CIRCLE MAPS HAVING AN INFINITE $\omega$-LIMIT SET WHICH CONTAINS A PERIODIC ORBIT HAVE POSITIVE TOPOLOGICAL ENTROPY

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Abstract. Let $f$ be a continuous map from the circle to itself. The main result of this paper is that the topological entropy of $f$ is positive if and only if $f$ has an infinite $\omega$-limit set which contains a periodic orbit.

1. Introduction

Let $f$ be a continuous map from a continuum $X$ to itself. We denote the $n$-fold composition $f^n$ of $f$ with itself by $f \circ \cdots \circ f$ and $f^0$ the identity map. Let $x$ be a point of $X$. We define the orbit $\text{Orb}(x; f)$ by $f \circ \cdots \circ f_n(x)$, and we define the $\omega$-limit set of $x$ to be the set $\omega(x; f) = \{y \in X\}$ for each neighborhood $V$ of $y$ and each positive integer $n$, $V \cap \text{Orb}(f^n(x); f)$ is nonempty}. It is known that $\omega(x; f)$ is nonempty and strongly invariant, i.e. $f(\omega(x; f)) = \omega(x; f)$. See [BC, p.72] for details. If $\omega(x; f)$ is finite, by [BC, Lemma IV4, p.72], $\omega(x; f)$ is a periodic orbit of some point.

Let $z$ be a periodic point of $f$. The unstable set of $z$ is defined to be the set $W(z; f) = \{x \in X\}$ for any neighborhood $V$ of $z$, $x \in f^k(V)$ for some $k > 0$. A point $y$ is homoclinic for $f$ if there exists a point $z \neq y$ such that $f^n(z) = z$ for some $n > 0$, $y \in W(z; f^n)$ and $f^k(y) = z$ for some $k > 0$. This definition of homoclinic points first appeared in [B]. A point $x \in X$ is a nonwandering point for $f$ if for any open set $U$ containing $x$ there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$.

The following theorem is well known.

Theorem 1.1. Let $f$ be a continuous map from a compact interval $I$ to itself. The following statements are equivalent:

(a) $f$ has positive topological entropy,
(b) $f^n$ is strictly turbulent for some positive integer $n$,
(c) $f$ has a nonwandering homoclinic point, and
(d) for some $c \in I$, $\omega(c; f)$ properly contains a periodic orbit.

We can consider the following theorem, analogous to the corresponding conditions in Theorem 1.1.

**Theorem 1.2.** Let \( f \) be a continuous map from the circle \( S^1 \) to itself. The following statements are equivalent:

(a) \( f \) has positive topological entropy,
(b) \( f^n \) is strictly turbulent for some positive integer \( n \),
(c) \( f \) has a nonwandering homoclinic point, and
(d) for some \( c \in S^1 \), \( \omega(c, f) \) properly contains a periodic orbit.

See [BC] p.229 for strictly turbulent of circle maps. Although the aim of this paper is to prove Theorem 1.2, it is known that (a), (b) and (c) are equivalent (see [BC] p.229 or [BCMN] Theorem B+, p.529) for details) and that conditions (a), (b) and (c) imply condition (d) (see [BC] p.230 for details). Therefore, in this paper, we prove that condition (d) implies condition (a). This is the answer of the question in [BC] p.230).

2. **Definitions**

**Notation 2.1.** Let \( Y \) be a subspace of a space \( X \), and let \( \text{int} \ Y \) and \( \overline{\text{Cl}} \ Y \) denote the interior and the closure of \( Y \) in a space \( X \), respectively.

**Definition 2.2.** Let \( f \) be a continuous map from a space \( X \) to itself. A point \( x \in X \) is a fixed point for \( f \) if \( f(x) = x \). A point \( x \in X \) is a periodic point of period \( n \geq 1 \) for \( f \) if \( f^n(x) = x \). We denote the sets of fixed points, periodic points and nonwandering points for \( f \) by \( \text{F}(f) \), \( \text{P}(f) \) and \( \Omega(f) \), respectively.

**Definition 2.3.** Let us denote a subspace \( \{z||z| = 1\} \) of the complex plane, i.e., the circle, by \( S^1 \). Let \( x \) and \( y \) be two distinct points of \( S^1 \). We denote the closed arc from \( x \) counterclockwise to \( y \) by \( [x, y] \), and we denote \( (x, y) = [x, y] \setminus \{x, y\} \), \( (x, y) = [x, y] \setminus \{x\} \) and \( (x, y) = [x, y] \setminus \{y\} \).

Let \( \pi \) be the canonical projection from the real line onto \( S^1 \) defined by \( \pi(t) = e^{2\pi ti} \), \( \tilde{x} \) and \( \tilde{y} \) two points of the real line such that \( \tilde{y} \in (\tilde{x}, \tilde{x}+1) \), \( \pi(\tilde{x}) = x \) and \( \pi(\tilde{y}) = y \). We see that \( \pi([\tilde{x}, \tilde{y}]: [\tilde{x}, \tilde{y}] \rightarrow [x, y] \) is a homeomorphism. Every continuous map \( f \) from the circle \( S^1 \) to itself has countable many lifts, i.e., continuous maps \( \tilde{f} \) from the real line to itself satisfying \( f \circ \pi = \pi \circ \tilde{f} \).

**Definition 2.4.** Let \( p \) be a fixed point of a continuous map \( f \) from \( S^1 \) to itself. If \( V \) is a neighborhood of \( p \) in \( [p, -p) \) (in \( (-p, p) \), respectively), we say \( V \) is an \( R \)-neighborhood of \( p \) (\( L \)-neighborhood of \( p \), respectively). Let \( S = R, L \). The \( S \)-sided unstable manifold of \( p \) is defined by

\[ W(p, f, S) = \{ x \in S^1 | \text{for any } S \text{-neighborhood } V \text{ of } p, x \in f^k(V) \text{ for some } k > 0 \} \]

Let \( \tilde{p} \) be a fixed point of a continuous map \( \tilde{f} \) from the real line to itself. If \( \tilde{V} \) is a neighborhood of \( \tilde{p} \) in \( [\tilde{p}, \tilde{p} + 1) \) (in \( (\tilde{p} - 1, \tilde{p}) \), respectively), we say \( \tilde{V} \) is an \( R \)-neighborhood of \( \tilde{p} \) (\( L \)-neighborhood of \( \tilde{p} \), respectively). The \( S \)-sided unstable manifold of \( \tilde{p} \) is defined by \( W(\tilde{p}, \tilde{f}, S) = \{ \tilde{x} \} \) for any \( S \)-neighborhood \( \tilde{V} \) of \( \tilde{p}, \tilde{x} \in \tilde{f}^k(\tilde{V}) \) for some \( k > 0 \).

We see that \( W(p, f) = W(p, f, R) \cup W(p, f, L) \) and that \( W(\tilde{p}, \tilde{f}) = W(\tilde{p}, \tilde{f}, R) \cup W(\tilde{p}, \tilde{f}, L) \).
3. Elementary lemmas

By [BC, Proposition II 2, p.48], we have the following lemma.

**Lemma 3.1.** Let $\tilde{f}$ be a continuous map from the real line to itself and $\tilde{z} \in F(\tilde{f})$.

1. If $W(\tilde{z}, \tilde{f}, L) \cap (\tilde{z}, \infty) \neq \emptyset$, then $W(\tilde{z}, \tilde{f}, R) \subset W(\tilde{z}, \tilde{f}, L)$.
2. If $W(\tilde{z}, \tilde{f}, L) \cap (\infty, \tilde{z}) = \emptyset$, then $W(\tilde{z}, \tilde{f}, L) = \{\tilde{z}\}$ or $W(\tilde{z}, \tilde{f}, R)$.
3. If $W(\tilde{z}, \tilde{f}, R) \cap (\infty, \tilde{z}) \neq \emptyset$, then $W(\tilde{z}, \tilde{f}, L) \subset W(\tilde{z}, \tilde{f}, R)$.
4. If $W(\tilde{z}, \tilde{f}, R) \cap (\tilde{z}, \infty) = \emptyset$, then $W(\tilde{z}, \tilde{f}, L) = \{\tilde{z}\}$ or $W(\tilde{z}, \tilde{f}, R)$.

By [BC, Proposition II 1 and 3] and [BCMN, Lemma 1], we have the following lemma.

**Lemma 3.2.** Let $X$ be either a compact interval or the circle or the real line, $f$ a continuous map from $X$ to itself, $z \in F(f)$ and $S = R, L$. Then $W(z, f, S)$ and $W(z, f)$ are connected, $f(W(z, f, S)) = W(z, f, S)$ and $f(W(z, f)) = W(z, f)$.

**Lemma 3.3.** Let $f$ be a continuous map from the circle $S^1$ to itself, $z \in F(f)$ and $S = R, L$. Also, let $\tilde{f}$ be the lift of $f$ with $\tilde{z} \in F(\tilde{f})$ satisfying $\pi(\tilde{z}) = z$. Then $\pi(W(z, f, S)) = W(z, f, S)$ and $\pi(W(z, f)) = W(z, f)$.

**Proof.** We give the proof for the first assertion. Let $\tilde{x} \in W(z, \tilde{f}, S)$ and $U$ a small $S$-neighborhood of $z$. There exists a small $S$-neighborhood $\tilde{V}$ of $\tilde{z}$ with $\pi(\tilde{V}) \subset U$. Since $\tilde{x} \in W(z, \tilde{f}, S)$, we have a positive integer $n$ such that $\tilde{x} \in f^n(\tilde{V})$, thus $\pi(\tilde{x}) \in f^n(\pi(\tilde{V})) \subset f^n(U)$. We conclude that $\pi(\tilde{x}) \in W(z, f, S)$ and that $\pi(W(z, f, S)) \subset W(z, f, S)$.

Let $x \in W(z, f, S) \setminus \{z\}$ and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of small connected $S$-neighborhoods of $z$ with $\bigcap_{n=1}^{\infty} U_n = \{z\}$. Since $\pi(U_n)$ is a small $S$-neighborhood of $z$ for each $n$, there exists a positive integer $k_n$ such that $x \in f^{k_n}(\pi(U_n))$. Since $x \in \pi(f^{k_n}(U_n))$, we see that $\pi^{-1}(x) \cap f^{k_n}(U_n) \neq \emptyset$. Thus there exist $\tilde{x} \in \pi^{-1}(x) \cap (\tilde{z} - 1, \tilde{z} + 1)$ and a sequence $\ell_1, \ell_2, \ldots$ of positive integers such that $\tilde{x} \in f^{\ell_n}(U_{\ell_n})$ for each $n$. This shows that $\tilde{x} \in W(z, \tilde{f}, S)$ and that $\pi(W(z, f, S)) \subset W(z, f, S)$. 

**Corollary 3.4.** Let $f$ be a continuous map from the circle $S^1$ to itself with lift $\tilde{f}$. If $\tilde{y}$ is a homoclinic point for $\tilde{f}$, then $y = \pi(\tilde{y})$ is a homoclinic point for $f$.

**Lemma 3.5.** Let $f$ be a continuous map from the circle $S^1$ to itself with lift $\tilde{f}$ and $z \in F(\tilde{f})$, $S, S' \in \{R, L\}$, and $\tilde{y}, \tilde{z}'$ two points of the real line satisfying that $\tilde{y} \in W(z, \tilde{f}, S)$, $\pi(\tilde{z}') = \pi(\tilde{z}) \neq \pi(\tilde{y})$ and $\tilde{f}^n(\tilde{y}) = \tilde{z}'$ for some $n \geq 1$. If $\tilde{f}^n(\tilde{U})$ contains an $S$-neighborhood of $\tilde{z}'$ for each $S'$-neighborhood $\tilde{U}$ of $\tilde{y}$, then $y = \pi(\tilde{y})$ is a nonwandering homoclinic point for $f$.

**Proof.** We notice that $z = \pi(\tilde{z}) \in F(f)$ and that $f^n(y) = \pi(\tilde{f}^n(\tilde{y})) = \pi(\tilde{z}') = z$. It follows from Lemma 3.3 that $y \in W(z, f, S) = W(z, f, S) \subset W(z, f)$. It suffices to show that $y \in \Omega(f)$. Let $U$ be a small $S'$-neighborhood of $y$ and $\tilde{U}$ an $S'$-neighborhood of $\tilde{y}$ with $\pi(\tilde{U}) = U$. From the assumption, $\tilde{f}^n(\tilde{U})$ contains some small $S$-neighborhood $\tilde{V}$ of $\tilde{z}'$. Since $\pi(\tilde{z}') = \pi(\tilde{z})$, there exists the integer $\delta$ such that $\tilde{z}' = \tilde{z} + \delta$. Set $\tilde{V} - \delta = \{\tilde{x} - \delta | \tilde{x} \in \tilde{V}\}$. We notice that $\tilde{V} - \delta$ is an $S$-neighborhood of $\tilde{z}$ and that $\pi(\tilde{V} - \delta) = \pi(\tilde{V})$ is an $S$-neighborhood of $z$. Also, since $\tilde{y} \in W(z, \tilde{f}, S)$, we have a positive integer $m$ such that $\tilde{y} \in \tilde{f}^m(\tilde{V} - \delta)$. This
shows that \( y \in \pi(\tilde{f}^n(\tilde{V} - \delta)) = f^m(\pi(\tilde{V} - \delta)) = f^m(\pi(\tilde{V})) \subset f^m(\pi(\tilde{f}^n(\tilde{U}))) = f^{m+n}(\pi(\tilde{U})) = f^{m+n}(U) \), thus, \( y \in \Omega(f) \).

\[ \square \]

4. A proof of Theorem 1.2

**Lemma 4.1.** Let \( f \) be a continuous map from the circle \( S^1 \) to itself. If there exists a point \( c \in S^1 \) such that \( \omega(c, f) \) is infinite containing some fixed point, then \( f \) has positive topological entropy.

**Proof.** Set \( c_m = f^m(c) \) for each \( m \). Choose \( z \in \omega(c, f) \cap F(f) \). We have an increasing sequence \( \{n_k\} \) of positive integers, \( S = R, L \) and an \( S \)-neighborhood \( U_z \) of \( z \) such that \( c_{n_k} \in U_z \setminus \{z\} \) for all \( k \) and that \( \lim_{k \to \infty} c_{n_k} = z \). Since \( \omega(c, f) \) is infinite, we see that \( W(z, f, S) \neq \emptyset \).

We show that \( \text{Orb}(c, f) \cap W(z, f, S) = \emptyset \). We suppose that \( \text{Orb}(c, f) \cap W(z, f, S) = \emptyset \). Since \( W(z, f, S) \) is connected by Lemma 3.2, there exist a point \( x \) of \( \omega(c, f) \) and a compact interval \( A \) such that \( \omega(c, f) \subset A \), \( A \setminus \text{int} A = \{z, x\} \) and \( W(z, f, S) \subset S^1 \setminus \text{int} A \). Since \( \omega(c, f) \) is finite, by the definition of \( W(z, f, S) \), there exists a point \( x' \in \text{CIW}(z, f, S) \cap \text{int} A \). This is a contradiction.

Since \( \text{Orb}(c, f) \cap W(z, f, S) \neq \emptyset \), Lemma 3.2 implies that \( \omega(c, f) \subset \text{CIW}(z, f, S) \). We suppose that \( \text{CIW}(z, f, S) \neq S^1 \). By Lemma 3.2, we see that \( \text{CIW}(z, f, S) \) is a compact interval. Since \( \text{Orb}(c, f) \cap W(z, f, S) \neq \emptyset \), we have \( c_m \in W(z, f, S) \) for some \( m \). Since \( \omega(c, f) = \omega(c, f) \) and \( f(\text{CIW}(z, f, S)) = \text{CIW}(z, f, S) \), we have \( \omega(c, f) = \omega(c_m, f| \text{CIW}(z, f, S)) \), where \( f| \text{CIW}(z, f, S) : \text{CIW}(z, f, S) \to \text{CIW}(z, f, S) \) is the restriction of \( f \). From Theorem 1.1, \( f| \text{CIW}(z, f, S) \) has positive topological entropy. We conclude from [BC] Proposition VIII 5, p.193 that \( f \) has positive topological entropy. We may assume that \( \text{CIW}(z, f, S) = S^1 \).

Since \((a)\) and \((c)\) in Theorem 1.2 are equivalent, we are going to show that \( f \) has a nonwandering homoclinic point.

Let \( \tilde{f} \) be the lift of \( f \) with \( \tilde{z} \in F(\tilde{f}) \) satisfying \( \pi(\tilde{z}) = z \). We suppose that \( \text{CIW}(\tilde{z}, \tilde{f}, S) \) is compact, i.e., bounded. Since \( \pi(\text{CIW}(\tilde{z}, \tilde{f}, S)) = \text{CIW}(z, f, S) = S^1 \) by Lemma 3.3, there exists a point \( \tilde{c} \in \text{CIW}(\tilde{z}, \tilde{f}, S) \) satisfying \( \pi(\tilde{c}) = c \). Set \( \tilde{c}_m = f^m(\tilde{c}) \) for each \( m \). Let \( \tilde{f} | \text{CIW}(\tilde{z}, \tilde{f}, S) : \text{CIW}(\tilde{z}, \tilde{f}, S) \to \text{CIW}(\tilde{z}, \tilde{f}, S) \) be the restriction of \( \tilde{f} \). Since \( \pi(\tilde{c}_m) = c_m \) for each \( m \), we have \( \pi(\omega(\tilde{c}, \tilde{f}| \text{CIW}(\tilde{z}, \tilde{f}, S))) \subset \omega(c, f) \). Let \( x \in \omega(c, f) \). We have a sequence \( n_1, n_2, \ldots \) of positive integers such that \( \lim_{k \to \infty} c_{n_k} = x \). Since \( \text{CIW}(\tilde{z}, \tilde{f}, S) \) is compact, there exist a subsequence \( n_{i_1}, n_{i_2}, \ldots \) and \( \tilde{x} \in \text{CIW}(\tilde{z}, \tilde{f}, S) \) such that \( \lim_{k \to \infty} \tilde{c}_{n_{i_k}} = \tilde{x} \). Since \( \tilde{x} \in \omega(\tilde{c}, \tilde{f}| \text{CIW}(\tilde{z}, \tilde{f}, S)) \) and \( \pi(\tilde{x}) = x \), we see that \( \pi(\omega(\tilde{c}, \tilde{f}| \text{CIW}(\tilde{z}, \tilde{f}, S))) \subset \omega(c, f) \) and conclude that \( \pi(\omega(\tilde{c}, \tilde{f}| \text{CIW}(\tilde{z}, \tilde{f}, S))) = \omega(c, f) \). This shows that \( \omega(\tilde{c}, \tilde{f}| \text{CIW}(\tilde{z}, \tilde{f}, S)) \) properly contains a periodic orbit. By Theorem 1.1, \( \tilde{f} | \text{CIW}(\tilde{z}, \tilde{f}, S) \) has a homoclinic point \( \tilde{y} \in \Omega(\tilde{f}) | \text{CIW}(\tilde{z}, \tilde{f}, S) \). Since \( \pi(\Omega(f| \text{CIW}(\tilde{z}, \tilde{f}, S))) \subset \pi(\Omega(f)) \subset \Omega(f) \), by Corollary 3.4, \( f \) has a homoclinic point \( \pi(\tilde{y}) \in \Omega(f) \). We may assume that \( \text{CIW}(\tilde{z}, \tilde{f}, S) \) is unbounded.

We suppose that \( W(\tilde{z}, \tilde{f}, S) \) contains some small open connected \( S \)-neighborhood \( \tilde{U} \) of \( \tilde{z} \). Since \( \text{CIW}(\tilde{z}, \tilde{f}, S) \) is unbounded containing \( \tilde{z} \), we have \( \delta = 1, -1 \) such that \( \tilde{z} + \delta \in W(\tilde{z}, \tilde{f}, S) \). We suppose that \( S = L \). Since \( \tilde{z} + \delta \in W(\tilde{z}, \tilde{f}, L) \), there exists a positive integer \( n \) such that \( \tilde{z} + \delta \in \tilde{f}^n(\tilde{U}) \). Set \( \tilde{y} = \max\{\tilde{y} \in \tilde{U} | \tilde{f}^n(\tilde{y}) = \tilde{z} + \delta\} \).
If $\delta = 1$, we see that $\tilde{f}^n(\tilde{V})$ is an $L$-neighborhood of $\tilde{z} + 1$ for each small $R$-neighborhood $\tilde{V}$ of $\tilde{y}$. It follows from Lemma 3.5 that $y = \pi(\tilde{y})$ is a nonwandering homoclinic point.

Next we suppose that $\delta = -1$ and $\tilde{z} + 1 \not\in W(\tilde{z}, \tilde{f}, L)$. Since $(-\infty, \tilde{z} - 1) \subset W(\tilde{z}, \tilde{f}, L)$, there exist a point $\tilde{x} \in (\tilde{z} - 1, \tilde{z})$ and a positive integer $m$ such that $\tilde{f}^m(\tilde{x}) < \tilde{z} - 1$. Thus, we have $\tilde{y}' = \min(\tilde{x}, \tilde{z}) \cap \tilde{f}^{-m}(\tilde{z} - 1)$. By the definition of $\tilde{y}'$, we see that $\tilde{f}^m(\tilde{V}')$ is an $L$-neighborhood of $\tilde{z} - 1$ for each small $L$-neighborhood $\tilde{V}'$ of $\tilde{y}'$. It follows from Lemma 3.5 that $y' = \pi(\tilde{y}')$ is a nonwandering homoclinic point.

We can prove $S = R$ by an argument similar to that for $S = L$. Thus, we may assume that $W(\tilde{z}, \tilde{f}, S)$ contains no $S$-neighborhood of $\tilde{z}$.

Without loss of generality, we may assume that $S = L$. We see from Lemma 3.1(2) that $W(\tilde{z}, \tilde{f}, R) = W(\tilde{z}, \tilde{f}, L) = [\tilde{z}, \infty)$. Let $\tilde{U}$ be a small connected $R$-neighborhood of $\tilde{z}$. Since $\tilde{z} + 1 \in W(\tilde{z}, \tilde{f}, R)$, there exists a positive integer $n$ such that $\tilde{z} + 1 \in \tilde{f}^n(\tilde{U})$. Set $\tilde{y} = \min\{\tilde{y}' \in \tilde{U} | \tilde{f}^n(\tilde{y}') = \tilde{z} + 1\}$. As above, we can show that $y = \pi(\tilde{y})$ is a nonwandering homoclinic point. \hfill \qed

**Theorem 4.2.** Condition (d) in Theorem 1.2 implies condition (a) in Theorem 1.2.

**Proof.** Let $f$ be a continuous map from the circle $S^1$ to itself. Let $c \in S^1$ such that $\omega(c, f)$ properly contains a periodic point of period $n$. By [BC, Lemma IV 4, p.72], $\omega(c, f)$ is infinite, thus, $\omega(f^n(c), f^n)$ is also infinite for all $j$ with $0 \leq j < n$ by [BC, p.70]. Since at least one of these $\omega$-limit sets contains a fixed point of $f^n$ by [BC, p.70], we see from Lemma 4.1 that $f^n$ has positive topological entropy. We conclude from [BC] Proposition VIII 2, p.191 that $f$ has positive topological entropy. \hfill \qed

**References**


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