

## A SIMPLE PROOF OF THE MODULAR IDENTITY FOR THETA FUNCTIONS

WIM COUWENBERG

(Communicated by Juha M. Heinonen)

*To A.C.M. van Rooij on the occasion of his 65th birthday*

ABSTRACT. The modular identity arises in the theory of theta functions in one complex variable. It states a relation between theta functions for parameters  $\tau$  and  $-1/\tau$  situated in the complex upper half-plane. A standard proof uses Poisson summation and hence builds on results from Fourier theory. This paper presents a simple proof using only a uniqueness property and the heat equation.

### 1. THE $\theta$ FUNCTION

Let  $\mathbb{H} \subset \mathbb{C}$  denote the upper half-plane of all complex numbers with a positive imaginary part. The following series converges locally uniformly in  $z \in \mathbb{C}$  and  $\tau \in \mathbb{H}$  and hence defines a holomorphic function on  $\mathbb{C} \times \mathbb{H}$ :

$$\theta(z, \tau) = \sum_{k \in \mathbb{Z}} e^{2\pi i k z + \pi i k^2 \tau}.$$

This function is often called the  $\theta_3$  function of Jacobi (some texts use  $q = e^{\pi i \tau}$  or replace  $\pi z$  by  $z$ ). It satisfies the shift relations in  $z$ ,

$$(1.1) \quad \theta(z + 1, \tau) = \theta(z, \tau)$$

and

$$(1.2) \quad \theta(z + \tau, \tau) = e^{-2\pi i z - \pi i \tau} \theta(z, \tau),$$

that can easily be verified from its definition. The following *heat equation* is also apparent from the definition of  $\theta$ :

$$(1.3) \quad \frac{d^2 \theta}{dz^2} = 4\pi i \frac{d\theta}{d\tau}.$$

Let  $\Lambda(\tau) = \mathbb{Z} + \mathbb{Z}\tau$  be the lattice spanned by 1 and  $\tau$ . For fixed parameter  $\tau$  the function  $\theta$  in  $z$  is the only entire function satisfying (1.1) and (1.2) up to complex multiples. This follows from the following theorem:

**Theorem 1.1.** *If  $f(z)$  is an entire function on  $\mathbb{C}$  satisfying the shift relations (1.1) and (1.2), then either  $f$  vanishes identically or all its roots equal  $(\tau + 1)/2$  modulo the lattice  $\Lambda(\tau)$ .*

---

Received by the editors July 6, 2001 and, in revised form, May 22, 2002.  
2000 *Mathematics Subject Classification.* Primary 14K25.

Suppose  $f$  does not vanish identically. The shift relations for  $f$  imply

$$\frac{f'(z+1)}{f(z+1)} = \frac{f'(z)}{f(z)} \quad \text{and} \quad \frac{f'(z+\tau)}{f(z+\tau)} = \frac{f'(z)}{f(z)} - 2\pi i.$$

For  $b \in \mathbb{C}$  define a closed fundamental domain  $P \subset \mathbb{C}$  by

$$P = \{b + x + y\tau \mid x, y \in [0, 1]\}.$$

The number of roots of  $f$  on  $P$  and the sum of its roots on  $P$  can be computed by the integrals

$$\frac{1}{2\pi i} \oint_{\partial P} \frac{f'(z)}{f(z)} dz$$

and

$$\frac{1}{2\pi i} \oint_{\partial P} \frac{zf'(z)}{f(z)} dz$$

respectively. By varying the number  $b$  we may assume that  $f$  has no roots on  $\partial P$ , so both integrals are well defined. Using the shift relations for  $f$  shows that the first integral evaluates to 1, so  $f$  has only one root on  $P$ . The second integral evaluates to a value equal to  $(\tau + 1)/2$  modulo the lattice  $\Lambda(\tau)$ . This proves the theorem.

**Corollary 1.2.** *If  $f$  is as in Theorem 1.1, then  $f(z) = c \cdot \theta(z, \tau)$  for some constant  $c \in \mathbb{C}$ .*

By Theorem 1.1 we find that  $\theta(0, \tau)f(z) - f(0)\theta(z, \tau)$  must vanish identically as it vanishes at  $z = 0$  as well as at  $(\tau + 1)/2$ . □

## 2. THE MODULAR IDENTITY

We are already in a position to prove the modular identity (2.3) for  $\theta$ . A very accessible treatment of this identity using Poisson summation can be found in [1]. For the proof given below, the heat equation suffices. In [2] (in particular, pages 69–70) a similar approach is taken, based on Fourier expansion in the parameter  $z$  rather than contour integration as in the previous section. Define an entire function  $\vartheta$  by

$$\vartheta(z) = e^{\pi i \tau z^2} \theta(\tau z, \tau).$$

Then  $\vartheta(z + 1) = \vartheta(z)$  and

$$\vartheta(z - 1/\tau) = e^{-2\pi i z + \pi i/\tau} \vartheta(z).$$

Hence  $\vartheta(z) = c(\tau) \cdot \theta(z, -1/\tau)$  for some function  $c$  on the upper half-plane by Corollary 1.2. Substituting  $\tau = i$  and  $z = 0$  shows that  $c(i) = 1$ . The heat equation for  $\theta$  will produce a simple differential equation for  $c$ . Elementary computations show that

$$(2.1) \quad \frac{d^2 \vartheta}{dz^2}(0) = 2\pi i \tau \theta(0, \tau) + \tau^2 \frac{d^2 \theta}{dz^2}(0, \tau) = c(\tau) \frac{d^2 \theta}{dz^2}(0, -1/\tau),$$

$$(2.2) \quad \frac{d\vartheta}{d\tau}(0) = \frac{d\theta}{d\tau}(0, \tau) = c'(\tau)\theta(0, -1/\tau) + c(\tau)\tau^{-2} \frac{d\theta}{d\tau}(0, -1/\tau).$$

Using the heat equation (1.3) on (2.1) yields

$$\frac{1}{2}\tau^{-1}\theta(0, \tau) + \frac{d\theta}{d\tau}(0, \tau) = c(\tau)\tau^{-2} \frac{d\theta}{d\tau}(0, -1/\tau)$$

and combining this with (2.2) leads to

$$\theta(0, \tau) = -2\tau c'(\tau)\theta(0, -1/\tau).$$

However, substituting  $z = 0$  in  $\vartheta(z)$  gives

$$\theta(0, \tau) = \vartheta(0) = c(\tau)\theta(0, -1/\tau)$$

and since  $\theta$  does not vanish at  $z = 0$ , we find that

$$-2\tau c'(\tau) = c(\tau).$$

Together with  $c(i) = 1$  we finally find

$$c(\tau) = \frac{1}{\sqrt{-i\tau}}$$

and thus the modular identity for the  $\theta$  function:

$$(2.3) \quad \theta(z, -1/\tau) = \sqrt{-i\tau} e^{\pi i \tau z^2} \theta(\tau z, \tau).$$

#### REFERENCES

1. R. Bellman, *A brief introduction to theta functions*, Holt, Rinehart and Winston, New York, 1961. MR **23**:A2556
2. L. Ehrenpreis, *Fourier analysis, partial differential equations and automorphic functions*, Theta Functions Bowdoin 1987 (L. Ehrenpreis and R. C. Gunning, eds.), Proceedings of Symposia in Pure Mathematics, vol. 49–Part 2, Amer. Math. Soc., 1989, pp. 45–100. MR **91k**:11036

UNIVERSITY NIJMEGEN, TOERNOOIVELD-1, 6525 ED NIJMEGEN, THE NETHERLANDS  
*E-mail address*: w.couwenberg@chello.nl