ON THE ALGEBRA RANGE OF AN OPERATOR ON A HILBERT $C^*$-MODULE OVER COMPACT OPERATORS

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ABSTRACT. Let $X$ be a Hilbert $C^*$-module over the $C^*$-algebra $K(H)$ of all compact operators on a complex Hilbert space $H$. Given an orthogonal projection $p \in K(H)$, we describe the set $V^n(A) = \{Ax, x \in X, \langle x, x \rangle = p\}$ for an arbitrary adjointable operator $A \in B(X)$. The relationship between the set $V^n(A)$ and the matricial range of $A$ is established.

1. Introduction and preliminaries

A left Hilbert $C^*$-module $X$ over a $C^*$-algebra $\mathcal{A}$ is by definition (see [6]) a linear space which is a left $\mathcal{A}$-module, together with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ on $X \times X$ that is linear in the first and conjugate-linear in the second variable. $X$ is also a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. Let $B(X)$ be the set of all maps $A : X \to X$ for which there is a map $A^* : X \to X$ such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all $x, y \in X$. Furthermore, let $K(X)$ be the closed linear subspace of $B(X)$ spanned by $\{\theta_{x,y} : x, y \in X\}$ where $\theta_{x,y}$ is a map in $B(X)$ defined by $\theta_{x,y}(z) = \langle z, y \rangle x$. It is well known that $B(X)$ is a $C^*$-algebra containing $K(X)$ as a two-sided ideal (for details see [6]).

By $B(H)$ and $K(H)$ we denote the algebra of all bounded operators and the ideal of all compact operators acting on a fixed complex Hilbert space $H$, respectively.

In the sequel $X$ will denote a left Hilbert $C^*$-module over the $C^*$-algebra $K(H)$. $X$ is assumed to be a full Hilbert $K(H)$-module which means that the ideal spanned by all inner products $\langle x, y \rangle$, $x, y \in X$, is dense in $K(H)$. (Otherwise, $X$ would be trivial, since $K(H)$ is a simple $C^*$-algebra.) It was shown in [5, Theorem 2] that $X$ possesses an orthonormal basis (i.e., an orthogonal system $(x_\lambda)$ that generates a dense submodule of $X$ such that $(x_\lambda, x_\lambda)$ is an orthogonal projection in $K(H)$ of rank 1). Furthermore, $X$ contains a Hilbert space $X_e$ with respect to the inner product $\langle \cdot, \cdot \rangle = \text{tr}(\langle \cdot, \cdot \rangle)$ where ‘tr’ means the trace. More precisely, for a fixed orthogonal projection $e$ in $K(H)$ of rank 1, $X_e$ is given as the set of all $e x$, $x \in X$.

It is known that $X$ and the Hilbert space $X_e$ have the same dimension (i.e., the cardinality of any orthonormal basis). (For all this see [3].) We shall assume that $X$ (and therefore $X_e$) is infinite dimensional.

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It was proved in [3] Remark 4(b), Theorem 5) that \( X_e \) is an invariant subspace for each \( A \) in \( B(X) \) and that the map \( A \mapsto A|X_e \) establishes an isomorphism between \( C^* \)-algebras \( B(X) \) and \( B(X_e) \) where \( B(X_e) \) denotes the algebra of all bounded operators on \( X_e \). This isomorphism enables us to describe the relationship between the matricial range of an operator \( A \) and the set \( V_p^n(A) = \{ \langle Ax, x \rangle : x \in X, \langle x, x \rangle = p \} \) where \( p \) is a fixed projection of rank \( n \). This is done in Section 3. In Section 2, \( V_p^n(A) \) is described in terms of isometries mapping the range of \( p \) into \( X_e \).

Before stating the results we establish some more notation as follows. First, a \( \mathbb{H} \)-valued vector \( h \) for each \( C \) in \( \mathbb{H} \) we get \( h \)-bounded operators on \( X \) by \( \mathbb{H} \)-denote by \( \mathbb{H} \). We now choose an orthonormal basis \( \{ \xi_1, \ldots, \xi_n \} \) for \( H_n \) which is to be held fixed for the rest of this paper. For \( \xi, \eta \in H, e_{\xi, \eta} \) in \( B(H) \) is defined by \( e_{\xi, \eta}(\nu) = \langle \nu | \eta \rangle \xi \). We introduce an inner product in \( H \) by \( \langle Ax, x \rangle = \langle Ax, x \rangle \langle x, x \rangle = p \).

2. Main result

**Definition 2.1.** For an operator \( A \in B(X) \) we define the set

\[
V_p^n(A) = \{ \langle Ax, x \rangle : x \in X, \langle x, x \rangle = p \}.
\]

**Remark 2.2.** Note that for \( n = 1 \), \( V_p^1(A) \) coincides (up to natural identification) with the classical numerical range of an operator \( A|X_{e_1} \) in \( B(X_{e_1}) \). Namely, \( \langle x, x \rangle = e_1 \) if and only if \( x \) is a unit vector in a Hilbert space \( X_{e_1} \). Furthermore, according to [2] Remark 4(b),(c) we get \( \langle Ax, x \rangle = (Ax, x)e_1 \).

The following lemma shows that \( V_p^n(A) \) is always non-empty.

**Lemma 2.3.** There exists a vector \( x \) in \( X \) such that \( \langle x, x \rangle = p \).

**Proof.** Since \( X \) is infinite dimensional we can choose an orthogonal set \( \{ y_1, \ldots, y_n \} \) in \( X \) such that \( \langle y_i, y_i \rangle = e \) for all \( i = 1, \ldots, n \) (see [3] Remark 4(d)). Let us denote \( x_i = e_{\xi_i, \xi_i} y_i \) for \( i = 1, \ldots, n \). Then we have

\[
\langle x_i, x_j \rangle = \langle e_{\xi_i, \xi_i} y_i, e_{\xi_j, \xi_j} y_j \rangle = e_{\xi_i, \xi_j} \langle y_i, y_j \rangle e_{\xi_i, \xi_j} = e_{\xi_i, \xi_j} \delta_{i,j} e_{\xi_i, \xi_i} = e_{\xi_i, \xi_i} \delta_{i,j} e_{\xi_i, \xi_i} = \delta_{i,j} e_{\xi_i, \xi_i}
\]

for all \( i, j = 1, \ldots, n \). It remains to define \( x = x_1 + x_2 + \cdots + x_n \).

**Remark 2.4.** It is clear that each \( x \in X \) such that \( \langle x, x \rangle = p \) satisfies \( \langle x - px, x - px \rangle = 0 \), so \( x = px \). Further, for every such vector \( x \) we have

\[
p\langle Ax, x \rangle = \langle A(px), x \rangle = \langle Ax, x \rangle = \langle Ax, px \rangle = \langle Ax, x \rangle p.
\]

Hence, the subspace \( H_n \) reduces \( \langle Ax, x \rangle \) for all \( A \in B(X) \). Moreover, for \( \eta \perp H_n \) we have \( \langle Ax, x \rangle \eta = \langle Ax, x \rangle p\eta = 0 \). This shows that the operator \( \langle Ax, x \rangle \) acts trivially on \( H_n \), so that, without loss of generality, \( \langle Ax, x \rangle \) can be regarded as an operator acting on the \( n \)-dimensional space \( H_n \).
Let $A$ be an arbitrary operator in $B(X)$. We shall see that the set $V^n_p(A)$ basically does not depend on the choice of the rank $n$ projection $p \in K(H)$. Namely, if $q \in K(H)$ is an arbitrary projection of rank $n$, then the sets $V^n_p(A)$ and $V^n_q(A)$ can be naturally identified, as shown in the following proposition.

**Proposition 2.5.** Let $A \in B(X)$ and let $p, q \in K(H)$ be projections of finite rank $n$. Let $\{\xi_1, \ldots, \xi_n\}$ and $\{\eta_1, \ldots, \eta_n\}$ be orthonormal bases for the ranges of $p$ and $q$, respectively. Then a map $\Phi : V^n_p(A) \to V^n_q(A)$ defined by $\Phi((Ax, x)) = \langle A(\sum_{i=1}^n e_{\eta_i, \xi_i} x), \sum_{i=1}^n e_{\eta_i, \xi_i} x \rangle$ is a bijection.

**Proof.** Let $\Phi$ be as in the statement of the proposition. Since

$$\sum_{i=1}^n e_{\eta_i, \xi_i} x, \sum_{i=1}^n e_{\eta_i, \xi_i} x = \sum_{i,j=1}^n e_{\eta_i, \xi_i} (x, x) e_{\xi_j, \eta_j} = \sum_{i,j=1}^n e_{\eta_i, \xi_i} p e_{\xi_j, \eta_j}$$

$$= \sum_{i,j=1}^n \delta_{i,j} e_{\eta_i, \xi_i} e_{\xi_j, \eta_j} = \sum_{i=1}^n e_{\eta_i, \xi_i} e_{\xi_i, \eta_i} = \sum_{i=1}^n e_{\eta_i, \eta_i} = q,$$

we conclude that $\Phi$ is a well-defined map.

To prove that $\Phi$ is injective, suppose that $\Phi((Ax, x)) = \Phi((Ay, y))$ for some $x, y \in X$, $(x, x) = (y, y) = p$. It follows that

$$\langle A(\sum_{i=1}^n e_{\eta_i, \xi_i} x), \sum_{i=1}^n e_{\eta_i, \xi_i} x \rangle = \langle A(\sum_{i=1}^n e_{\eta_i, \xi_i} y), \sum_{i=1}^n e_{\eta_i, \xi_i} y \rangle,$$

which implies

$$\sum_{i,j=1}^n e_{\eta_i, \xi_i} (Ax, x) e_{\xi_j, \eta_j} = \sum_{i,j=1}^n e_{\eta_i, \xi_i} (Ay, y) e_{\xi_j, \eta_j}.$$

Multiplying the above equality on its left side by $e_{\xi_i, \eta_i}$ and on its right side by $e_{\eta_i, \xi_i}$, we obtain

$$e_i \langle Ax, x \rangle e_j = e_i \langle Ay, y \rangle e_j$$

for all $i, j = 1, \ldots, n$. Thus we have

$$\langle Ax, x \rangle = p(\langle Ax, x \rangle) = \sum_{i,j=1}^n e_i \langle Ax, x \rangle e_j$$

$$= \sum_{i,j=1}^n e_i \langle Ay, y \rangle e_j = p(\langle Ay, y \rangle) = \langle Ay, y \rangle.$$

It remains to show that $\Phi$ is surjective. To see this, take any $\langle Ay, y \rangle \in V^n_q(A)$. We define $x = \sum_{i=1}^n e_{\xi_i, \eta_i} y$. Then $\langle Ax, x \rangle \in V^n_p(A)$ since

$$\langle x, x \rangle = \sum_{i,j=1}^n e_{\xi_i, \eta_i} \langle y, y \rangle e_{\xi_j, \xi_j} = \sum_{i,j=1}^n e_{\xi_i, \eta_i} q e_{\eta_j, \xi_j}$$

$$= \sum_{i,j=1}^n \delta_{i,j} e_{\xi_i, \eta_i} e_{\eta_j, \xi_j} = \sum_{i=1}^n e_{\xi_i, \eta_i} e_{\eta_i, \xi_i} = \sum_{i=1}^n e_{\xi_i, \xi_i} = p.$$
Finally,

\[
\Phi(\langle Ax, x \rangle) = \langle A(\sum_{i=1}^{n} e_{n,i} x), \sum_{i=1}^{n} e_{n,i} x \rangle = \langle A(\sum_{i=1}^{n} e_{n,i} y), \sum_{i=1}^{n} e_{n,i} y \rangle = \langle Ay, y \rangle.
\]

This completes the proof. \qed

If an arbitrary rank \(n\) projection \(p\) is fixed, then according to the identification from the above proposition, we can write \(V^n_p(A) = V^n(A)\). Remark 2.4 shows us now that the set \(V^n(A)\) can be considered as a subset of \(B(H_n)\) where \(H_n\) denotes, as before, the range of \(p\).

In the following theorem we give an alternative description of the set \(V^n(A)\). To do this, we have to introduce a “transposed” operator on \(B(H_n)\).

**Definition 2.6.** Let \(\{\xi_1, \ldots, \xi_n\}\) be the fixed orthonormal basis for \(H_n\). We define a “transposed” operator \(\tau : B(H_n) \to B(H_n)\) by the formula \(t \mapsto \tau(t)\) where \(\tau(t)\) is given by its action on the basis \(\{\xi_1, \ldots, \xi_n\}\):

\[
\tau(t)\xi_j = \sum_{i=1}^{n} (t\xi_i | \xi_j)\xi_i.
\]

**Remark 2.7.** According to the above definition, for \(t \in B(H_n)\) and \(\eta = \sum_{j=1}^{n} \alpha_j \xi_j \in H_n\), it follows that

\[
\tau(t)\eta = \sum_{j=1}^{n} \alpha_j \tau(t)\xi_j = \sum_{j=1}^{n} \alpha_j (\sum_{i=1}^{n} (t\xi_i | \xi_j)\xi_i).
\]

Further, let us denote by \([t_{ij}]\) and \([\tau(t)_{ij}]\) the matrix representations of the linear operators \(t\) and \(\tau(t)\) with respect to the orthonormal basis \(\{\xi_1, \ldots, \xi_n\}\). Then we get

\[
\tau(t)_{kj} = (\tau(t)\xi_j | \xi_k) = (\sum_{i=1}^{n} (t\xi_i | \xi_j)\xi_i) = (t\xi_k | \xi_j) = t_{jk}
\]

for all \(k, j = 1, \ldots, n\). This shows that the matrix of \(\tau(t)\) is obtained by transposing the matrix of \(t\), hence the map \(\tau\) is a linear operator on \(B(H_n)\).

In our next proposition some elementary properties of the map \(\tau\) are collected.

**Proposition 2.8.** The operator \(\tau\) from Definition 2.6 has the following properties:

(i) \(\tau(t^*) \geq 0\) for every \(t \in B(H_n)\),
(ii) \(\tau^2(t) = t\) for every \(t \in B(H_n)\),
(iii) \(\tau(t^*) = \tau(t)^*\) for every \(t \in B(H_n)\),
(iv) \(\tau(ts) = \tau(s)\tau(t)\) for all \(t, s \in B(H_n)\).

Since all assertions are clear, the proof is omitted.
We now state our theorem.

**Theorem 2.9.** Let \( A \) be an operator in \( B(X) \). Then

\[
\tau(V^n(A)) := \{\tau((Ax, x)) : x \in X, \langle x, x \rangle = p\} = \{v^*Ax : v : H_n \to X_e \text{ is an isometry}\}.
\]

**Remark 2.10.** Note that in the assertion of this theorem we use the fact that \( X_e \) is an invariant subspace for each \( A \) in \( B(X) \) (see [3, Remark 4(b)]).

**Proof of Theorem 2.9.** Given an isometry \( v : H_n \to X_e \), we define for \( i = 1, \ldots, n \) the vector \( x_i = e_{x_i} v \xi_i \). Then \( \{x_1, \ldots, x_n\} \) is an orthogonal set in \( X \) such that \( \langle x_i, x_j \rangle = \delta_{i,j} e_i \). Indeed, for \( 1 \leq i, j \leq n \) we have

\[
\langle x_i, x_j \rangle = e_{x_i} (v \xi_i, v \xi_j) e_{x_j} = e_{x_i} e_{x_j} = \delta_{i,j} e_i,
\]

since the equality \( \langle y, z \rangle = \langle y, z \rangle e \) is satisfied for all \( y, z \) from \( X_e \) (see [3, Remark 4(c)]). The vector \( x = x_1 + \cdots + x_n \) clearly satisfies \( \langle x, x \rangle = p \). Furthermore, for \( 1 \leq i, j \leq n \), we obtain

\[
\tau((Ax, x)) \xi_j | \xi_i) = \tau((Ax, x)) \xi_i | \xi_j).
\]

Therefore, we have shown that \( v^*Axv = \tau((Ax, x)). \)

Conversely, let \( x \) be a vector in \( X \) such that \( \langle x, x \rangle = p \). We define an operator \( v : H_n \to X_e \) on the orthonormal basis \( \{\xi_1, \ldots, \xi_n\} \) by putting \( v \xi_i = e_{x_i} x \) for \( i = 1, \ldots, n \). Observe that the operator \( v \) takes its values in \( X_e \), since

\[
\langle v \xi_i, v \xi_j \rangle = e_{x_i} \langle x, x \rangle e_{x_j} = e_{x_i} e_{x_j} e = e_{x_i} = e_{x_j}.
\]

for \( i = 1, \ldots, n \). Moreover, \( v \) is an isometry since

\[
\langle v \xi_i, v \xi_j \rangle = \text{tr}(v \xi_i, v \xi_j) = \text{tr}(e_{x_i} \langle x, x \rangle e_{x_j} = \text{tr}(e_{x_i} e_{x_j} e) = \delta_{i,j}
\]

for all \( i, j = 1, \ldots, n \). If we put \( x_i = e_{x_i} x \) for \( i = 1, \ldots, n \), then \( \langle x, x \rangle = p \) implies \( x = px = (e_1 + \cdots + e_n)x = x_1 + \cdots + x_n \). Observe that \( \langle x_i, x_j \rangle = \delta_{i,j} e_i \) and also that \( x_i = e_{x_i} x = e_{x_i} \xi_i x = e_{x_i} v \xi_i \) for \( i = 1, \ldots, n \). Thus, as in the proof of the first part, we conclude that \( \tau((Ax, x)) = v^*Axv \). This completes the proof. \( \square \)

3. Relation between \( V^n(A) \) and \( W^n(A) \)

Let \( A \) be a unital \( C^* \)-algebra. Given an element \( a \) in \( A \), we shall denote by \( C^*(a) \) the \( C^* \)-algebra generated by \( a \) and the identity. Let \( CP(C^*(a), C^n, 1) \) be the set of all completely positive maps of \( C^*(a) \) into \( B(C^n) \) which preserve the identity. (The reader is referred to [1] or [7] for the definition and more details about completely positive maps.)

Furthermore, given a subset \( S \) of a unital \( C^* \)-algebra \( A \), we denote by \( mconv(S) \) the matricial convex hull of \( S \), i.e., the set of all finite sums of the type \( \sum t^*_i a_i t_i \), where each \( a_i \in S \) and where the elements \( t_i \in A \) are such that \( \sum t^*_i t_i = 1 \).
By $S^-$ we denote the topological closure of a set $S$.

In what follows, let us fix a unitary operator $u : C^n \to H_n$. If $\{e_1, \ldots, e_n\}$ and $\{\xi_1, \ldots, \xi_n\}$ denote the standard orthonormal basis for $C^n$ and our fixed orthonormal basis for $H_n$, respectively, then $u$ can be chosen by its action on the basis $\{e_1, \ldots, e_n\}$, i.e., $ue_j = \xi_j$ for $j = 1, \ldots, n$. In the sequel $\psi : B(H_n) \to B(C^n)$ will denote an isomorphism between $C^*$-algebras $B(H_n)$ and $B(C^n)$ defined by $\psi(t) = u^*tu$, $t \in B(H_n)$.

We begin by recalling the definition of the matricial range of an element in a unital $C^*$-algebra (see [2], [4] or [9]).

**Definition 3.1.** Let $A$ be a unital $C^*$-algebra. For an element $a \in A$, the matricial range of $a$ is the set

$$W^n(a) = \{ \varphi(a) : \varphi \in CP(C^*(a), C^n), 1 \}.$$  

**Remark 3.2.** It is clear that the corresponding elements of $*$-isomorphic $C^*$-algebras have the same matricial range. In particular, for all $A \in B(X)$ we have $W^n(A) = W^n(A|X_e)$.

In [3, Theorem 3.5] J. Bunce and N. Salinas showed that for a given operator $A|X_e$ in $B(X_e)$ it holds that

$$W^n(A|X_e) = \text{mconv}\{ v^*A|X_ev : v : C^n \to X_e \text{ is an isometry} \}^-.$$  

Theorem 2.9 immediately implies that

$$\psi(\tau(V^n(A))) = \{ \psi(v^*A|X_ev) : v : H_n \to X_e \text{ is an isometry} \} = \{ v^*A|X_ev : v : C^n \to X_e \text{ is an isometry} \}.$$  

Thus we have the following result:

**Theorem 3.3.** If $A \in B(X)$, then $W^n(A) = \text{mconv}(\psi(\tau(V^n(A))))^-.$

Further, for an operator $T \in B(H)$, recall that the essential matricial range of $T$ is the set

$$W^n_e(T) = \{ \varphi(T) : \varphi \in CP(C^*(T), C^n), 1, \varphi[C^*(T) \cap K(H) = 0] \}.$$  

We now introduce the definition of the essential matricial range of $A \in B(X)$ as follows:

**Definition 3.4.** For an operator $A \in B(X)$ the essential matricial range of $A$ is the set

$$W^n_e(A) = \{ \varphi(A) : \varphi \in CP(C^*(A), C^n, 1), \varphi[C^*(A) \cap K(X) = 0] \}.$$  

**Remark 3.5.** Since $A \in K(X)$ if and only if $A|X_e \in K(X_e)$ (see [3, Theorem 6]), it follows that $W^n_e(A) = W^n_e(A|X_e)$.

Given an operator $A \in B(X)$, there is an interesting relationship between the sets $W^n(A)$, $\psi(\tau(V^n(A)))$ and $W^n_e(A)$, as an immediate consequence of Theorem 2.9 and Theorem 3.7 of [5].

**Theorem 3.6.** If $A \in B(X)$, then $W^n(A) = \text{mconv}(\psi(\tau(V^n(A))) \cup W^n_e(A)).$

Finally, as a consequence of the equivalence of the conditions (a) and (c) in Theorem 3.1 of [7], we get a description of the essential matricial range of an operator $A$ in $B(X)$.
Theorem 3.7. Let $A$ be in $B(X)$. Then $l \in W_n(A)$ if and only if there exists an orthogonal sequence $(x_k)$ in $X$ such that $\langle x_k, x_k \rangle = p$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} \psi((Ax_k, x_k)) = l$.

Proof. As in the proof of Theorem 2.9 we conclude that $v : H_n \to X_e$ is an isometry if and only if there exists a vector $x$ in $X$ such that $\langle x, x \rangle = p$. Thereby $v^* A | X_e v = \tau(\langle Ax, x \rangle)$ and the vectors $x_i = e_i x$ satisfy $\langle x_i, x_j \rangle = \delta_{i,j} e_i, x_1 + \cdots + x_n = x, x_i = e_{i, f} v \xi_i$ and $v \xi_i = e_{\xi, \xi} x$ for all $i, j = 1, \ldots, n$. Observe that isometries $v u, v' u : C^n \to X_e$ have mutually orthogonal ranges if and only if isometries $v, v' : H_n \to X_e$ have mutually orthogonal ranges, that is, if and only if $\langle x, x' \rangle = 0$ is satisfied for the corresponding vectors $x$ and $x'$. To complete the proof, it remains to apply Theorem 3.1 ((a) $\iff$ (c)) of $[3]$. □

Remark 3.8. We provide here an alternative proof for the sufficiency part. To do this, we need the following lemma concerning the characterisation of an operator in $K(X)$. Notice that if $n = \text{rank}(p) = 1$, our lemma reduces to Theorem 7 ((a) $\iff$ (c)) from $[3]$. 

Lemma 3.9. For $A \in B(X)$ the following statements are mutually equivalent:

1. $A \in K(X)$.
2. $\lim_{k \to \infty} \langle Ax_k, x_k \rangle = 0$ for each orthogonal system $(x_k)$ in $X$ such that $\langle x_k, x_k \rangle = p$ for all $k \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). Since $A = B + IC$, where $B, C \in K(X)$ are self-adjoint operators, we may assume that $A$ is self-adjoint. First, observe that $\|\langle Ax_k, x_k \rangle\| \leq \|A\| \|x_k\|^2 = \|A\|p = \|A\|$ for all $k \in \mathbb{N}$. If $l$ is a cluster point of the norm bounded sequence $(\langle Ax_k, x_k \rangle)$ in $B(H_n)$, then there exists a subsequence $(\langle Ax_k, x_k \rangle)$ of $(\langle Ax_k, x_k \rangle)$ converging to $l$. Obviously, $l$ is a self-adjoint operator in $B(H_n)$. We shall show that $l$ must be zero. Let $(\eta_1, \ldots, \eta_n)$ be an orthonormal basis for $H_n$ consisting of eigenvectors of $l$. Put $f_j = e_{\eta_j, \eta_j}$ for $j = 1, \ldots, n$. Then $(f_j x_k)_i$ is an orthonormal system in $X$ (i.e., an orthonormal system of vectors whose inner squares are orthogonal projections of rank 1). Indeed,

$$\langle f_j x_k, f_s x_k \rangle = \langle f_j x_k, x_k \rangle f_j = f_j \delta_{k,i} p f_j = \delta_{s,t} f_j$$

for all $s, t \in \mathbb{N}$. Now, for $j = 1, \ldots, n$, we have

$$f_j f_j = \lim_{i \to \infty} f_j \langle Ax_k, x_k \rangle f_j = \lim_{i \to \infty} \langle Af_j x_k, f_j x_k \rangle = 0,$$

where the last equality follows from Theorem 7 of $[3]$. We conclude that $l = 0$.

(ii) $\Rightarrow$ (i). Let $(y_k)$ be an arbitrary orthonormal system in a Hilbert space $X_e$. We define $x_k = e_{\xi_1, \xi} y_{(k-1)n+1} + e_{\xi_2, \xi} y_{(k-1)n+2} + \cdots + e_{\xi_n, \xi} y_{kn}$ for $k \in \mathbb{N}$. It is easy to see that $(x_k)$ is an orthogonal system in $X$ such that $\langle x_k, x_k \rangle = p$ for all $k \in \mathbb{N}$. By the hypothesis it follows that

$$\lim_{k \to \infty} \langle Ae_i x_k, e_i x_k \rangle = \lim_{k \to \infty} e_i (Ax_k, x_k) e_i = 0$$

for $i = 1, \ldots, n$. Now we use the facts that $X_e$ is invariant for $A$ and that $\langle x, y \rangle = \langle x, y \rangle e$ for all $x, y \in X_e$ (see $[3]$ Remark 4(b),(c)) to obtain

$$\langle Ae_i x_k, e_i x_k \rangle = e_{\xi, \xi} \langle Ay_{(k-1)n+i}, y_{(k-1)n+i} \rangle e_{\xi, \xi} = (A | X_e y_{(k-1)n+i}, y_{(k-1)n+i}) e_i$$
for \( i = 1, \ldots, n, k \in \mathbb{N} \). Hence, \( \lim_{\lambda \to -\infty} (A|x_{\lambda y_{(k-1)n+i}}, y_{(k-1)n+i}) = 0 \) for \( i = 1, \ldots, n \). It obviously follows that \( \lim_{\lambda \to -\infty} (A|x_{\lambda y_k}, y_k) = 0 \), so by Theorem 1.8.7 of [3], \( A|X_\varepsilon \) is a compact operator on \( X_\varepsilon \). Therefore, Theorem 6 of [3] implies that \( A \in K(X) \).

An alternative proof of the sufficiency part of Theorem 3.7. For \( k \in \mathbb{N} \) we define a map \( \varphi_k : C^*(A) \to B(H_n) \) by setting

\[
\varphi_k(T) = \tau((Tx_k, x_k)) \quad (T \in C^*(A)).
\]

Then, \( \varphi_k \) is a completely positive map which takes \( I \) to \( p \). Namely, since every positive element of \( M_r(C^*(A)) \) is a finite sum of the elements of the type \( [T_i^* T_j] \) where \( T_i \in C^*(A) \), it is sufficient to show that \( \varphi_k(T_i^* T_j) \) is a positive element of \( M_r(B(H_n)) \) for all \( T_1, \ldots, T_r \in C^*(A) \). We have

\[
[\varphi_k(T_i^* T_j)] = [\tau(T_i^* T_j x_k, x_k)] = [\tau(T_j x_k, T_i x_k)] = \tau((T_i x_k, T_j x_k))
\]

where \( \tau \) also stands for the “transposed” operator on \( M_r(B(H_n)) \). By Lemma 4.2 of [3] (the assertion is also true for left Hilbert \( C^* \)-modules), we have \( [T_i x_k, T_j x_k] \geq 0 \), so \( [\varphi_k(T_i^* T_j)] \geq 0 \). Since the set of all completely positive maps of \( C^*(A) \) into \( B(H_n) \) which take \( I \) to \( p \) is BW-compact (the BW-topology is introduced in [3]), there exists a subsequence \( (\varphi_{k_j}) \) of \( (\varphi_k) \) and a completely positive map \( \varphi : C^*(A) \to B(H_n) \) such that \( \varphi(T) = \lim_{j \to \infty} \varphi_{k_j}(T) \) for all \( T \in C^*(A) \). In particular, by the preceding lemma, for \( K \in C^*(A) \cap K(X) \) it holds that \( \varphi(K) = \lim_{j \to \infty} \tau((K x_{k_j}, x_{k_j})) = 0 \). Hence, \( \psi \circ \varphi \) is a completely positive map of \( C^*(A) \) into \( B(C^n) \) satisfying (\( \psi \circ \varphi)(I) = \psi(p) = u^* p u = 1 \) and such that \( (\psi \circ \varphi)(K) = 0 \) for all \( K \in C^*(A) \cap K(X) \). Therefore,

\[
l = \lim_{i \to \infty} \psi(\tau((Ax_{k_i}, x_{k_i}))) = \lim_{i \to \infty} (\psi \circ \varphi_{k_i})(A) = (\psi \circ \varphi)(A) \in W^e(A),
\]

as desired. \( \square \)

In conclusion, let us consider the case when a given operator \( A \) in \( B(X) \) is normal. W. B. Arveson proved in [2] Proposition 2.4.1] that the matricial range of a normal operator \( T \in B(H) \) is the set

\[
W^n(T) = \left\{ \sum_{i=1}^r \lambda_i k_i : r \geq 1, \lambda_i \in \sigma(T), k_i \in B(C^n), k_i \geq 0, \sum_{i=1}^r k_i = 1 \right\},
\]

where \( \sigma(T) \) denotes the spectrum of \( T \).

Notice that Proposition 2.8 (i) implies \( l \in W^n(T) \) if and only if \( \psi(\tau(\psi^{-1}(l))) \in W^n(T) \).

We need only to apply Proposition 2.8, Theorem 3.3, Theorem 3.6 and Theorem 3.7 to obtain

**Corollary 3.10.** Let \( A \) be a normal operator in \( B(X) \). Then the following statements hold:

(i) \( W^n(A) = \mathrm{mconv}(\psi(V^n(A))) \).

(ii) \( W^n(A) = \mathrm{mconv}(\psi(V^n(A)) \cup W^e(A)) \).

(iii) \( l \in W^n(A) \) if and only if there is an orthogonal sequence \( (x_k) \) in \( X \) such that \( \langle x_k, x_k \rangle = p \) for all \( k \in \mathbb{N} \) and \( \lim_{k \to \infty} \psi(\langle Ax_k, x_k \rangle) = l \).
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