ON A SUBSPACE PERTURBATION PROBLEM

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Abstract. We discuss the problem of perturbation of spectral subspaces for linear self-adjoint operators on a separable Hilbert space. Let $A$ and $V$ be bounded self-adjoint operators. Assume that the spectrum of $A$ consists of two disjoint parts $\sigma$ and $\Sigma$ such that $d = \text{dist}(\sigma, \Sigma) > 0$. We show that the norm of the difference of the spectral projections $E_A(\sigma)$ and $E_{A+V}(\{\lambda \mid \text{dist}(\lambda, \sigma) < d/2\})$ for $A$ and $A + V$ is less than one whenever either (i) $\|V\| < \frac{d^2}{4 + d}$ or (ii) $\|V\| < \frac{1}{2}d$ and certain assumptions on the mutual disposition of the sets $\sigma$ and $\Sigma$ are satisfied.

1. Introduction

It is well known (see, e.g., [10]) that if $A$ and $V$ are bounded self-adjoint operators on a separable Hilbert space $H$, then (the perturbation) $V$ does not close gaps of length greater than $2\|V\|$ in the spectrum of $A$. More precisely, if $(a, b)$ is a finite interval and $(a, b) \subset \rho(A)$, the resolvent set of $A$, then

$(a + \|V\|, b - \|V\|) \subset \rho(A + sV)$ for all $s \in [-1, 1]$

whenever $2\|V\| < b - a$. Hence, under the assumption that $A$ has an isolated part $\sigma$ of the spectrum separated from its remainder by gaps of length greater than or equal to $d > 0$, the spectrum of the operators $A + sV$, $s \in [-1, 1]$, will also have separated components, provided that the condition

$$\|V\| < \frac{d}{2}$$

holds.

Our main concern is to study the variation of the corresponding spectral subspace associated with the isolated part $\sigma$ of the spectrum of $A$ under perturbations satisfying (1.1).

For notational setup we assume the following hypothesis.

Hypothesis 1. Assume that $A$ and $V$ are bounded self-adjoint operators on a separable Hilbert space $H$. Suppose that the spectrum of $A$ has a part $\sigma$ separated from the remainder of the spectrum $\Sigma$ in the sense that

$$\text{spec}(A) = \sigma \cup \Sigma \quad \text{and} \quad \text{dist}(\sigma, \Sigma) = d > 0.$$
Introduce the orthogonal projections \( P = E_A(\sigma) \) and \( Q = E_{A+V}(U_{d/2}(\sigma)) \), where \( U_\varepsilon(\sigma), \varepsilon > 0 \), is the open \( \varepsilon \)-neighborhood of the set \( \sigma \). Here \( E_A(\Delta) \) and \( E_{A+V}(\Delta) \) denote the spectral projections for operators \( A \) and \( A+V \), respectively, corresponding to a Borel set \( \Delta \subset \mathbb{R} \).

In this note we address the following question: Assuming Hypothesis 1, does condition (1.1) imply \( \| P - Q \| < 1? \)

We give a partially affirmative answer to this question. The precise statement reads as follows.

**Theorem 1.** Assume Hypothesis 1 and suppose that either (i) \( \| V \| < \frac{2}{2 + \pi} d \) or (ii) \( \| V \| < \frac{1}{2} d \) and

\[
\text{conv.hull}(\sigma) \cap \Sigma = \emptyset \quad \text{or} \quad \text{conv.hull}(\Sigma) \cap \sigma = \emptyset.
\]

Then \( \| P - Q \| < 1. \)

Our strategy of the proof of Theorem 1 does not allow us to relax the condition

\[
\| V \| < \frac{2}{2 + \pi} d
\]

and just assume the natural condition (1.1) with no additional hypotheses. It is an open problem whether Hypothesis 1 alone and the bounds

\[
\frac{2}{2 + \pi} \leq \frac{\| V \|}{d} < \frac{1}{2}
\]

on the perturbation \( V \) imply \( \| P - Q \| < 1. \)

For compact perturbations \( V \) satisfying inequality (1.1) we can however state that the pair \((P, Q)\) of the orthogonal projections is a Fredholm pair with zero index. Recall that the pair \((P, Q)\) of orthogonal projections is called Fredholm if the operator \( PQ \) viewed as a map from \( \text{Ran} P \) to \( \text{Ran} Q \) is a Fredholm operator. The index of this operator is called the index of the pair \((P, Q)\).

**Theorem 2.** Assume Hypothesis 1 and suppose that \( V \) is a compact operator satisfying (1.1). Then the pair \((P, Q)\) is Fredholm with zero index. In particular, the subspaces \( \text{Ker}(PQ^\perp - I) \) and \( \text{Ker}(P^\perp Q - I) \) are finite-dimensional and

\[
\dim \text{Ker}(PQ^\perp - I) = \dim \text{Ker}(P^\perp Q - I).
\]

In the “overcritical” case \( \| V \| > d/2 \), the perturbed operator \( A + V \) may not have separated parts of the spectrum at all. In this case we give an example where the spectral measure of the perturbed operator \( A + V \) is “concentrated” on the unit sphere in the space of bounded operators \( \mathcal{B}(\mathcal{H}) \) centered at the point \( P = E_A(\sigma) \), with the norm of the perturbation being arbitrarily close to \( d/2 \). That is, given \( d > 0 \), for any \( \varepsilon > 0 \) one can find a self-adjoint operator \( A \) satisfying Hypothesis 1 and a self-adjoint perturbation \( V \) with \( \| V \| = d/2 + \varepsilon \) such that

\[
\| E_A(\sigma) - E_{A+V}(\Delta) \| = 1
\]

for any Borel set \( \Delta \subset \mathbb{R} \).
2. Proof of Theorem 1

Our proof of Theorem 1 is based on the following sharp result (see [9] and references cited therein) taken from geometric perturbation theory initiated by C. Davis [6] and developed further in [4], [5], [7], [8], [10].

Proposition 2.1. Let $A$ and $B$ be bounded self-adjoint operators and $\delta$ and $\Delta$ two Borel sets on the real axis $\mathbb{R}$. Then

$$\text{dist}(\delta, \Delta) \|e_A(\delta) e_B(\Delta)\| \leq \frac{\pi}{2} \|A - B\|.$$ 

If, in addition, the convex hull of the set $\delta$ does not intersect the set $\Delta$, or the convex hull of the set $\Delta$ does not intersect the set $\delta$, then one has the stronger result

$$\text{dist}(\delta, \Delta) \|e_A(\delta) e_B(\Delta)\| \leq \|A - B\|.$$ 

We split the proof of Theorem 1 into the following two lemmas.

Lemma 2.2. Assume Hypothesis 1. Assume, in addition, that (1.3) holds. Then

$$\|P - Q\| < 1.$$ 

Proof. Clearly $\text{spec}(A + V) \subset U_{\|V\|}(\sigma \cup \Sigma)$, where the bar denotes the (usual) closure in $\mathbb{R}$, and then

$$Q^\perp = E_{A + V}(U_{\|V\|}(\Sigma)).$$

By the first claim of Proposition 2.1

(2.1) $$\|PQ^\perp\| \leq \frac{\pi}{2} \|V\| \text{dist}(\sigma, U_{\|V\|}(\Sigma)),$$

The distance between the set $\sigma$ and the $\|V\|$-neighborhood of the set $\Sigma$ can be estimated from below as follows:

$$\text{dist}(\sigma, U_{\|V\|}(\Sigma)) \geq d - \|V\| > 0.$$ 

Then (2.1) implies the inequality

$$\|PQ^\perp\| \leq \frac{\pi}{2} \frac{\|V\|}{d - \|V\|}.$$ 

Hence, from inequality (1.3) it follows that

(2.2) $$\|PQ^\perp\| \leq \frac{\pi}{2} \frac{\|V\|}{d - \|V\|} < 1.$$ 

Interchanging the roles of $\sigma$ and $\Sigma$ one obtains the analogous inequality

(2.3) $$\|P^\perp Q\| < 1.$$ 

Since

(2.4) $$\|P - Q\| = \max\{\|PQ^\perp\|, \|P^\perp Q\|\}$$

(see, e.g., [2, Ch. III, Section 39]), inequalities (2.2) and (2.3) prove the assertion.

Under additional assumptions on mutual disposition of the parts $\sigma$ and $\Sigma$ of the spectrum of $A$ one can relax the condition (1.3) on the norm of perturbation and replace it by the natural condition (1.1).
Lemma 2.3. Assume Hypothesis 1 and suppose that condition (1.1) holds.

(i) If either $\sigma \cap \text{conv.hull}(\Sigma) = \emptyset$ or $\text{conv.hull}(\sigma) \cap \Sigma = \emptyset$, then

$$\|P - Q\| < 1.$$  \hfill (2.5)

(ii) If in addition the sets $\sigma$ and $\Sigma$ are subordinated, that is,

$$\text{conv.hull}(\sigma) \cap \text{conv.hull}(\Sigma) = \emptyset,$$

then the following sharp estimate holds:

$$\|P - Q\| < \frac{\sqrt{2}}{2}.$$  \hfill (2.6)

Proof. (i) The proof follows that of Lemma 2.2. Applying the second assertion of Proposition 2.1 instead of inequality (2.1), one derives the estimates

$$\|PQ^\perp\| \leq \frac{\|V\|}{\text{dist}(\sigma, U_{\|V\|}(\Sigma))} \leq \frac{\|V\|}{d - \|V\|} < 1,$$  \hfill (2.7)

under hypothesis (1.4), and then the inequality $\|P^\perp Q\| < 1$, proving assertion (2.5) using (2.4).

(ii) First assume that $V$ is off-diagonal, that is,

$$E_A(\sigma)V E_A(\sigma) = E_A(\sigma)^\perp V E_A(\sigma)^\perp = 0.$$  

Then the inequality $\|P - Q\| < \frac{\sqrt{2}}{2}$ follows from the tan $2\Theta$-Theorem proven first by C. Davis (see, e.g., [8])

$$\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \frac{2\|V\|}{d} \right) < \frac{\sqrt{2}}{2}.$$  

A related result can be found in [1].

The general case can be reduced to the off-diagonal one by the following trick. Assume that $V$ is not necessarily off-diagonal. Decomposing the perturbation $V$ into the diagonal $V_{\text{diag}}$ and off-diagonal $V_{\text{off}}$ parts with respect to the orthogonal decomposition $H = \text{Ran} E_A(\sigma) \oplus \text{Ran} E_A(\sigma)^\perp$ associated with the range of the projection $E_A(\sigma)$

$$V = V_{\text{diag}} + V_{\text{off}},$$

one concludes that

$$E_{A + V_{\text{diag}}}(U_{d/2}(\sigma)) = E_A(\sigma).$$

Moreover, the distance between the spectrum of the part of $A + V_{\text{diag}}$ associated with the invariant subspace $\text{Ran} E_{A + V_{\text{diag}}}(U_{d/2}(\sigma))$ and the remainder of the spectrum of $A + V_{\text{diag}}$ does not exceed $d - 2\|V_{\text{diag}}\| > 0$. Using the tan $2\Theta$-Theorem then yields

$$\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \frac{2\|V_{\text{off}}\|}{d - 2\|V_{\text{diag}}\|} \right) \leq \sin \left( \frac{1}{2} \arctan \frac{2\|V\|}{d - 2\|V\|} \right) < \frac{\sqrt{2}}{2},$$

completing the proof. \qed
The sharpness of estimate \(2.6\) is shown by the following example.

**Example 2.4.** Let \( \mathcal{H} = \mathbb{C}^2 \). For an arbitrary \( \varepsilon \in (0, 3/4) \) consider the \( 2 \times 2 \) matrices

\[
A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1/2 - \varepsilon & \sqrt{\varepsilon}/2 \\ \sqrt{\varepsilon}/2 & -1/2 + \varepsilon \end{pmatrix}.
\]

Let \( \sigma = \{0\} \) and \( \Sigma = \{1\} \). Obviously, \( \text{dist}(\sigma, \Sigma) = 1 \). Since

\[
\|V\| = \frac{1}{2}\sqrt{1 - 3\varepsilon + 4\varepsilon^2} < \frac{1}{2},
\]

the perturbation \( V \) satisfies the hypotheses of Lemma 2.3. Simple calculations yield

\[
Q = E_{A+V}(U_{1/2}(\sigma)) = E_{A+V}((-1/2, 1/2)) = \frac{1}{1 + (2\sqrt{\varepsilon} + \sqrt{1 + 4\varepsilon})^2} \begin{pmatrix} (2\sqrt{\varepsilon} + \sqrt{1 + 4\varepsilon})^2 & -2\sqrt{\varepsilon} - \sqrt{1 + 4\varepsilon} \\ -2\sqrt{\varepsilon} - \sqrt{1 + 4\varepsilon} & 1 \end{pmatrix},
\]

and hence,

\[
\|P - Q\| = \left[1 + (2\sqrt{\varepsilon} + \sqrt{1 + 4\varepsilon})^2\right]^{-1/2} < \frac{\sqrt{2}}{2}.
\]

Taking \( \varepsilon \) sufficiently small, the norm \( \|P - Q\| \) can be made arbitrarily close to \( \sqrt{2}/2 \).

### 3. Proof of Theorem 2

**Lemma 3.1.** Assume Hypothesis 7 and suppose, in addition, that \( V \) is a compact operator satisfying condition \(1.1\). Then there is a unitary \( W \) such that \( Q = WPW^* \) and \( W - I \) is compact.

**Proof.** Fix \( \varepsilon > 0 \) such that \( (1 + \varepsilon)\|V\| < d/2 \) and introduce the family of spectral projections

\[
\mathcal{P}(s) = E_{A+\varepsilon V}(U_{d/2}(\sigma)), \quad s \in (-\varepsilon, 1 + \varepsilon).
\]

Clearly, \( \mathcal{P}(0) = P \) and \( \mathcal{P}(1) = Q \). From the analytical perturbation theory (see [10]) one concludes that the operator-valued function \( \mathcal{P}(s) \) is real-analytic on \((-\varepsilon, 1 + \varepsilon)\). Moreover (see [10], Section II.4.2)),

\[
\mathcal{P}(s) = X(s)\mathcal{P}(0)X(s)^*, \quad s \in [0, 1],
\]

where \( X(s) \) is the unique unitary solution to the initial value problem

\[
X'(s) = H(s)X(s), \quad s \in [0, 1],
\]

\[
X(0) = I,
\]

with \( H(s) = \mathcal{P}'(s)\mathcal{P}(s) - \mathcal{P}(s)\mathcal{P}'(s) \).

Let \( \Gamma \) be a Jordan counterclockwise oriented contour encircling \( U_{d/2}(\sigma) \) in a way such that no point of \( U_{d/2}(\Sigma) \) lies within \( \Gamma \). Then

\[
\mathcal{P}(s) = -\frac{1}{2\pi i} \int_{\Gamma} (A + sV - z)^{-1}dz, \quad s \in [0, 1],
\]

and hence,

\[
\mathcal{P}'(s) = \frac{1}{2\pi i} \int_{\Gamma} (A + sV - z)^{-1}V(A + sV - z)^{-1}dz, \quad s \in [0, 1].
\]

By the hypothesis \( V \) is compact, and hence, \( \mathcal{P}'(s), s \in [0, 1], \) is also compact, which implies that \( H(s) \) is a compact operator for \( s \in [0, 1] \).
Applying the successive approximation method
\[ X_n(s) = I + \int_0^s H(t)X_{n-1}(t)\,dt, \quad X_0(s) = I, \]
yields that \( X_n(s) \) converges to \( X(s) \), \( s \in [0, 1] \), in the norm topology and \( X_n(s) - I \) is compact for all \( n \in \mathbb{N} \). Thus, \( X(s) - I \) is a compact operator for all \( s \in [0, 1] \). Taking \( W = X(1) \) yields \( Q = WPW^* \), completing the proof. \( \square \)

Lemma 3.1 implies that the operator \( PWP \) viewed as a map from \( \text{Ran} \, P \) to \( \text{Ran} \, P \) is Fredholm with zero index. By Theorem 5.2 of [3] it follows that the pair \((P, Q)\) is Fredholm and \( \text{index}(P, Q) = \text{index}(PW^*|\text{Ran} \, P) = 0 \), proving Theorem 2.

### 4. Overcritical perturbations

If the perturbation \( V \) closes a gap between the separated parts \( \sigma \) and \( \Sigma \) of the spectrum of the unperturbed operator \( A \), then, necessarily, we are dealing with the case \( \|V\| \geq d/2 \). In this case one encounters a new phenomenon: It may happen that any invariant subspace of the operator \( A + V \) contains a nontrivial element orthogonal to \( \text{Ran} \, P = \text{Ran} \, E_A(\sigma) \).

To illustrate this phenomenon we need the following abstract result.

**Lemma 4.1.** Let \( A \) and \( V \) be bounded self-adjoint operators and \( \sigma \neq \emptyset \) be a finite set consisting of isolated eigenvalues of \( A \) of finite multiplicity. Assume that the spectrum of the operator \( A + V \) has no pure point component. Then for the orthogonal projection \( Q \) onto an arbitrary invariant subspace of the operator \( A + V \), the subspace \( \ker(P^+Q - I) \), where \( P = E_A(\sigma) \), is infinite-dimensional. In particular,

\[
\|P - Q\| = 1.
\]

**Proof.** Since \( A + V \) has no eigenvalues, \( \text{Ran} \, Q \) is an infinite-dimensional subspace. By hypothesis, \( \text{Ran} \, P \) is a finite-dimensional subspace. Thus, there exists an orthonormal system \( \{f_n\}_{n \in \mathbb{N}} \) in \( \text{Ran} \, Q \) such that \( f_n \) is orthogonal to \( \text{Ran} \, P \) for any \( n \in \mathbb{N} \) and hence \( P^+Qf_n = f_n \), \( n \in \mathbb{N} \), proving \( \dim(\ker(P^+Q - I)) = \infty \). Now equality \((4.1)\) follows from representation \((2.4)\). \( \square \)

The next lemma shows that an isolated *eigenvalue* of the unperturbed operator \( A \) separated from the remainder of the spectrum of \( A \) by a gap of length 1 may “dissolve” in the essential spectrum of the perturbed operator \( A + V \) turning into a “resonance”, with the norm of the perturbation being larger but arbitrarily close to 1/2.

**Lemma 4.2.** Let \( \varepsilon > 0 \). Let \( A \) and \( V \) be \( 2 \times 2 \) operator matrices in \( \mathcal{H} = L^2(0, 1) \oplus \mathbb{C} \),

\[
A = \begin{pmatrix} M & 0 \\ 0 & -I_C \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -\left(\frac{1}{2} + \varepsilon\right)I_{L^2(0, 1)} & \varepsilon v \\ \sqrt{\varepsilon}v^* & \left(\frac{1}{2} + \varepsilon\right)I_C \end{pmatrix}
\]

with respect to the decomposition \( \mathcal{H} = L^2(0, 1) \oplus \mathbb{C} \). Here \( M \) denotes the multiplication operator in \( L^2(0, 1) \),

\[
(Mf)(\mu) = \mu f(\mu), \quad 0 < \mu < 1, \quad f \in L^2(0, 1),
\]

and \( v \in \mathcal{B}(\mathbb{C}, L^2(0, 1)) \)

\[
(vg)(\mu) = w(\mu)g, \quad \mu \in (0, 1), \quad g \in \mathbb{C}, \quad w(\mu) = \sqrt{\mu(1 - \mu)}.
\]

If \( \varepsilon < 2/5 \), then the operator \( A + V \) has no eigenvalues.
Proof. Assume to the contrary that \( \lambda \in \mathbb{R} \) is an eigenvalue of the perturbed operator \( A + V \), that is,

\[
(\mu - 1/2 - \varepsilon)f(\mu) + \sqrt{\varepsilon}w(\mu)g = \lambda f(\mu) \quad \text{a.e. } \mu \in (0, 1)
\]

and

\[
\sqrt{\varepsilon} \int_0^1 d\mu f(\mu)w(\mu) + (-1/2 + \varepsilon)g = \lambda g
\]

for some \( f \in L^2(0, 1) \) and \( g \in \mathbb{C} \). In particular,

\[
f(\mu) = \sqrt{\varepsilon} \frac{w(\mu)}{\lambda - (\mu - 1/2 - \varepsilon)}g,
\]

and hence \( f \notin L^2(0, 1) \) whenever \( \lambda \in [-1/2 - \varepsilon, 1/2 - \varepsilon] \) (unless \( f = 0 \) and \( g = 0 \)). Thus, the interval \([-1/2 - \varepsilon, 1/2 - \varepsilon]\) does not intersect the point spectrum of \( A + V \). Moreover, \( \lambda \in (-\infty, -1/2 - \varepsilon) \cup (1/2 - \varepsilon, \infty) \) is an eigenvalue of \( A + V \) if and only if

\[
\lambda + \frac{1}{2} - \varepsilon + \varepsilon \int_0^1 d\mu \frac{\mu(1 - \mu)}{\mu - 1/2 - \varepsilon - \lambda} = 0.
\]

Elementary analysis of the graph of the function on the left-hand side of (4.2) then yields that under the condition \( 0 < \varepsilon < 2/5 \) there is no solution of equation (4.2) in \((-\infty, -1/2 - \varepsilon) \cup (1/2 - \varepsilon, \infty) \). Thus, the point spectrum of \( A + V \) is empty.

Remark 4.3. We note that \( \text{spec}(A) = \{-1\} \cup [0, 1] \) and hence \( \text{spec}(A) \) has two components separated by a gap of length one, and the norm of the perturbation \( V \) may be arbitrarily close to 1/2 (from above): \( \|
V\| \in (1/2, 1) \).

Using scaling arguments, Remark 4.3 combined with the result of Lemma 4.1 shows that given \( d > 0 \), for any \( \varepsilon > 0 \) one can find a self-adjoint operator \( A \) satisfying Hypothesis 1 and a self-adjoint perturbation \( V \) with \( \|
V\| = d/2 + \varepsilon \) such that

\[
\|E_A(\sigma) - Q\| = 1
\]

for the orthogonal projection \( Q \) onto an arbitrary invariant subspace of the operator \( A + V \).

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