PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 10, Pages 2977–2985
S 0002-9939(03)07009-6
Article electronically published on April 21, 2003

LOCAL COHOMOLOGY OVER HOMOGENEOUS RINGS WITH ONE-DIMENSIONAL LOCAL BASE RING

M. BRODMANN, S. FUMASOLI, AND R. TAJAROD

(Communicated by Bernd Ulrich)

Abstract. Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring with local base ring $(R_0, \mathfrak{m}_0)$ and let M be a finitely generated graded R-module. Let $H^i_{R_+}(M)$ be the i-th local cohomology module of M with respect to $R_+ := \bigoplus_{n>0} R_n$. If $\dim R_0 \leq 1$, the R-modules $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$, $(0 :_{H^i_{R_+}(M)} \mathfrak{m}_0)$ and $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$ are Artinian for all $i \in \mathbb{N}_0$. As a consequence, much can be said on the asymptotic behaviour of the R_0-modules $H^i_{R_+}(M)_n$ for $n \to -\infty$.

1. Introduction

Let $R = \bigoplus_{n\geq 0} R_n$ be a homogeneous Noetherian ring and let M be a finitely generated and graded R-module. Moreover let $R_+ := \bigoplus_{n>0} R_n$ be the irrelevant ideal of R. For each $i \in \mathbb{N}_0$ let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to R_+, furnished with its natural grading (cf. [B-S, chap. 12]). For each $n \in \mathbb{Z}$ let $H^i_{R_+}(M)_n$ denote the n-th graded component of $H^i_{R_+}(M)$.

The modules $H^i_{R_+}(M)$ and their graded components are closely related to sheaf cohomology over projective schemes (cf. [B-S, chap. 20]) and this turns them into objects of particular interest. As shown by recent investigations, the modules $H^i_{R_+}(R)$ may have an astonishingly complicated structure — even for nice rings R (cf. [B-K-S], [K]). On the other hand, if the base ring R_0 is Artinian much can be said on the R-modules $H^i_{R_+}(M)$, mainly as they are Artinian (cf. [B-M-M]). Also, the "tameness" shown in [B-He, 4.3] gives a first hint, that the modules $H^i_{R_+}(M)$ do not behave too badly if the base ring R_0 is local and of dimension 1.

Our first main result gives an explanation for the nice behaviour of the modules $H^i_{R_+}(M)$ if the base ring R_0 is local (with maximal ideal $\mathfrak{m}_0$) and of dimension ≤ 1: We namely show that the graded R-modules $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$, $(0 :_{H^i_{R_+}(M)} \mathfrak{m}_0)$, $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ and $H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ are Artinian in this case (cf. Theorem 2.5).

Received by the editors April 16, 2002.
2000 *Mathematics Subject Classification.* Primary 13D45, 13E10.
Key words and phrases. Local cohomology modules, Artinian modules, graded components.
The third author thanks the University of Zürich for the hospitality offered during the preparation of this paper.

This allows us to draw conclusions on the asymptotic behaviour of the n-th graded component $H^i_{R_+}(M)_n$ of $H^i_{R_+}(M)$ for $n \to -\infty$. More precisely (cf. Theorem 3.5): If $\dim R_0 \leq 1$, if $i \in \mathbb{N}_0$ and if $\mathfrak{q}_0 \subset R_0$ is an $\mathfrak{m}_0$-primary ideal the lengths of the R_0-modules $H^i_{R_+}(M)_n/\mathfrak{q}_0 H^i_{R_+}(M)_n$, $(0 :_{H^i_{R_+}(M)_n} \mathfrak{q}_0)$ and $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)_n$ as well as the multiplicity $\mathrm{e}_{\mathfrak{q}_0}(H^i_{R_+}(M)_n)$ of $H^i_{R_+}(M)_n$ with respect to $\mathfrak{q}_0$ depend on n polynomially of degree $< i$ if $n \ll 0$. Moreover, the set $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$, a result which improves the "asymptotic gap freeness" shown in [B-He, 4.2]. Observe also that the above polynomial dependencies allow us to define various types of cohomological deficiency functions and postulation numbers which extend the corresponding concepts from the case of Artinian base rings (cf. [B-M-M]) to the case of one-dimensional local base rings.

Also, we provide examples showing that:

$H^1_{R_+}(R)/\mathfrak{m}_0 H^1_{R_+}(R)$ and $(0 :_{H^2_{R_+}(R)} \mathfrak{m}_0)$ need not be Artinian if $(R_0, \mathfrak{m}_0)$ is (regular and) of dimension 2 (cf. Examples 4.1 resp. 4.2);

$\mathrm{Ass}_{R_0}(H^3_{R_+}(R)_n)$ need not be asymptotically stable for $n \to -\infty$ even if $\bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}(H^3_{R_+}(R)_n)$ is finite (cf. Example 4.3);

$\bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}(H^2_{R_+}(R)_n)$ need not be finite (cf. Example 4.4).

In the last two examples, the base ring $(R_0, \mathfrak{m}_0)$ is regular local and of dimension 4. To give these examples, the results of [B-K-S] resp. [K] shall be used.

2. Artinian quotients and submodules

We keep the previous notations and hypotheses. In this section we prove that certain quotients and submodules of the local cohomology modules $H^i_{R_+}(M)$ are Artinian, provided the base ring R_0 is of dimension ≤ 1.

Remark 2.1. If $\dim R_0 = 0$, R_0 is Artinian and hence $H^i_{R_+}(M)$ is Artinian for all $i \in \mathbb{N}_0$ [B-S, 7.1.4].

Lemma 2.2. *Let H be an Artinian R-module and M_0 a finitely generated R_0-module. Then for any $i \in \mathbb{N}_0$ the R-module $\mathrm{Tor}_i^{R_0}(M_0, H)$ is Artinian.*

Proof. Let

$$\ldots \to F_{i+1} \to F_i \to F_{i-1} \to \ldots \to F_1 \to F_0 \to M_0 \to 0$$

be a minimal free resolution of M_0. Then $\mathrm{Tor}_i^{R_0}(M_0, H)$ is an R-subquotient of $F_i \otimes_{R_0} H \cong H^{\mathrm{rk}(F_i)}$, and hence is Artinian. □

Lemma 2.3. *For any $i \in \mathbb{N}_0$ the R-module $H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M))$ is Artinian.*

Proof. Since $\Gamma_{\mathfrak{m}_0 R}(M)$ is a finitely generated $\mathfrak{m}_0 R$-torsion module, there exists an $m \in \mathbb{N}$ such that ${\mathfrak{m}_0}^m \Gamma_{\mathfrak{m}_0 R}(M) = 0$. Therefore $\Gamma_{\mathfrak{m}_0 R}(M)$ is a graded $R/{\mathfrak{m}_0}^m R$-module. Let $i \in \mathbb{N}_0$. By the graded Independence Theorem [B-S, 13.1.6] there is an isomorphism of graded R-modules $H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)) \cong H^i_{(R/{\mathfrak{m}_0}^m R)_+}(\Gamma_{\mathfrak{m}_0 R}(M))$ where the latter is Artinian as $\dim(R/{\mathfrak{m}_0}^m R)_0 = \dim R_0/{\mathfrak{m}_0}^m = 0$ (cf. Remark 2.1). □

Lemma 2.4. *Let $i \in \mathbb{N}_0$. Then, if $R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M/\Gamma_{\mathfrak{m}_0 R}(M))$ is Artinian, so is $R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M)$.*

Proof. Let $\bar{M} := M/\Gamma_{\mathfrak{m}_0 R}(M)$. There is a commutative diagram with exact rows:

$$\begin{tikzcd}
H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)) \arrow[r, "u"] & H^i_{R_+}(M) \arrow[r, "v"] \arrow[d, two heads] & H^i_{R_+}(\bar{M}) \arrow[r] \arrow[d, equal] & H^{i+1}_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)) & \\
0 \arrow[r] & H^i_{R_+}(M)/\operatorname{Im}(u) \arrow[r] & H^i_{R_+}(\bar{M}) \arrow[r] & H^i_{R_+}(\bar{M})/\operatorname{Im}(v) \arrow[r] & 0 \\
& & & H^{i+1}_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)) \arrow[u, hook] &
\end{tikzcd}$$

Being a submodule of $H^{i+1}_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M))$, the module $T := H^i_{R_+}(\bar{M})/\operatorname{Im}(v)$ is Artinian (cf. Lemma 2.3). So by Lemma 2.2 the R-module $\operatorname{Tor}_1^{R_0}(R_0/\mathfrak{m}_0, T)$ is Artinian. From the lower row of the diagram we deduce the exact sequence

$$\operatorname{Tor}_1^{R_0}(R_0/\mathfrak{m}_0, T) \to R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M)/\operatorname{Im}(u) \to R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(\bar{M})$$

where the last module is Artinian by assumption. Hence the last module in the exact sequence

$$R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(\Gamma_{\mathfrak{m}_0 R}(M)) \to R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M) \to R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M)/\operatorname{Im}(u)$$

is Artinian, too. Now we conclude by Lemma 2.3. □

Theorem 2.5. *Let* $\dim R_0 \leq 1$ *and let* $i \in \mathbb{N}_0$. *Then*

a) *the R-module* $R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M)$ *is Artinian;*

b) *the R-modules* $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ *and* $H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$ *are Artinian.*

Proof. a) By Lemma 2.4 we may assume that $\Gamma_{\mathfrak{m}_0 R}(M) = 0$. Also we may assume that $\dim R_0 = 1$ (cf. Remark 2.1). As $\dim R_0 > 0$ there exists an M-regular element $x \in \mathfrak{m}_0$ which avoids all minimal primes of R_0. The exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ induces an exact sequence

$$H^{i-1}_{R_+}(M/xM) \to H^i_{R_+}(M) \xrightarrow{x} H^i_{R_+}(M) \xrightarrow{u} H^i_{R_+}(M/xM).$$

As x avoids all minimal primes of R_0, $\dim R_0/xR_0 = 0$. Therefore $H^i_{R_+}(M/xM)$ is Artinian by the Independence Theorem, and so is $\operatorname{Im}(u)$. The above sequence gives an exact sequence

$$R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M) \xrightarrow{x} R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M) \to R_0/\mathfrak{m}_0 \otimes_{R_0} \operatorname{Im}(u) \to 0.$$

As $x \in \mathfrak{m}_0$, the multiplication map "x" is zero. So there is an isomorphism

$$R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M) \cong R_0/\mathfrak{m}_0 \otimes_{R_0} \operatorname{Im}(u).$$

b) (cf. [B-He, 4.2]) We choose $x \in \mathfrak{m}_0$ such that $\sqrt{xR_0} = \mathfrak{m}_0$. Then, for each $i \in \mathbb{N}_0$, there is an exact sequence of graded R-modules

$$H^{i-1}_{R_+}(M) \xrightarrow{\eta_x^{i-1}} H^{i-1}_{R_+}(M)_x \to H^i_{(R_+,x)}(M) \to H^i_{R_+}(M) \xrightarrow{\eta_x^i} H^i_{R_+}(M)_x$$

in which η_x^{i-1} and η_x^i are the natural homomorphisms (cf. [B-S, 13.1.12]). As

$$\operatorname{Coker}(\eta_x^{i-1}) \cong H^1_{xR}(H^{i-1}_{R_+}(M)) = H^1_{\mathfrak{m}_0 R}(H^{i-1}_{R_+}(M))$$

and

$$\operatorname{Ker}(\eta_x^i) = \Gamma_{xR}(H^i_{R_+}(M)) = \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$$

we obtain an exact sequence

$$0 \to H^1_{\mathfrak{m}_0 R}(H^{i-1}_{R_+}(M)) \to H^i_{(R_+,x)}(M) \to \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) \to 0.$$

As $\sqrt{(R_+, x)} = \mathfrak{m}_0 R + R_+ \subset R$ is a maximal ideal, the R-module $H^i_{(R_+,x)}(M)$ is Artinian. This proves our claim if we make i run through $\mathbb{N}_0$. □

Corollary 2.6. *Let* $\dim R_0 \leq 1$*, let* $i \in \mathbb{N}_0$ *and let* $\mathfrak{q}_0 \subset R_0$ *be an* $\mathfrak{m}_0$*-primary ideal. Then*

a) *the* R*-module* $R_0/\mathfrak{m}_0/\mathfrak{q}_0 H^i_{R_+}(M)$ *is Artinian;*

b) *the* R*-module* $(0 :_{H^i_{R_+}(M)} \mathfrak{q}_0)$ *is Artinian.*

Proof. a) As $\mathfrak{q}_0$ is $\mathfrak{m}_0$-primary, there is some $m \in \mathbb{N}$ such that ${\mathfrak{m}_0}^m \subset \mathfrak{q}_0$. So, it suffices to show that $R_0/{\mathfrak{m}_0}^m \otimes_{R_0} H^i_{R_+}(M)$ is Artinian. For $m = 1$, this is clear by Theorem 2.5 a). If $m > 1$ we consider the exact sequence

$$ {\mathfrak{m}_0}^{m-1}/{\mathfrak{m}_0}^m \otimes_{R_0} H^i_{R_+}(M) \to R_0/{\mathfrak{m}_0}^m \otimes_{R_0} H^i_{R_+}(M) \to R_0/{\mathfrak{m}_0}^{m-1} \otimes_{R_0} H^i_{R_+}(M) $$

and the natural isomorphism

$$ {\mathfrak{m}_0}^{m-1}/{\mathfrak{m}_0}^m \otimes_{R_0} H^i_{R_+}(M) \cong {\mathfrak{m}_0}^{m-1} \otimes_{R_0} (R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M)). $$

Now, by using Theorem 2.5 a) and Lemma 2.2 we may conclude by induction.

b) As $(0 :_{H^i_{R_+}(M)} \mathfrak{q}_0)$ is a submodule of $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$, we conclude by Theorem 2.5 b). □

3. Asymptotic behaviour of graded components

In this section we use our previous results to draw conclusions on the behaviour of the R_0-modules $H^i_{R_+}(M)_n$ for $n \ll 0$.

Remark 3.1. Keep our previous notations. Let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a graded Artinian R-module. Then $\mathrm{Att}_R(T)$ is a finite set of graded primes and T admits a minimal secondary representation by graded secondary submodules (cf. [B-M-M, sec. 2]).

Lemma 3.2. *Let* $W = \bigoplus_{n\in\mathbb{Z}} W_n$ *be a graded* R*-module such that* $W/\mathfrak{m}_0 W$ *is Artinian and such that* W_n *is a finitely generated* R_0*-module for all* $n \in \mathbb{Z}$*. Then, there is some* $\tau \in \mathbb{Z} \cup \{\infty\}$ *such that for each* $x \in R_1 \setminus \bigcup_{\mathfrak{p}\in \mathrm{Att}_R(W/\mathfrak{m}_0 W)\setminus \mathrm{Var}(R_+)} \mathfrak{p}$ *and each* $n < \tau$ *the multiplication map* $W_n \xrightarrow{x} W_{n+1}$ *is surjective.*

Proof. Let $\mathcal{P} := \mathrm{Att}_R(W/\mathfrak{m}_0 W) \setminus \mathrm{Var}(R_+)$ and let $\mathcal{U} := R_1 \setminus \bigcup_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}$. Also, keep in mind that the R_0-modules W_n are finitely generated. So, by Nakayama, it suffices to find some $\tau \in \mathbb{Z} \cup \{\infty\}$ such that the multiplication maps $W_n/\mathfrak{m}_0 W_n \xrightarrow{x} W_{n+1}/\mathfrak{m}_0 W_{n+1}$ are surjective for all $n < \tau$.

Let $W/\mathfrak{m}_0 W = S^1 + \cdots + S^t$ be a graded secondary presentation with $\mathfrak{p}_j := \sqrt{0 :_R S^j}$ for all $1 \leq j \leq t$. If $\mathfrak{p}_j \in \mathcal{P}$ and $x \in \mathcal{U}$, then $x \notin \mathfrak{p}_j$ so that $S^j_n \xrightarrow{x} S^j_{n+1}$ is surjective for all $n \in \mathbb{Z}$. As $\mathrm{Att}_R(W/\mathfrak{m}_0 W) \subset \mathrm{Var}(\mathfrak{m}_0 R)$, there is at most one $\mathfrak{p} \in \mathrm{Att}_R(W/\mathfrak{m}_0 W)$ not belonging to $\mathcal{P}$. If there is such a $\mathfrak{p}$, say $\mathfrak{p}_j$, we have $\mathfrak{p}_j = \mathfrak{m}_0 R + R_+$ and hence S^j is concentrated in finitely many degrees. So in this case take $\tau = \operatorname{beg} S^j - 1$, where $\operatorname{beg} S^j$ denotes the beginning degree of S^j. Otherwise take $\tau = \infty$. □

Lemma 3.3. *Let* $\mathbb{S} \subset \mathbb{N}_0$*, let* $R_0/\mathfrak{m}_0$ *be infinite and assume that the* R*-module* $R_0/\mathfrak{m}_0 \otimes_{R_0} H^i_{R_+}(M)$ *is Artinian for each* $i \in \mathbb{S}$*. Then, there is some* $\sigma \in \mathbb{Z} \cup \{\infty\}$ *and an* $M/\Gamma_{R_+}(M)$*-regular element* $x \in R_1$ *such that the multiplication maps* $H^i_{R_+}(M)_n \xrightarrow{x} H^i_{R_+}(M)_{n+1}$ *are surjective for all* $i \in \mathbb{S}$ *and for all* $n < \sigma$*.*

Proof. As $H^i_{R_+}(M) = 0$ for all $i \gg 0$, we may assume that $\mathbb{S}$ is finite. Therefore

$$\mathcal{P} := \Big(\mathrm{Ass}_R(M) \cup \bigcup_{i\in\mathbb{S}} \mathrm{Att}_R\big(H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)\big)\Big) \setminus \mathrm{Var}(R_+)$$

is a finite set of graded primes in R, none of which contains R_1.

As $R_0/\mathfrak{m}_0$ is infinite, the set $\mathcal{U} := R_1 \setminus \bigcup_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}$ is not empty (cf. [B-H, 1.5.12]). Let $x \in \mathcal{U}$. As $\mathrm{Ass}_R(M/\Gamma_{R_+}(M)) = \mathrm{Ass}_R(M) \setminus \mathrm{Var}(R_+) \subset \mathcal{P}$ we see that x is $M/\Gamma_{R_+}(M)$-regular. Moreover, by Lemma 3.2 there is some $\sigma \in \mathbb{Z}\cup\{\infty\}$ such that the multiplication maps $H^i_{R_+}(M)_n \xrightarrow{x} H^i_{R_+}(M)_{n+1}$ are surjective for all $i \in \mathbb{S}$ and all $n < \sigma$. □

Notation and Convention 3.4. A) For a finitely generated R_0-module N and an $\mathfrak{m}_0$-primary ideal $\mathfrak{q}_0 \subset R_0$ we use $\mathrm{e}_{\mathfrak{q}_0}(N)$ to denote the multiplicity of N with respect to $\mathfrak{q}_0$ so that

$$\mathrm{e}_{\mathfrak{q}_0}(N) = \lim_{n\to\infty} \frac{(\dim N)!}{n^{\dim N}} \,\mathrm{length}_{R_0} N/{\mathfrak{q}_0}^n N$$

if $N \neq 0$ and $\mathrm{e}_{\mathfrak{q}_0}(0) = 0$.

B) If $(N_n)_{n\in\mathbb{Z}}$ is a family of R_0-modules, we say that the set $\mathrm{Ass}_{R_0}(N_n)$ is asymptotically stable for $n \to -\infty$ if there is some $n_0 \in \mathbb{Z}$ such that $\mathrm{Ass}_{R_0}(N_n) = \mathrm{Ass}_{R_0}(N_{n_0})$ for all $n \le n_0$.

Now, we are ready to present the main result of this section.

Theorem 3.5. *Let $i \in \mathbb{N}_0$ and assume that $\dim R_0 \le 1$. Let $\mathfrak{q}_0 \subset R_0$ be an $\mathfrak{m}_0$-primary ideal. Then*

a) *there exists a numerical polynomial $P \in \mathbb{Q}[t]$ of degree less than i such that $\mathrm{length}_{R_0}(H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M))_n = P(n)$ for all $n \ll 0$;*

b) *there exists a numerical polynomial $Q \in \mathbb{Q}[t]$ of degree less than i such that $\mathrm{e}_{\mathfrak{q}_0}(H^i_{R_+}(M)_n) = Q(n)$ for all $n \ll 0$;*

c) *there exists a numerical polynomial $\bar{P} \in \mathbb{Q}[t]$ of degree less than i such that $\mathrm{length}_{R_0}(0 :_{H^i_{R_+}(M)} \mathfrak{q}_0)_n = \bar{P}(n)$ for all $n \ll 0$;*

d) *there exists a numerical polynomial $\bar{Q} \in \mathbb{Q}[t]$ of degree less than i such that $\mathrm{length}_{R_0} \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))_n = \bar{Q}(n)$ for all $n \ll 0$;*

e) *the set $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ is asymptotically stable for $n \to -\infty$.*

Proof. Let y be an indeterminate and consider the at most one-dimensional local ring ${R_0}' := R_0[y]_{\mathfrak{m}_0 R_0[y]}$ with maximal ideal ${\mathfrak{m}_0}' = \mathfrak{m}_0 {R_0}'$, the ${\mathfrak{m}_0}'$-primary ideal ${\mathfrak{q}_0}' := \mathfrak{q}_0 {R_0}' \subset {R_0}'$, the Noetherian homogeneous ${R_0}'$-algebra $R' := R \otimes_{R_0} {R_0}'$ and the finitely generated and graded R'-module $M' := M \otimes_{R_0} {R_0}'$. By the flat base change property of local cohomology we have natural isomorphisms of ${R_0}'$-modules $H^i_{R'_+}(M')_n \cong H^i_{R_+}(M)_n \otimes_{R_0} {R_0}'$ for all $n \in \mathbb{Z}$. As $R_0/\mathfrak{m}_0 \otimes_{R_0} {R_0}' \cong {R_0}'/{\mathfrak{m}_0}'$ we may conclude that the numerical invariants occurring in statements a)–d) remain the same if we replace $R, M, \mathfrak{q}_0$ and $\mathfrak{m}_0$ by $R', M', {\mathfrak{q}_0}'$ and ${\mathfrak{m}_0}'$, respectively. A similar conclusion holds for the claim made in statement e). Therefore, we may assume that $R_0/\mathfrak{m}_0$ is infinite.

As $H^0_{R_+}(M)_n = 0$ for all $n \ll 0$, the statements of our theorem are obvious for $i = 0$. So let $i > 0$. In view of the natural isomorphism $H^i_{R_+}(M/\Gamma_{R_+}(M)) \cong H^i_{R_+}(M)$ we also assume that $\Gamma_{R_+}(M) = 0$. So, by Lemma 3.3 there is some $\sigma \in \mathbb{Z} \cup \{\infty\}$ and an M-regular element $x \in R_1$ such that the multiplication maps

$H^j_{R_+}(M)_n \xrightarrow{x} H^j_{R_+}(M)_{n+1}$ are surjective for $j = i-1, i$ and for all $n < \sigma$. If we apply cohomology to the exact sequence $0 \to M \xrightarrow{x} M(1) \to M/xM(1) \to 0$ we thus get a short exact sequence of R_0-modules

$$0 \to H^{i-1}_{R_+}(M/xM)_{n+1} \to H^i_{R_+}(M)_n \xrightarrow{x} H^i_{R_+}(M)_{n+1} \to 0$$

for each $n < \sigma$. So, for each $n < \sigma$ we have

$$\begin{aligned}\operatorname{Ass}_{R_0}(H^{i-1}_{R_+}(M/xM)_{n+1}) &\subset \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)\\ &\subset \operatorname{Ass}_{R_0}(H^{i-1}_{R_+}(M/xM)_{n+1}) \cup \operatorname{Ass}_{R_0}(H^i_{R_+}(M)_{n+1}).\end{aligned}$$

From this statement e) follows immediately by induction on i.

As a consequence of e) there are integers $d, \bar{d} \leq 1$ such that $\dim_{R_0} H^i_{R_+}(M)_n = d$ and $\dim_{R_0} H^{i-1}_{R_+}(M/xM)_n = \bar{d}$ for all $n \ll 0$. So, by the above exact sequences and on use of the additivity of multiplicities, we get

$$\mathrm{e}_{\mathfrak{q}_0}(H^i_{R_+}(M)_n) - \mathrm{e}_{\mathfrak{q}_0}(H^i_{R_+}(M)_{n+1}) = \varepsilon \mathrm{e}_{\mathfrak{q}_0}(H^{i-1}_{R_+}(M/xM)_{n+1})$$

for all $n \ll 0$, where $\varepsilon = 0$ if $\bar{d} \neq d$ and $\varepsilon = 1$ if $\bar{d} = d$. From this, statement b) follows easily by induction on i.

a) By Corollary 2.6 a) the graded R-module $H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M)$ is Artinian. So, by [Ki], there is a numerical polynomial $P \in \mathbb{Q}[t]$ such that

$$\operatorname{length}_{R_0}(H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M))_n = P(n)$$

for all $n \ll 0$. It remains to show that P is of degree less than i. The above short exact sequence gives an exact sequence

$$R_0/\mathfrak{m}_0/\mathfrak{q}_0 H^{i-1}_{R_+}(M/xM)_{n+1} \to R_0/\mathfrak{m}_0/\mathfrak{q}_0 H^i_{R_+}(M)_n \xrightarrow{x} R_0/\mathfrak{m}_0/\mathfrak{q}_0 H^i_{R_+}(M)_{n+1},$$

whence

$$\begin{aligned}\operatorname{length}_{R_0}(H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M))_n &- \operatorname{length}_{R_0}(H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M))_{n+1}\\ &\leq \operatorname{length}_{R_0}(H^{i-1}_{R_+}(M/xM)/\mathfrak{q}_0 H^{i-1}_{R_+}(M/xM))_{n+1}\end{aligned}$$

for each $n < \sigma$. This allows us to conclude by induction on i.

c), d) By Theorem 2.5 b) and Corollary 2.6 b) the graded R-modules

$$\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) \text{ and } (0 \underset{H^i_{R_+}(M)}{:} \mathfrak{q}_0)$$

are both Artinian. So, the numerical polynomials $\bar{P}, \bar{Q} \in \mathbb{Q}[t]$ of statements c) and d) exist again by [Ki]. It remains to show that these polynomials are of degree less than i. If we apply the left exact functors $\operatorname{Hom}_{R_0}(R_0/\mathfrak{q}_0, \bullet)$, resp. $\Gamma_{\mathfrak{m}_0 R}$, to our original short exact sequences, we get this by induction on i. □

4. A few examples

In this final section we present some examples related to our previous results. Our first example shows that statement a) of Theorem 2.5 need not hold if $\dim R_0 > 1$, even if $i = 1$ and R_0 is regular local and of dimension 2.

Example 4.1. Let K be a field, let x, y, t be indeterminates, let $R_0 := K[x,y]_{(x,y)}$ and $\mathfrak{m}_0 := (x,y)R_0$. Moreover let $R := R_0[\mathfrak{m}_0 t]$ be the (truncated) Rees ring of $\mathfrak{m}_0$. As $R_+ = (xt, yt)$, the R_+-transform of R is given by

$$D_{R_+}(R) = \bigcup_{n\in\mathbb{N}} \big(R \underset{R_{xtyt}}{:} (xt,yt)^n\big) = R_{xt} \cap R_{yt} \subset R_0[t,t^{-1}]$$

(cf. [B-S, 2.2.15]) so that

$$D_{R_+}(R)_n = \begin{cases} R_n, & \text{if } n \geq 0; \\ R_0 t^n, & \text{if } n < 0. \end{cases}$$

In view of the standard exact sequence $0 \to R \to D_{R_+}(R) \to H^1_{R_+}(R) \to 0$ we thus have

$$H^1_{R_+}(R)_n = \begin{cases} 0, & \text{if } n \geq 0; \\ R_0 t^n, & \text{if } n < 0. \end{cases}$$

It follows that $R_1 H^1_{R_+}(R)_{n-1} = \mathfrak{m}_0 t \cdot R_0 t^{n-1} = \mathfrak{m}_0 t^n \subsetneqq H^1_{R_+}(R)_n$ for all $n < 0$, so that the multiplication map $w : H^1_{R_+}(R)_{n-1} \to H^1_{R_+}(R)_n$ is not surjective whenever $w \in R_1$ and $n < 0$. In view of Lemma 3.3 it follows that $R_0/\mathfrak{m}_0 \otimes_{R_0} H^1_{R_+}(R)$ is not Artinian.

Essentially the same example allows us to show that statement b) of Theorem 2.5 need not hold if the base ring R_0 is regular local and of dimension 2.

Example 4.2. Let $R = R_0[\mathfrak{m}_0 t]$ be as in Example 4.1. Let u, v be two further indeterminates and let $S = R_0[u,v]$. Let $f := xv - yu$. Then, we have an exact sequence of graded S-modules

$$0 \to S(-1) \xrightarrow{f} S \longrightarrow R \to 0.$$

Fix $n \in \mathbb{Z}$. If we apply cohomology to the above exact sequence and on use of the graded base ring independence of local cohomology, we get an exact sequence of R_0-modules:

$$H^2_{S_+}(S)_{n-1} \xrightarrow{f} H^2_{S_+}(S)_n \longrightarrow H^2_{R_+}(R)_n \to 0$$

On use of graded local duality, the flat base change property of local cohomology and as $\operatorname{Ext}^1_{R_0}(S_{-n-1}, R_0) = 0$, we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^2_{S_+}(S)_{n-1} \arrow[r, "f"] \arrow[d, "\cong"] & H^2_{S_+}(S)_n \arrow[r] \arrow[d, "\cong"] & H^2_{R_+}(R)_n \arrow[r] & 0 \\
\operatorname{Hom}_{R_0}(S_{-n-1}, R_0) \arrow[r, "\operatorname{Hom}_{R_0}(f,R_0)"] & \operatorname{Hom}_{R_0}(S_{-n-2}, R_0) \arrow[r] & \operatorname{Ext}^1_{R_0}(R_{-n-1}, R_0) \arrow[r] & 0
\end{tikzcd}$$

Therefore $H^2_{R_+}(R)_n \cong \operatorname{Ext}^1_{R_0}(R_{-n-1}, R_0)$ so that $H^2_{R_+}(R)_n = 0$ if $n \geq -1$ and $H^2_{R_+}(R)_n \cong \operatorname{Ext}^1_{R_0}({\mathfrak{m}_0}^{-n-1}, R_0) \cong \operatorname{Ext}^2_{R_0}(R_0/{\mathfrak{m}_0}^{-n-1}, R_0)$ if $n \leq -2$. In particular we have ${\mathfrak{m}_0}^{-n-1} H^2_{R_+}(R)_n = 0$ for all $n \leq -2$. If E denotes the injective hull of the R_0-module $R_0/\mathfrak{m}_0$, local duality gives (for all $n \leq -2$)

$$\begin{aligned} R_0/{\mathfrak{m}_0}^{-n-1} \cong H^0_{\mathfrak{m}_0}(R_0/{\mathfrak{m}_0}^{-n-1}) &\cong \operatorname{Hom}_{R_0}(\operatorname{Ext}^2_{R_0}(R_0/{\mathfrak{m}_0}^{-n-1}, R_0), E) \\ &\cong \operatorname{Hom}_{R_0}(H^2_{R_+}(R)_n, E). \end{aligned}$$

Hence $(0 :_{R_0} H^2_{R_+}(R)_n) = \mathfrak{m}_0{}^{-n-1}$ for all $n \leq -2$. This shows that $H^2_{R_+}(R) = \Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(R))$ and that $(\mathfrak{m}_0{}^n H^2_{R_+}(R))_{n\in\mathbb{N}}$ is a strictly descending sequence of submodules, so that $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(R))$ is not Artinian. Hence statement b) of Theorem 2.5 does not hold in our situation. Moreover, by Melkerssons Lemma (cf. [B-S, 7.1.2]) it follows that $(0 :_{H^2_{R_+}(R)} \mathfrak{m}_0)$ is not Artinian, so that statement b) of Corollary 2.6 does not hold either.

In [B-K-S] a modified version of an example of Singh [S] is studied, which allows to show that statements c), d) and e) of Theorem 3.5 need not hold over a regular local base ring of dimension 4.

Example 4.3. Let x, y, z, u, v, w be indeterminates and fix a prime number $p \in \mathbb{N}$. Let $R_0 = \mathbb{Z}[x, y, z]_{(p,x,y,z)}$. Also, let $\mathfrak{p}_0 := (x, y, z)R_0$ and $\mathfrak{m}_0 := (x, y, z, p)R_0$. Consider $R_0[u, v, w]$ in the natural way as a homogeneous R_0-algebra and let $R := R_0[u, v, w]/(xu + yv + zw)R_0[u, v, w]$. Then, by [B-K-S, 2.17] and on use of the flat base change property of local cohomology we get for each $n \leq -3$:

$$\operatorname{Ass}_{R_0}(H^3_{R_+}(R)_n) = \begin{cases} \{\mathfrak{p}_0, \mathfrak{m}_0\}, & \text{if } p \mid \prod_{i=1}^{-n-3} \binom{-n-2}{i}; \\ \{\mathfrak{p}_0\}, & \text{otherwise.} \end{cases}$$

In particular, the two sets

$$\mathbb{A} := \{n \in \mathbb{Z} \mid \operatorname{Ass}_{R_0}(H^3_{R_+}(R)_n) = \{\mathfrak{p}_0, \mathfrak{m}_0\}\},$$
$$\mathbb{B} := \{n \in \mathbb{Z} \mid \operatorname{Ass}_{R_0}(H^3_{R_+}(R)_n) = \{\mathfrak{p}_0\}\}$$

are both infinite, as $p \mid \binom{pm}{1}$ for any $m \in \mathbb{Z}$ and $p \nmid \binom{p^k-1}{i}$ for any $k \in \mathbb{N}$ and any $i \in \{1, \ldots, p^k - 2\}$ (cf. also [B-K-S, 2.16]). So statement e) of Theorem 3.5 does not hold in our situation. As $(0 :_{H^3_{R_+}(R)_n} \mathfrak{m}_0) = \Gamma_{\mathfrak{m}_0 R}(H^3_{R_+}(R))_n = 0$ if $n \in \mathbb{B}$ and $0 \neq (0 :_{H^3_{R_+}(R)_n} \mathfrak{m}_0) \subset \Gamma_{\mathfrak{m}_0 R}(H^3_{R_+}(R))_n$ if $n \in \mathbb{A}$, statements c) and d) of Theorem 3.5 also fail.

In the previous example $\bigcup_{n\in\mathbb{Z}} \operatorname{Ass}_{R_0}(H^3_{R_+}(R)_n)$ is finite. The example presented in [K] allows us to show that this latter set need not be finite — again in the case of a base ring R_0 which is regular, local and of dimension 4.

Example 4.4. Let K be a field and let x, y, s, t, u, v be indeterminates. Let $\bar{R}_0 := K[x, y, s, t]$, consider $\bar{R}_0[u, v]$ as a homogenous $\bar{R}_0$-algebra in the natural way and let $\bar{R} := \bar{R}_0[u, v]/(sx^2v^2-(t+s)xyuv+ty^2u^2)\bar{R}_0[u, v]$. Finally let $\bar{\mathfrak{m}} := (x, y, s, t, u, v)\bar{R}$. Then, according to [K] the set $\operatorname{Ass}_{\bar{R}_{\bar{\mathfrak{m}}}}(H^2_{(\bar{R}_+)_{\bar{\mathfrak{m}}}}(\bar{R}_{\bar{\mathfrak{m}}}))$ is infinite. By the graded flat base change property of local cohomology it follows that the set

$$\{\bar{\mathfrak{p}} \in \operatorname{Ass}_{\bar{R}}(H^2_{\bar{R}_+}(\bar{R})) \mid \bar{\mathfrak{p}} \subset \bar{\mathfrak{m}}\}$$

is not finite. Writing $\bar{\mathfrak{m}}_0 = \bar{\mathfrak{m}} \cap \bar{R}_0 = (x, y, s, t)\bar{R}_0$ we may conclude that the set

$$\{\bar{\mathfrak{p}} \in \operatorname{Ass}_{\bar{R}}(H^2_{\bar{R}_+}(\bar{R})) \mid \bar{\mathfrak{p}} \cap \bar{R}_0 \subset \bar{\mathfrak{m}}_0\}$$

is infinite.

Now, let $R_0 := (\bar{R}_0)_{\bar{\mathfrak{m}}_0} = K[x, y, s, t]_{(x,y,s,t)}$ and consider the homogeneous Noetherian R_0-algebra $R := R_0 \otimes_{\bar{R}_0} \bar{R} = (\bar{R}_0 \setminus \bar{\mathfrak{m}}_0)^{-1}\bar{R}$. On use of the graded flat

base change property of local cohomology it follows that the set $\mathrm{Ass}_R(H^2_{R_+}(R))$ is infinite. So, in view of the natural bijection (cf. [B-He, 5.5])

$$\mathrm{Ass}_R(H^2_{R_+}(R)) \longrightarrow \bigcup_{n\in\mathbb{Z}} \mathrm{Ass}_{R_0}(H^2_{R_+}(R)_n) : \mathfrak{p} \mapsto \mathfrak{p} \cap R_0,$$

the latter set is infinite, too.

References

[B-He] BRODMANN, M., HELLUS, M., Cohomological patterns of coherent sheaves over projective schemes, Journal of Pure and Applied Algebra 172 (2002), 165–182.

[B-K-S] BRODMANN, M., KATZMAN, M., SHARP, R.Y., Associated primes of graded components of local cohomology modules, Transactions of the AMS 354 (2002), 4261–4283.

[B-M-M] BRODMANN, M., MATTEOTTI, C., MINH, N.D., Bounds for cohomological deficiency functions of projective schemes over artinian rings, to appear in Vietnam Journal of Mathematics.

[B-S] BRODMANN, M., SHARP, R.Y., Local cohomology: an algebraic introduction with geometric applications, Cambridge studies in advanced mathematics 60, Cambridge University Press (1998). MR **99h:**13020

[B-H] BRUNS, W., HERZOG, J., Cohen Macaulay rings, Cambridge studies in advanced mathematics 39, Cambridge University Press (1993). MR **95h:**13020

[K] KATZMAN, M., An example of an infinite set of associated primes of a local cohomology module, Journal of Algebra 252 (2002), 161–166.

[Ki] KIRBY, D., Artinian modules and Hilbert polynomials, Quart. J. Math. Oxford Ser. (2) 24 (1973) 47–57. MR **47:**4993

[S] SINGH, A.K., p-torsion elements in local cohomology modules, Math. Res. Lett. 7 (2000) 165–176. MR **2001g:**13039

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, CH – 8057 Zürich, Switzerland

E-mail address: brodmann@math.unizh.ch

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, CH – 8057 Zürich, Switzerland

E-mail address: fumasoli@math.unizh.ch

Institute of Mathematics, University for Teacher Education, 599 Taleghani Avenue, Tehran 15614, Iran

E-mail address: roshan@iranpasargad.net