THE $p$-EXPONENT OF THE $K(1)_s$-LOCAL SPECTRUM $\Phi SU(n)$

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Abstract. Let $p$ be a fixed odd prime. In this paper we prove an exponent conjecture of Bousfield, namely that the $p$-exponent of the spectrum $\Phi SU(n)$ is $( n - 1 ) + \nu_p(( n - 1 )!)$ for $n \geq 2$. It follows from this result that the $p$-exponent of $\Omega^i SU(n)(i)$ is at least $( n - 1 ) + \nu_p(( n - 1 )!)$ for $n \geq 2$ and $i, q \geq 0$, where $SU(n)(i)$ denotes the $i$-connected cover of $SU(n)$.

1. Introduction

Let $p$ be a prime number and $A$ be an object in an additive category. We define the $p$-exponent of $A$ to be the smallest non-negative integer $e$ such that the morphism $p^e A : A \to A$ is the zero morphism.

The purpose of this paper is to give a proof of a conjecture of Bousfield, namely that the $p$-exponent of the spectrum $\Phi SU(n)$ is $( n - 1 ) + \nu_p(( n - 1 )!)$ for $n \geq 2$ and for $p$ an odd prime. Here and throughout $\nu_p$ denotes the exponent of $p$ in an integer and $\Phi$ is a $v_1$ telescope functor from the homotopy category of pointed CW-complexes to the category of $K(1)_s$-local spectra.

The functor $\Phi$ was introduced by Bousfield and is described in [1, 2, 6]. A similar functor can also be found in [5]. Among the many intriguing properties of $\Phi$ are the following: (i) for any spectrum $E$, there is a natural equivalence $\Phi(\Omega^\infty E) \simeq E_{K/p}$, (ii) $\Phi$ preserves fibrations, and (iii) $v_1^{-1} \pi_*(X; p) \cong \pi_*(\Phi X)$.

The functor $\Phi$ is complicated enough to make actual calculations somewhat onerous. However, the following example is well known. It was shown in [4] that $\Phi S^{2n+1} = v_1^{-1} M(p^n)$, where $M(p^n)$ is the mod $p^n$ Moore space.

One can also obtain $\Phi^i S^{2n}$ from the fibration

$$S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1}.$$ 

From here, using towers of fibrations with products of loop spaces on spheres, various Lie groups can be computed. The Lie group $SU(n)$ is a natural first choice; it is interesting, yet tractable.

Given a 1-connected finite $H$-space $X$, let $M \cong \tilde{Q} K^1(X; \tilde{Z}_p) \cong P K^1(X; \tilde{Z}_p)$, the $p$-adic Adams module of indecomposables or primitives. In [3], Bousfield proves, among other things, that if $H_*(X; \mathbb{Q})$ is associative and $H_*(X; \mathbb{Z}(p))$ is finitely generated over $\mathbb{Z}(p)$, then $M/\psi^\infty$ and $\Phi X$ have the same $p$-exponent. For the case
X = SU(n) we have, via a result of Hodgkin [8],

\[ M_n \cong \tilde{Q}K^1(SU(n); \tilde{Z}_p) \cong K^1(\Sigma CP^{n-1}; \tilde{Z}_p) \cong \tilde{K}^0(\Sigma CP^{n-1}; \tilde{Z}_p). \]

Now, because \( \tilde{K}^0(\Sigma CP^{n-1}; \tilde{Z}_p) = \tilde{Z}_p[x]/(1, x^n) \) where \( x = \xi - 1 \) and \( \xi \) is the canonical line bundle on \( CP^{n-1} \), we have \( M_n \{ x, x^2, \ldots, x^{n-1} \} \) with \( \psi^p x = \sum_{i=1}^{n-1} (p)^i x^i \) and \( \psi^p x^m = (\psi^p x)^m \) for \( 2 \leq m \leq n - 1 \). Hence to prove Bousfield's conjecture, it suffices to prove the following lemma.

**Lemma 1.1.** The \( p \)-exponent of \( M_n / \psi^p \) is \( (n - 1) + \nu_p((n - 1)!) \) for \( n \geq 2 \).

From this we deduce our main theorem.

**Theorem 1.2.** The \( p \)-exponent of \( \Phi SU(n) \) is \( (n - 1) + \nu_p((n - 1)!) \) for \( n \geq 2 \).

Additionally, we obtain the following corollary since the functor \( \Phi \) preserves loopings and since \( \Phi \) carries \( i \)-connected coverings to equivalences.

**Corollary 1.3.** The \( p \)-exponent of \( \Phi^i SU(n) / i \) is at least \( (n - 1) + \nu_p((n - 1)!) \) for \( n \geq 2 \) and \( i, q \geq 0 \), where \( SU(n) / i \) denotes the \( i \)-connected cover of \( SU(n) \).

2. **Proof of Lemma 1.1**

The proof of Lemma 1.1 will proceed in two steps. Let \( e_i \) denote the \( p \)-exponent of \( x^i \) in \( M_n / \psi^p \), and let \( b = (n - 1) + \nu_p((n - 1)!) \). We will show \( e_1 = b \) and \( e_i \leq b \) for all \( i, 2 \leq i \leq n - 1 \).

**Lemma 2.1.** Let \( a_1 = p^{b-1} \) and, for \( k > 1 \),

\[ a_k = \frac{(-1)^{k+1}}{k!} p^{b-k}(p-1)(2p-1)(3p-1) \cdots ((k-1)p-1). \]

Then \( \psi^p(\sum_{k=1}^{n-1} a_k x^k) = p^b x \) and \( \sum_{k=1}^{n-1} a_k x^k \) is the unique element of \( M_n \) taken to \( p^b x \) under the action of \( \psi^p \). Moreover \( e_1 = b \).

**Proof.** Consider the matrix of \( \psi^p \) (over \( \tilde{Z}_p \)) with respect to the basis \( \{ x, x^2, \ldots, x^{n-1} \} \):

\[
[\psi^p] = \begin{bmatrix}
c_{1,1} & 0 & 0 & \cdots & 0 \\
c_{2,1} & c_{2,2} & 0 & \cdots & 0 \\
c_{3,1} & c_{3,2} & c_{3,3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \cdots & c_{n-1,n-1}
\end{bmatrix}
\]

where \( c_{i,j} \) is the coefficient of \( x^i \) in \( ((1+x)^p-1)^j \). Note that

\[
\sum_{i_1 + i_2 + \cdots + i_k = i} \binom{p}{i_1} \binom{p}{i_2} \cdots \binom{p}{i_k} = \binom{kp}{i}.
\]

Thus, by the principle of inclusion and exclusion (see [9] for example),

\[
c_{i,j} = \sum_{k=0}^{j-1} (-1)^k \binom{j}{j-k} \binom{(j-k)p}{i}.
\]

For the time being, view \([\psi^p]\) as a linear transformation from \( \mathbb{Q}^{n-1} \) to \( \mathbb{Q}^{n-1} \). Then for \( m \geq 0 \), let \( a'_i = p^{m-1} \) and, for \( k > 1 \),

\[
a'_k = \frac{(-1)^{k+1}}{k!} p^{m-k}(p-1)(2p-1)(3p-1) \cdots ((k-1)p-1).
\]
We will show that
\[
\begin{bmatrix}
  c_{1,1} & 0 & 0 & \cdots & 0 \\
  c_{2,1} & c_{2,2} & 0 & \cdots & 0 \\
  c_{3,1} & c_{3,2} & c_{3,3} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \cdots & c_{n-1,n-1}
\end{bmatrix}
\begin{bmatrix}
  a_1' \\
  a_2' \\
  a_3' \\
  \vdots \\
  a_{n-1}'
\end{bmatrix} = \begin{bmatrix}
p^m \\
  0 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}.
\]

Clearly \( \sum_{j=1}^{n-1} c_{i,j}a_j' = p^m \) and \( \sum_{j=1}^{n-1} c_{2,j}a_j' = 0 \). We are left to show that \( \sum_{j=1}^{n-1} c_{i,j}a_j' \) \((i \geq 3)\) yields

\[
\frac{p^m}{i!} (p-1)(p-2) \cdots (p-i+1) = 0.
\]

So it suffices to show that

\[
\sum_{i=1}^{n-1} (-1)^{i+1} \binom{i}{l} \left( \frac{lp}{i} \right) \frac{1}{lp-1} = 0.
\]

Notice that

\[
\sum_{i=1}^{n-1} (-1)^{i+1} \binom{i}{l} \left( \frac{lp}{i} \right) \frac{1}{lp-1} = \frac{p}{(i-1)!} \sum_{i=1}^{n-1} (-1)^{i+1} \binom{i-1}{l-1} (lp-2) \cdots (lp-i+1).
\]

Let \( f(t) = \sum_{i=1}^{n-1} (-1)^{i+1} \binom{i}{l} (lp-2) \cdots (lp-i+1) t^{lp-i} \). Then

\[
f(t) = \sum_{i=1}^{n-1} (-1)^{i-1} \binom{i-1}{l-1} (lp-2) \cdots (lp-i+1) t^{lp-i} = \left( \frac{d}{dt} \right)^{i-2} t^{lp-2} \sum_{i=1}^{n-1} (-1)^{i-1} \binom{i-1}{l-1} (lp-2) \cdots (lp-i+1) t^{lp-i}.
\]

Hence \( f(t) = (\frac{d}{dt})^{i-2} (1-t)^{-l+1} \). Thus \( f(1) = 0 \) since all terms will be divisible by \((1-t)^{lp}\). Therefore \( \sum_{i=1}^{n-1} c_{i,j}a_j' = 0 \) for \( i \geq 3 \).

Note that \( \ker[\nu^p] = \emptyset \) over \( \mathbb{Q} \). Thus \( \langle a_1', a_2', a_3', \ldots, a_{n-1}' \rangle \) is the unique vector in \( \mathbb{Q}^{n-1} \) that is taken to \( \langle p^m, 0, 0, \ldots, 0 \rangle \) by the transformation \([\nu^p] \).

Now notice that the \( a_k', 1 \leq k \leq n-1 \), are integral, hence also elements of \( \mathbb{Z}_p \), only when \( m-k \geq \nu_p(k!) \), i.e., \( m \geq n-1 + \nu_p((n-1)!)) = b \).
Let $a_1 = p^{b-1}$ and, for $k > 1$,
\[ a_k = \frac{(-1)^{k+1}}{k!} p^{b-k}(p-1)(2p-1)(3p-1) \cdots ((k-1)p-1). \]

Then, since ker[$\psi^p$] = 0 over $\hat{\mathbb{Z}}_p$, $(a_1, a_2, a_3, \ldots, a_{n-1}) = \sum_{k=1}^{n-1} a_k x^k$ is the unique element of $M_n$ such that $\psi^p(\sum_{k=1}^{n-1} a_k x^k) = p^b x$.

To see that there does not exist $w \in M_n$ such that $\psi^p(w) = p^{b-\epsilon}x$, $\epsilon \in \mathbb{Z}^+$, consider the following. Suppose such a $w = \sum_{k=1}^{n-1} q_k x^k$ existed. Then at least one of the $q_k$ has to be in $\hat{\mathbb{Z}}_p - \mathbb{Z}$. But then $\psi^p(p^\epsilon w) = p^\epsilon \psi^p(w) = p^b x$. Since $p^\epsilon q_k = a_k$ by uniqueness, we get the contradiction $p^\epsilon q_k \in \hat{\mathbb{Z}}_p - \mathbb{Z}$ and $p^\epsilon q_k \in \mathbb{Z}$.

The next lemma will finish the proof of Lemma 1.1.

**Lemma 2.2.** For $2 \leq i \leq n - 1$, let $e_i$ denote the $p$-exponent of $x^i$ in $M_n/\psi^p$. Then $e_i \leq b$.

**Proof.** First note that the relations of $M_n/\psi^p$ are given by the following equations:
\[
\begin{align*}
\alpha_{1,1} x + \alpha_{1,2} x^2 + \alpha_{1,3} x^3 + \cdots + \alpha_{1,n-2} x^{n-2} + \alpha_{1,n-1} x^{n-1} &= 0, \\
\alpha_{2,2} x^2 + \alpha_{2,3} x^3 + \cdots + \alpha_{2,n-2} x^{n-2} + \alpha_{2,n-1} x^{n-1} &= 0, \\
\alpha_{3,3} x^3 + \cdots + \alpha_{3,n-2} x^{n-2} + \alpha_{3,n-1} x^{n-1} &= 0, \\
&\vdots \\
\alpha_{n-2,n-2} x^{n-2} + \alpha_{n-2,n-1} x^{n-1} &= 0, \\
\alpha_{n-1,n-1} x^{n-1} &= 0,
\end{align*}
\]

where $\alpha_{i,j} = \sum_{k=0}^{i-1} (-1)^k \binom{i}{k} \binom{j}{k} p^k$ (these relations can be obtained from the transpose of the matrix $[\psi^p]$). Notice that $\alpha_{i,i} = \binom{i}{1}^2$ and $\alpha_{i,i+1} = \binom{i}{1} \binom{i}{2} (p)^{-1}$.

Since $\alpha_{n-1,n-1} = p^{n-1}$ we know that the $p$-exponent of $x^{n-1}$ is $n - 1$. Via back-substitution, we are then able to find the $p$-exponent of $x^{n-2}$, $x^{n-3}$, and so on, all the way up to $x$. This line of thinking leads us to the formula
\[ e_i = d_{i,i} + \max\{e_j - d_{i,j} : j = i + 1, \ldots, n - 1\}, \]

where $d_{i,j} = \nu_p(\alpha_{i,j})$. (Note: $d_{i,i} = i$.) It follows that $e_i \geq e_{i+1} + (i - d_{i,i+1})$.

Next we see that $i - d_{i,i+1} = -\nu_p(i)$, since $d_{i,i+1} = \nu_p(\alpha_{i,i+1}) = \nu_p(\binom{i}{1} \binom{i}{2} (p)^{-1}) = \nu_p(\frac{p^{i-1}}{2} (p)^{i-1})$. Thus $e_i \geq e_{i+1} - \nu_p(i)$. Hence if we can show that $e_{ip} \leq b - \nu_p(ip)$ for all $i$ such that $1 \leq i \leq q$, where $qp \leq n - 1 < (q + 1)p$, we will be done.

By Lemma 2.1 and the relation
\[ \alpha_{1,1} x + \alpha_{1,2} x^2 + \cdots + \alpha_{1,n-1} x^{n-1} = \sum_{i=1}^{n-1} \binom{p}{i} x^i = 0 \]
we have $e_p \leq b - 1$. Now choose the smallest $k$ such that $e_{kp} > b - \nu_p(kp)$. Then the relation
\[ \alpha_{k,k} x^k + \alpha_{k,k+1} x^{k+1} + \cdots + \alpha_{k,k-p} x^{k-p} + \alpha_{k,k-p} x^{kp} = p^k x^k + \cdots + x^{kp} = 0 \]
implies that $e_k \geq k + b - \nu_p(kp) + 1 = (b + 1) + (k - \nu_p(kp))$. 


Since \( k - \nu_p(kp) \geq 0 \) for all \( k \) and \( p \), we choose \( i \) so that \((i + 1)p > k \geq ip\) and get the contradiction
\[
b + (1 + k - \nu_p(kp)) \leq e_k \leq e_{k-1} \leq \cdots \leq e_{ip} \leq b.
\]
Therefore it must be the case that \( e_{ip} \leq b - \nu_p(ip) \) for all \( 1 \leq i \leq q \). \( \square \)

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References


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