THE p-EXponent OF THE K(1),-LOCAL SPECTRUM ΦSU(n)

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Abstract. Let p be a fixed odd prime. In this paper we prove an exponent conjecture of Bousfield, namely that the p-exponent of the spectrum ΦSU(n) is \((n-1)+\nu_p((n-1)!)\) for \(n \geq 2\). It follows from this result that the p-exponent of Ω^iSU(n)(i) is at least \((n-1)+\nu_p((n-1)!)\) for \(n \geq 2\) and \(i, q \geq 0\), where SU(n)(i) denotes the i-connected cover of SU(n).

1. Introduction

Let p be a prime number and A be an object in an additive category. We define the p-exponent of A to be the smallest non-negative integer e such that the morphism \(p^e 1_A : A \to A\) is the zero morphism.

The purpose of this paper is to give a proof of a conjecture of Bousfield, namely that the p-exponent of the spectrum ΦSU(n) is \((n-1)+\nu_p((n-1)!)\) for \(n \geq 2\) and for p an odd prime. Here and throughout \(\nu_p\) denotes the exponent of p in an integer and Φ is a \(v_1\) telescope functor from the homotopy category of pointed CW-complexes to the category of K(1),-local spectra.

The functor Φ was introduced by Bousfield and is described in [1, 2, 6]. A similar functor can also be found in [5]. Among the many intriguing properties of Φ are the following: (i) for any spectrum E, there is a natural equivalence \(\Phi(\Omega\infty E) \simeq E_{K/p}\), (ii) Φ preserves fibrations, and (iii) \(v_1^{-1}\pi_*(X; p) \cong \pi_*(FX)\).

The functor Φ is complicated enough to make actual calculations somewhat onerous. However, the following example is well known. It was shown in [4] that \(\Phi S^{2n+1} = v_1^{-1}M(p^n)\), where \(M(p^n)\) is the mod \(p^n\) Moore space.

One can also obtain \(\Phi S^{2n}\) from the fibration \(S^{2n-1} \to \Omega^2 S^{2n} \to \Omega^4 S^{4n-1}\).

From here, using towers of fibrations with products of loop spaces on spheres, various Lie groups can be computed. The Lie group \(SU(n)\) is a natural first choice; it is interesting, yet tractable.

Given a 1-connected finite H-space X, let \(M \cong \hat{Q}K^1(X; \hat{Z}_p) \cong PK^1(X; \hat{Z}_p)\), the p-adic Adams module of indecomposables or primitives. In [3], Bousfield proves, among other things, that if \(H_*(X; \mathbb{Q})\) is associative and \(H_*(X; \mathbb{Z}(p))\) is finitely generated over \(\mathbb{Z}(p)\), then \(M/\psi^p\) and \(\Phi X\) have the same p-exponent. For the case

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$X = SU(n)$ we have, via a result of Hodgkin \[5\],

$$M_n \cong \tilde{Q}K^1(SU(n);\tilde{Z}_p) \cong K^1(\Sigma \mathbb{CP}^{n-1};\tilde{Z}_p) \cong \tilde{K}^0(\mathbb{CP}^{n-1};\tilde{Z}_p).$$

Now, because $\tilde{K}^0(\mathbb{CP}^{n-1};\tilde{Z}_p) = \tilde{Z}_p[x]/(1, x^n)$ where $x = \xi - 1$ and $\xi$ is the canonical line bundle on $\mathbb{CP}^{n-1}$, we have $M_n[x, x^2, \ldots, x^{n-1}]$ with $\psi^p x = \sum_{i=1}^{n-1} (p_i)x^i$ and $\psi^p x^m = (\psi^p x)^m$ for $2 \leq m \leq n - 1$. Hence to prove Bousfield’s conjecture, it suffices to prove the following lemma.

**Lemma 1.1.** The $p$-exponent of $M_n/\psi^p$ is $(n - 1) + \nu_p((n - 1)!!)$ for $n \geq 2$.

From this we deduce our main theorem.

**Theorem 1.2.** The $p$-exponent of $\Phi SU(n)$ is $(n - 1) + \nu_p((n - 1)!!)$ for $n \geq 2$.

Additionally, we obtain the following corollary since the functor $\Phi$ preserves loopings and since $\Phi$ carries $i$-connected coverings to equivalences.

**Corollary 1.3.** The $p$-exponent of $\Phi^i SU(n)$ is at least $(n - 1) + \nu_p((n - 1)!!)$ for $n \geq 2$ and $i, q \geq 0$, where $SU(n)(i)$ denotes the $i$-connected cover of $SU(n)$.

2. Proof of Lemma 1.1

The proof of Lemma 1.1 will proceed in two steps. Let $e_i$ denote the $p$-exponent of $x^i$ in $M_n/\psi^p$, and let $b = (n - 1) + \nu_p((n - 1)!!)$. We will show $e_1 = b$ and $e_i \leq b$ for all $i$, $2 \leq i \leq n - 1$.

**Lemma 2.1.** Let $a_1 = p^{b-1}$ and, for $k > 1$,

$$a_k = \frac{(-1)^{k+1}}{k!}p^{b-k}(p-1)(2p-1)(3p-1)\cdots((k-1)p-1).$$

Then $\psi^p(\sum_{k=1}^{n-1} a_kx^k) = p^p x$ and $\sum_{k=1}^{n-1} a_kx^k$ is the unique element of $M_n$ taken to $p^p x$ under the action of $\psi^p$. Moreover $e_1 = b$.

**Proof.** Consider the matrix of $\psi^p$ (over $\tilde{Z}_p$) with respect to the basis $\{x, x^2, \ldots, x^{n-1}\}$:

$$[\psi^p] = \begin{bmatrix}
c_{11} & 0 & 0 & \ldots & 0 
c_{21} & c_{22} & 0 & \ldots & 0 
c_{31} & c_{32} & c_{33} & \ldots & 0 
\vdots & \vdots & \vdots & \ddots & \vdots 
c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \cdots & c_{n-1,n-1}
\end{bmatrix}$$

where $c_{i,j}$ is the coefficient of $x^i$ in $((1 + x)^p - 1)^j$. Note that

$$\sum_{i_1+i_2+\cdots+i_k=i} \binom{p}{i_1} \binom{p}{i_2} \cdots \binom{p}{i_k} = \binom{kp}{i}.$$ 

Thus, by the principle of inclusion and exclusion (see \[7\] for example),

$$c_{i,j} = \sum_{k=0}^{j-1} (-1)^k \binom{j}{j-k} \binom{(j-k)p}{i}.$$ 

For the time being, view $[\psi^p]$ as a linear transformation from $\mathbb{Q}^{n-1}$ to $\mathbb{Q}^{n-1}$. Then for $m \geq 0$, let $a'_i = p^{m-1} x^i$ and, for $k > 1$,

$$a'_k = \frac{(-1)^{k+1}}{k!}p^{m-k}(p-1)(2p-1)(3p-1)\cdots((k-1)p-1).$$
We will show that
\[
\begin{bmatrix}
c_{1,1} & 0 & 0 & \cdots & 0 \\
c_{2,1} & c_{2,2} & 0 & \cdots & 0 \\
c_{3,1} & c_{3,2} & c_{3,3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{n,1} & c_{n,2} & c_{n,3} & \cdots & c_{n,n-1}
\end{bmatrix}
\begin{bmatrix}
a'_1 \\
a'_2 \\
a'_3 \\
\vdots \\
a'_{n-1}
\end{bmatrix}
= \begin{bmatrix}
p^m \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]
Clearly \(\sum_{j=1}^{n-1} c_{i,j}a'_j = p^m\) and \(\sum_{j=1}^{n-1} c_{2,j}a'_j = 0\). We are left to show that \(\sum_{j=1}^{n-1} c_{i,j}a'_j (i \geq 3)\) yields
\[
\begin{align*}
= & \binom{p}{i} \left( \binom{1}{1} a_1' - \binom{2}{1} a_2' + \binom{3}{1} a_3' + \cdots + (-1)^{i-1} \binom{i}{1} a_i' \right) \\
+ & \binom{2p}{i} \left( \binom{2}{1} a_2' - \binom{3}{2} a_3' + \binom{4}{2} a_4' + \cdots + (-1)^{i-2} \binom{i}{2} a_i' \right) \\
& + \cdots + \binom{kp}{i} \left( \binom{k}{1} a_k' - \binom{k+1}{k} a_{k+1}' + \cdots + (-1)^{i-k} \binom{i}{k} a_i' \right) \\
& + \cdots + \binom{ip}{i} \binom{i}{i} a_i'.
\end{align*}
\]
By induction one can see that for \(l = 1, \ldots, i,\)
\[
\sum_{k=l}^{i} (-1)^{k-l} \binom{k}{l} a_k' = \frac{(-1)^{i+1}}{i!} p^{m-i} \binom{i}{l} (p-1)(2p-1) \cdots (lp-1) \cdots (ip-1)
\]
where \(\sim\) means leave out. Therefore (2.1) becomes
\[
\left( \frac{p^{m-i}}{i!} (p-1)(2p-1) \cdots (ip-1) \right) \sum_{l=1}^{i} (-1)^{l+1} \binom{i}{l} \binom{lp}{i} \frac{1}{lp-1}.
\]
So it suffices to show that
\[
\sum_{l=1}^{i} (-1)^{l+1} \binom{i}{l} \binom{lp}{i} \frac{1}{lp-1} = 0.
\]
Notice that
\[
\sum_{l=1}^{i} (-1)^{l+1} \binom{i}{l} \binom{lp}{i} \frac{1}{lp-1} = \frac{p}{(i-1)!} \sum_{l=1}^{i} (-1)^{l+1} \binom{i-1}{l-1} (lp-2) \cdots (lp-i+1).
\]
Let \(f(t) = \sum_{l=1}^{i} (-1)^{l+1} \binom{i-1}{l-1} (lp-2) \cdots (lp-i+1)t^{l-1-i}.\) Then
\[
f(t) = \sum_{l=1}^{i} (-1)^{l+1} \binom{i-1}{l-1} \frac{d}{dt} t^{l-2} = \binom{d}{dt} t^{i-2} t^{p-2} \sum_{l=1}^{i} (-1)^{l-1} \binom{i-1}{l-1} t^{l(p-1)}.
\]
Hence \(f(t) = \binom{d}{dt} t^{i-2} t^{p-2} (1 - t^p)^{i-1}.\) Thus \(f(1) = 0\) since all terms will be divisible by \((1 - t^p).\) Therefore \(\sum_{j=1}^{n-1} c_{i,j}a'_j = 0\) for \(i \geq 3.\)

Note that \(\ker [\psi^p] = 0\) over \(\mathbb{Q}^p.\) Thus \(\langle a'_1, a'_2, a'_3, \ldots, a'_{n-1} \rangle\) is the unique vector in \(\mathbb{Q}^{i-1} - \text{local spectrum, SU}(n)\) that is taken to \(\langle p^m, 0, 0, \ldots, 0 \rangle\) by the transformation \([\psi^p].\)

Now notice that the \(a'_k, 1 \leq k \leq n-1,\) are integral, hence also elements of \(\mathbb{Z}.\) Consequently, only when \(m - k \geq \nu_p (k!),\) i.e., \(m \geq n - 1 + \nu_p (n-1)! = b.\)
Let $a_1 = p^{b-1}$ and, for $k > 1,$

$$a_k = \frac{(-1)^{k+1}}{k!} p^{b-k} (p-1)(2p-1)(3p-1) \cdots ((k-1)p-1).$$

Then, since $\ker[\psi^p] = 0$ over $\widehat{\mathbb{Z}}_p$, $(a_1, a_2, a_3, \ldots, a_{n-1}) = \sum_{k=1}^{n-1} a_k x^k$ is the unique element of $M_n$ such that $\psi^p(\sum_{k=1}^{n-1} a_k x^k) = p^b x$.

To see that there does not exist $w \in M_n$ such that $\psi^p(w) = p^{b-\epsilon} x$, $\epsilon \in \mathbb{Z}^+$, consider the following. Suppose such a $w = \sum_{k=1}^{n-1} q_k x^k$ existed. Then at least one of the $q_k$ has to be in $\widehat{\mathbb{Z}}_p - \mathbb{Z}$. But then $\psi^p(p^j w) = p^j \psi^p(w) = p^b x$. Since $p^j q_k = a_k$ by uniqueness, we get the contradiction $p^j q_k \in \widehat{\mathbb{Z}}_p - \mathbb{Z}$ and $p^j q_k \in \mathbb{Z}$. \hfill \Box

The next lemma will finish the proof of Lemma 1.1.

**Lemma 2.2.** For $2 \leq i \leq n - 1$, let $e_i$ denote the $p$-exponent of $x^i$ in $M_n/\psi^p$. Then $e_i \leq b$.

**Proof.** First note that the relations of $M_n/\psi^p$ are given by the following equations:

$$\begin{align*}
\alpha_{1,1} x + \alpha_{1,2} x^2 + \alpha_{1,3} x^3 + \cdots + \alpha_{1,n-2} x^{n-2} + \alpha_{1,n-1} x^{n-1} &= 0, \\
\alpha_{2,2} x^2 + \alpha_{2,3} x^3 + \cdots + \alpha_{2,n-2} x^{n-2} + \alpha_{2,n-1} x^{n-1} &= 0, \\
\alpha_{3,3} x^3 + \cdots + \alpha_{3,n-2} x^{n-2} + \alpha_{3,n-1} x^{n-1} &= 0, \\
& \vdots \\
\alpha_{n-2,n-2} x^{n-2} + \alpha_{n-2,n-1} x^{n-1} &= 0, \\
\alpha_{n-1,n-1} x^{n-1} &= 0,
\end{align*}$$

where $\alpha_{i,j} = \sum_{k=0}^{i-1} (-1)^k \binom{i}{k} \binom{j}{k} p^k$ (these relations can be obtained from the transpose of the matrix $[\psi^p]$). Notice that $\alpha_{i,i} = \binom{i}{i}$ and $\alpha_{i,i+1} = \binom{i}{2} \binom{i}{i} (\binom{i}{i} p)^{i-1}$.

Since $\alpha_{n-1,n-1} = p^{n-1}$ we know that the $p$-exponent of $x^{n-1}$ is $n - 1$. Via back-substitution, we are then able to find the $p$-exponent of $x^{n-2}$, $x^{n-3}$, and so on, all the way up to $x$. This line of thinking leads us to the formula

$$e_i = d_{i,i} + \max\{e_j - d_{i,j} : j = i + 1, \ldots, n - 1\},$$

where $d_{i,j} = \nu_p(\alpha_{i,j})$. (Note: $d_{i,i} = i$.) It follows that $e_i \geq e_{i+1} + (i - d_{i,i+1})$.

Next we see that $i - d_{i,i+1} = -\nu_p(i)$, since $d_{i,i+1} = \nu_p(\alpha_{i,i+1}) = \nu_p(\binom{i}{2} \binom{i}{i} (\binom{i}{i} p)^{i-1}) = \nu_p(\frac{p^i}{2} ip^i)$. Thus $e_i \geq e_{i+1} - \nu_p(i)$. Hence if we can show that $e_{ip} \leq b - \nu_p(ip)$ for all $i$ such that $1 \leq i \leq q$, where $qp \leq n - 1 < (q + 1)p$, we will be done.

By Lemma 2.1 and the relation

$$\alpha_{1,1} x + \alpha_{1,2} x^2 + \cdots + \alpha_{1,n-1} x^{n-1} = \sum_{i=1}^{n-1} \binom{p}{i} x^i = 0$$

we have $e_p \leq b - 1$. Now choose the smallest $k$ such that $e_{kp} > b - \nu_p(kp)$. Then the relation

$$\alpha_{k,k} x^k + \alpha_{k,k+1} x^{k+1} + \cdots + \alpha_{k,kp-1} x^{kp-1} + \alpha_{k,kp} x^{kp} = p^k x^k + \cdots + x^{kp} = 0$$

implies that $e_k \geq k + b - \nu_p(kp) + 1 = (b + 1) + (k - \nu_p(kp))$. 


Since \( k - \nu_p(kp) \geq 0 \) for all \( k \) and \( p \), we choose \( i \) so that \((i+1)p > k \geq ip\) and get the contradiction
\[
b + (1 + k - \nu_p(kp)) \leq e_k \leq e_{k-1} \leq \cdots \leq e_{ip} \leq b.
\]
Therefore it must be the case that \( e_{ip} \leq b - \nu_p(ip) \) for all \( 1 \leq i \leq q \). \( \square \)

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