\textbf{π}_1 \text{ OF HAMILTONIAN S}^1 \text{ MANIFOLDS}

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Abstract. Let \((M, \omega)\) be a connected, compact symplectic manifold equipped with a Hamiltonian \(S^1\) action. We prove that, as fundamental groups of topological spaces, \(\pi_1(M) = \pi_1(\text{minimum}) = \pi_1(\text{maximum}) = \pi_1(M_{\text{red}})\), where \(M_{\text{red}}\) is the symplectic quotient at any value in the image of the moment map \(\phi\).

Let \((M, \omega)\) be a connected, compact symplectic manifold equipped with a circle action. If the action is Hamiltonian, then the moment map \(\phi : M \to \mathbb{R}\) is a perfect Bott-Morse function. Its critical sets are precisely the fixed point sets \(M_{S^1}\) of the \(S^1\) action, and \(M_{S^1}\) is a disjoint union of symplectic submanifolds. Each fixed point set has even index. By [1], \(\phi\) has a unique local minimum and a unique local maximum. We will use Morse theory to prove

\textbf{Theorem 0.1.} Let \((M^{2n}, \omega)\) be a connected, compact symplectic manifold equipped with a Hamiltonian \(S^1\) action. Then as fundamental groups of topological spaces, \(\pi_1(M) = \pi_1(\text{minimum}) = \pi_1(\text{maximum}) = \pi_1(M_{\text{red}})\), where \(M_{\text{red}}\) is the symplectic quotient at any value in the image of the moment map \(\phi\).

Remark 0.2. The theorem is not true for orbifold \(\pi_1\) of \(M_{\text{red}}\), as shown in the example below. (See [5] or [11] for the definition of orbifold \(\pi_1\).)

Let \(a \in \text{im}(\phi)\), and \(\phi^{-1}(a) = \{x \in M \mid \phi(x) = a\}\) be the level set. Define \(M_a = \phi^{-1}(a)/S^1\) to be the symplectic quotient.

Note that if \(a\) is a regular value of \(\phi\), and if the circle action on \(\phi^{-1}(a)\) is not free, then \(M_a\) is an orbifold, and we have an orbibundle:

\[
\begin{array}{ccc}
S^1 & \hookrightarrow & \phi^{-1}(a) \\
\downarrow & & \downarrow \\
M_a & & \\
\end{array}
\]  

(0.1)

If \(a\) is a critical value of \(\phi\), then \(M_a\) is a stratified space ([11]).

Now, let \(S^1\) act on \((S^2 \times S^2, 2\rho \oplus \rho)\) (where \(\rho\) is the standard symplectic form on \(S^2\)) by \(\lambda(z_1, z_2) = (\lambda^2 z_1, \lambda z_2)\). Let 0 be the minimal value of the moment map. Then for \(a \in (1, 2)\), \(M_a\) is an orbifold which is homeomorphic to \(S^2\) and has two \(Z_2\) singularities. The orbifold \(\pi_1\) of \(M_a\) is \(Z_2\), but the \(\pi_1\) of \(M_a\) as a topological space is trivial.

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Let \( a \) be a regular or a critical value of \( \phi \). Define
\[
M^a = \{ x \in M \mid \phi(x) \leq a \}.
\]

By Morse theory, we have the following lemmas about how \( M^a \) and \( \phi^{-1}(a) \) change when \( \phi \) doesn't cross or crosses a critical level.

**Lemma 0.3** (Theorem 3.1 in [7]). Assume \([a, b] \subset \text{im}(\phi)\) is an interval consisting of regular values. Then \( \phi^{-1}(a) \) is diffeomorphic to \( \phi^{-1}(b) \).

**Lemma 0.4** (See [7] and [8]). Let \( c \in (a, b) \) be the only critical value of \( \phi \) in \([a, b]\), \( F \subset \phi^{-1}(c) \) the fixed point set component, \( D^- \) the negative disk bundle of \( F \), and \( S(D^-) \) its sphere bundle. Then \( M^b \) is homotopy equivalent to \( M^a \cup_{S(D^-)} D^- \).

**Lemma 0.5.** Under the same hypothesis of Lemma 0.4, \( \phi^{-1}(a) \cup_{S(D^-)} D^- \) has the homotopy type of \( \phi^{-1}(c) \).

**Proof.** If \( F \) is a point, then from the proof of Theorem 3.2 in [7], we can see that the region between \( \phi^{-1}(a) \cup_{S(D^-)} D^- \) and \( \phi^{-1}(c) \) is homotopy equivalent to both \( \phi^{-1}(a) \cup_{S(D^-)} D^- \) and \( \phi^{-1}(c) \). (See pp. 18 and 19 in [7].)

The same idea applies when \( F \) is a submanifold. \( \square \)

This lemma immediately implies the following

**Lemma 0.6.** Under the same hypothesis of Lemma 0.4, \( M_c \) has the homotopy type of \( M_a \cup_{S(D^-)} D^- / S^1 \).

We will also need

**Lemma 0.7.** Assume \( F \) is a critical set, \( \phi(F) \in (a, b) \) and there are no other critical sets in \( \phi^{-1}([a, b]) \). If \( \text{index}(F) = 2 \), then there is an embedding \( i \) from \( F \) to \( M_a \) such that \( S(D^-) \) can be identified with the restriction of \( \phi^{-1}(a) \) to \( F \), i.e., we have the following bundle identification:
\[
\begin{array}{ccc}
S^1 & \hookrightarrow & S(D^-) \\
\downarrow & & \downarrow \\
F & \xrightarrow{i} & M_a
\end{array}
\]

**Proof.** Assume that the positive normal bundle \( D^+ \) of \( F \) has complex rank \( m \). We may assume \( \phi(F) = 0 \). By Lemma 0.3, we can assume \( a = -\epsilon \) and \( b = +\epsilon \) for \( \epsilon \) small. By the equivariant symplectic embedding theorem ([8]), a tubular neighborhood of \( F \) is equivariantly diffeomorphic to \( P \times G(\mathbb{C} \times \mathbb{C}^m) \), where \( G = S^1 \times U(m) \) and \( P \) is a principal \( G \)-bundle over \( F \). The moment map can be written
\[
\phi = -p_0 |z_0|^2 + p_1 |z_1|^2 + \cdots + p_m |z_m|^2,
\]
where \( p_0, p_1, ..., p_m \) are positive integers. Then \( \phi^{-1}(-\epsilon) = P \times G(S^1 \times \mathbb{C}^m) \), \( M_{-\epsilon} = P \times G(S^1 \times \mathbb{C}^m)/S^1 \), \( F = P \times G(S^1 \times 0)/S^1 \subset M_{-\epsilon} \), and \( S(D^-) = P \times G S^1 \) is the restriction of \( \phi^{-1}(\epsilon) \) to \( F \). \( \square \)

We are now ready to prove the theorem.

**Proof.** Let us put the critical values of \( \phi \) in the order
\[
\text{minimal} = 0 < a_1 < a_2 < \cdots < a_k = \text{maximal}.
\]
First, we prove \( \pi_1(\text{minimum}) = \pi_1(M_{\text{red}}) \).

For \( a \in (0, a_1) \), by the equivariant symplectic embedding theorem, \( \phi^{-1}(a) \) is a sphere bundle over the minimum. Assume the fiber of this sphere bundle is \( S^{2l+1} \) with \( l \geq 0 \). Then \( M_a \) is diffeomorphic to a weighted \( CP^l \) bundle over the minimum.
(possibly an orbifold). The weighted $\mathbb{C}P^l$ is the symplectic reduction of $S^{2l+1}$ by
the $S^1$ action with different weights. We can easily see that $S^{2l+1} \to$ weighted $\mathbb{C}P^l$
induces a surjection in $\pi_1$ since the inverse image of each point is connected. So
the weighted $\mathbb{C}P^l$ is simply connected, hence $\pi_1(M_a) = \pi_1(\text{minimum})$.

Next, let $b \in (a_1, a_2)$, and let $F \subset \phi^{-1}(a_1)$ be the critical set. (If there are other
critical sets on the same level, argue similarly for each connected component.)

By Lemma 0.6 and the Van-Kampen theorem, we have

$$\pi_1(M_a) = \pi_1(M_a) \ast_{\pi_1(S(D^-)/S^1)} \pi_1(D^-/S^1) = \pi_1(M_a),$$

since $S(D^-)/S^1$ is a weighted projectivized bundle over $F$, and $D^-/S^1$ is homotopy
equivalent to $F$, so $\pi_1(S(D^-)/S^1)$ is isomorphic to $\pi_1(D^-/S^1)$.

Similarly, using $-\phi$, we can obtain $\pi_1(M_b) = \pi_1(M_{a_1})$.

By induction on the critical values, and by repeating the argument each time $\phi$
crosses a critical level, we see that if $a' \in (a_k-1, a_k)$, then $\pi_1(M_a') = \pi_1(\text{minimum})$.
Similarly to the proof of $\pi_1(M_a) = \pi_1(\text{minimum})$ when $a \in (0, a_1)$, we have
$\pi_1(M_a') = \pi_1(\text{minimum})$.

Therefore we have proved that $\pi_1(M_{\text{red}}) = \pi_1(\text{minimum}) = \pi_1(\text{maximum})$.

Next, we prove $\pi_1(M) = \pi_1(\text{minimum})$.

Consider $M^a$, for $a \in (0, a_1)$. Since $M^a$ is a complex disk bundle over the
minimum, $\pi_1(M^a) = \pi_1(\text{minimum}) = \pi_1(M_a)$.

Consider $b \in (a_1, a_2)$, and let $F \subset \phi^{-1}(a_1)$ be the critical set.
First assume $\text{index}(F) = 2$. By Lemma 0.4 and the Van-Kampen theorem,

$$\pi_1(M^b) = \pi_1(M^a) \ast_{\pi_1(S(D^-))} \pi_1(D^-) = \pi_1(M^a) \ast_{\pi_1(S(D^-))} \pi_1(F).$$

Consider the fibration

$$S^1 \hookrightarrow S(D^-) \xrightarrow{F}$$

and its homotopy exact sequence

$$\cdots \to \pi_1(S^1) \xrightarrow{f} \pi_1(S(D^-)) \xrightarrow{F} \pi_1(F) \to 0.$$

The map $f$ is surjective. By Lemma 0.7, the image of $\ker(f) = \text{im}(j)$ in $\pi_1(M_a)$
is $0$. By induction, $\pi_1(M^a) = \pi_1(M_a)$. So the image of $\ker(f)$ in $\pi_1(M^a)$ is $0$.
Hence, $\pi_1(M^b) = \pi_1(M^a) = \pi_1(\text{minimum})$.

If $\text{index}(F) > 2$, then the corresponding map $\pi_1(S(D^-)) \to \pi_1(F)$ is an isomorphism.
So we also have $\pi_1(M^b) = \pi_1(M^a)$.

By induction, we see that $\pi_1(M) = \pi_1(\text{minimum})$.

Remark 0.8. The proof that $\pi_1(\text{minimum}) = \pi_1(M_{\text{red}})$ can be achieved by using
known results about how the reduced space changes after $\phi$ crosses a critical level.
(See [4], for instance, or [6] where the action is semi-free.) After the first induction
step, when $\phi$ crosses a critical set $F$, if $\text{index}(F) = 2$, then $M_a$ is homeomorphic to
$M_{a_1}$; if $\text{index}(F) > 2$, then $M_{a_1}$ can be obtained from $M_a$ by a blow-up followed
by a blow-down. $M_b$ and $M_{a_1}$ are similarly related. Then we modify the proof of
D. McDuff’s (9) result.

Proposition 0.9. If $\tilde{X}$ is the blow-up of $X$ along a submanifold $N$, then $\pi_1(\tilde{X}) = \pi_1(X)$.  

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References


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