}|_{G_L}$ is abelian for all $\ell$ for a fixed finite extension $L$ of $K$. These cases are also markedly different from the point of view of the distribution of the eigenvalues of the Frobenius (see I-25 of [Se]).

Yet another notable difference is in the behaviour of the compositum of the splitting fields of the Frobenius polynomials attached to $E$, i.e., characteristic polynomials of $\rho_{\ell}|_{G_L}(Frob_r)$ (for primes $r$ of $L$ at which $E_L$ has good reduction: these characteristic polynomials depend only on $r$ and not on $\ell$, and here again $L$ is a fixed number field which contains $K$. When $E$ does not have CM, this compositum is always an infinite extension of $\mathbb{Q}$ (see exercise on page IV-13 of [Se]). If $E$ has CM, this compositum is a finite extension whenever all the endomorphisms (i.e., CM) of $E$ are defined over $L$.

This difference is the most pertinent to this short note. We study here in a more abstract setting the relationship between a compatible system of $\ell$-adic Galois representations being potentially abelian and the nature of the field generated by the splitting fields of the Frobenius polynomials.

Definition 1. Let $E, F$ be number fields and fix embeddings of $E, F$ in each completion of $\overline{\mathbb{Q}}$.

(1) Consider $(\rho_{\lambda})$ a strictly compatible system of $E$-rational, continuous, semisimple, $n$-dimensional, $\lambda$-adic representations $\rho_{\lambda} : G_K \to GL_n(E_\lambda)$ for a number field $K$ and $\lambda$ running through the places of $E$ (see I-11 and I-13 of [Se]). We say that $(\rho_{\lambda})$ is $F$-split if for almost all places $r$ of $L$ the characteristic polynomial of $\rho_{\lambda}(Frob_r)$ (which is defined when $r$ and $\lambda$ are of coprime residue characteristics, and then is independent of $\lambda$) splits over

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F. We say that \((\rho_\lambda)\) is potentially \(F\)-split if \((\rho_\lambda|_{GL})\) is \(F\)-split for a finite extension \(L\) of \(K\).

(2) We say that a continuous, semisimple representation \(\rho_\lambda : G_K \to GL_n(E_\lambda)\)
is \(F\)-split if \(\phi_\lambda\) is \(\mathbb{Q}\)-rational and \(\phi_\lambda\) is \(F\)-split for a finite extension \(L\) of \(K\).

\[\text{Definition 2.}\]
We say that either a strictly compatible system \((\rho_\lambda)\) of \(E\)-rational, continuous, semisimple, \(n\)-dimensional \(\mathbb{Q}\)-adic representations of \(G_K\) or an \(E\)-rational, continuous, semisimple representation \(\rho_\lambda : G_K \to GL_n(E_\lambda)\)
is abelian (resp., potentially abelian) if each \(\rho_\lambda\) has abelian image (resp., if for some finite extension \(L\) of \(K\) each \(\rho_\lambda|_{GL}\) has abelian image).

\[\text{Theorem 1.}\]
A strictly compatible system \((\rho_\lambda)\) of \(E\)-rational, continuous, semisimple, \(n\)-dimensional \(\mathbb{Q}\)-adic representations of \(G_K\) is potentially \(F\)-split for some number field \(F\) if and only if it is potentially abelian.

\[\text{Proof.}\]
We first prove the “only if” statement of the theorem. By restriction of scalars, it is enough to prove that a \(\mathbb{Q}\)-rational, continuous, semisimple, \(n\)-dimensional \(\ell\)-adic system of representations of \(G_K\) that is potentially \(F\)-split for some number field \(F\) is potentially abelian. (We reduce to this case for ease of comparison with \([LP]\).) Consider a prime \(\mathfrak{p}\) that splits completely in \(F\). Let \(L\) be a finite extension of \(K\) such that the compatible system \((\rho_\ell|_{GL})\) is \(F\)-split and such that the Zariski closure of the image \(\rho_\mathfrak{p}(GL)\) is connected (such an \(L\) exists!). From the assumption of \(F\)-splitness and the Cebotarev density theorem we deduce that the subgroup \(\rho_\mathfrak{p}(GL)\) of \(GL_n(\mathbb{Q}_\mathfrak{p})\) contains no non-split torus. Since \(\rho_\ell\) is semisimple this implies that the \(\ell\)-adic representation has toral image (and in particular is abelian). This together with the \(\mathbb{Q}\)-rationality of \(\rho_\mathfrak{p}\), and a consequence of a result of Waldschmidt in transcendental number theory (see Theorem 2 of \([H]\)), implies that \(\rho_\mathfrak{p}\) arises as the direct sum of \(1\)-dimensional representations arising from algebraic Hecke characters \(\chi_i\) of \(L\), \(i = 1, \ldots, n\). The \(\chi_i\)’s give rise (see \(\Pi\) of \([H]\)) to a strictly compatible system of \((\text{continuous, semisimple, } n\text{-dimensional})\) \(\mathbb{Q}\)-adic representations. Comparing this with \((\rho_\ell|_{GL})\), we deduce that \((\rho_\ell|_{GL})\) itself “arises” from the direct sum of the algebraic Hecke characters \(\chi_i\) proving the proposition. (Note that without using \([H]\) it follows from Proposition 6.14 of \([LP]\) that there is a finite extension \(L\) of \(K\) such that for a density \(1\) set of primes \(\ell\), \(\rho_\ell|_{GL}\) has abelian image, and thus \((\rho_\ell|_{GL})\) is an “abelian system” with \(\ell\) running through a density \(1\) set of primes. But we do not get the stronger assertion that \((\rho_\ell|_{GL})\) itself is an abelian system.)

We now prove the other direction of the statement of the theorem. Consider a strictly compatible system \((\rho_\lambda)\) that is potentially abelian and consider an extension \(L\) of \(K\) such that \((\rho_\lambda|_{GL})\) is abelian. Choose any place \(\lambda\) of \(E\). By appealing to \([H]\) again, and using that \(\rho_\lambda\) is rational over \(E\), semisimple and abelian, we deduce that \(\rho_\lambda|_{GL}\) arises as the direct sum of \(1\)-dimensional representations arising from algebraic Hecke characters \(\chi_i\) of \(L\). Then by standard properties of algebraic Hecke characters (see \(\Pi\) of \([H]\)), we conclude that \(\rho_\lambda|_{GL}\) is \(F\)-split for some number field \(F\), which in its turn implies that \((\rho_\lambda|_{GL})\) is an \(F\)-split strictly compatible system.

\[\text{Remark.}\]
While it is true that an \(E\)-rational abelian, semisimple representation \(\rho_\lambda\) (or compatible system \((\rho_\lambda))\) is always \(F\)-split for some number field \(F\) (as follows
from the proof, it is not true that an $F$-split representation $\rho_\lambda$ (or compatible system $(\rho_\lambda)$) is abelian (consider the “constant”, compatible systems arising from Artin representations).

The case of a single, $F$-split, $\lambda$-adic representation of $G_K$ we cannot settle, even in the case when the image is in $GL_2(\mathbb{Q}_\ell)$ (if the completions of $F$ contain all the quadratic extensions of $\mathbb{Q}_\ell$ we do not know how to proceed). We end with a question.

**Question 1.** Is a continuous, semisimple, $F$-split $\lambda$-adic representation $\rho : G_K \to GL_n(E_\lambda)$ potentially abelian?

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**References**


