BANACH SPACES HAVING THE RADON-NIKODÝM PROPERTY AND NUMERICAL INDEX 1

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(Communicated by Jonathan M. Borwein)

Abstract. Let $X$ be a Banach space with the Radon-Nikodým property. Then, the following are equivalent.
(i) $X$ has numerical index 1.
(ii) $|x^*(Tx)| = 1$ for all $x^* \in \text{ex}(B_{X^*})$ and $x^{**} \in \text{ex}(B_{X^{**}})$.
(iii) $X$ is an almost-CL-space.
(iv) There are a compact Hausdorff space $K$ and a linear isometry $J : X \to C(K)$ such that $|x^{**}(J^*s)| = 1$ for all $s \in K$ and $x^{**} \in \text{ex}(B_{X^{**}})$.

If $X$ is a real space, the above conditions are equivalent to being semi-nicely embedded in some space $C(K)$.

The numerical index of a Banach space is a constant relating the norm and the numerical radius of operators on the space. Let us present the relevant definitions. For a Banach space $X$, we write $B_X$ for the closed unit ball and $S_X$ for the unit sphere. We denote by $X^*$ the dual space and by $L(X)$ the Banach algebra of all bounded linear operators on $X$. For such an operator $T$, the numerical radius of $T$ is

$$v(T) = \sup \{|x^*(Tx)| : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1\}.$$  

The numerical index of the space $X$ is then given by

$$n(X) = \max \{k \geq 0 : k \|T\| \leq v(T) \ \forall T \in L(X)\}.$$  

We refer the reader to the books [3, 4] and to the expository paper [13] for general information and background. Recent results can be found in [7, 12, 14, 15].

Let us mention here some facts concerning the numerical index which will be relevant to our discussion. First, one has $v(T^*) = v(T)$ for every $T \in L(X)$, where $T^*$ is the adjoint operator of $T$ (see [5, §9]), and it clearly follows that $n(X^*) \leq n(X)$. The question whether this is actually an equality seems to be open. Second, it is a classical result that $L$- and $M$-spaces have numerical index 1 [6].

The aim of this paper is to characterize Banach spaces with numerical index 1 among those having the Radon-Nikodým property (RNP for short; see [5] for background). To this end, we prove that some general sufficient conditions for a Banach space to have numerical index 1 are also necessary if the space has the RNP.
Let us introduce some definitions and comments. A Banach space $X$ is an almost-CL-space if $B_X$ is the absolutely closed convex hull of every maximal convex subset of $S_X$. This notion was introduced by A. Lima [10], generalizing the concept of CL-space given by R. Fullerton [8] in 1960. Real and complex almost-CL-spaces have numerical index 1 (see [12, §4] or [1]), but it is not known whether the reciprocal result is true. As usual, we write $C(K)$ for the Banach space of all continuous functions from the compact Hausdorff space $K$ into the scalar field. Given $s \in K$, $\delta_s$ stands for the functional $f \mapsto f(s)$ on $C(K)$. Finally, we write $\text{co}(B)$ for the convex-hull of $B$ and, given a convex subset $A$ of $X$, $\text{ex}(A)$ and $\text{dent}(A)$ are, respectively, the set of extreme points and the set of denting points of $A$.

The main result of the paper is the following.

**Theorem 1.** Let $X$ be a Banach space having the RNP. Then the following are equivalent.

(i) $n(X) = 1$.

(ii) $|x^{**}(x^*)| = 1$ for all $x^* \in \text{ex}(B_{X^*})$ and $x^{**} \in \text{ex}(B_{X^{**}})$.

(iii) $X$ is an almost-CL-space.

(iv) There are a compact Hausdorff space $K$ and a linear isometry $J : X \to C(K)$ such that $|x^{**}(J^*\delta_s)| = 1$ for all $s \in K$ and $x^{**} \in \text{ex}(B_{X^{**}})$.

Proof. The implication $(ii) \Rightarrow (i)$ is straightforward. Just use that $v(T) = v(T^*)$ for every $T \in L(X)$. The implication $(iii) \Rightarrow (i)$ has been discussed above.

$(iv) \Rightarrow (i)$. We fix $T \in L(X)$ and we have to prove that $v(T) = ||T||$. For each $s \in K$, we take $x^{**} \in \text{ex}(B_{X^{**}})$ such that

$$|x^{**}(T^*(J^*\delta_s))| = ||T^*(J^*\delta_s)||.$$  

Since $|x^{**}(J^*\delta_s)| = 1$, we have

$$v(T) = v(T^*) \geq ||T^*(J^*\delta_s)||$$

for each $s \in K$. It is clear that

$$||T|| = ||T^*|| = \sup \{ ||T^*(J^*\delta_s)|| \ : \ s \in K \}$$

and, so, $v(T) = ||T||$ and $n(X) = 1$.

$(i) \Rightarrow (ii)$. We use [12, Lemma 1] to get $|x^*(x)| = 1$ for all $x^* \in \text{dent}(B_X)$ and $x^* \in \text{ex}(B_{X^*})$. Since $X$ has the RNP, it may be concluded that

$$\text{ex}(B_{X^{**}}) \subseteq \text{dent}(B_X)^{**},$$

so $|x^{**}(x^*)| = 1$ for all $x^* \in \text{ex}(B_{X^*})$ and $x^{**} \in \text{ex}(B_{X^{**}})$.

$(i) \Rightarrow (iii)$. Let $F$ be a maximal convex subset of $S_X$. By using Hahn-Banach and Krein-Milman Theorems, it is easily checked that there exists $x^* \in \text{ex}(B_{X^*})$ such that

$$F = \{ x \in B_X : x^*(x) = 1 \}.$$  

So, by [12, Lemma 1], $\text{dent}(B_X)$ is contained in the absolutely convex hull of $F$. On the other hand, since $X$ has the RNP, $B_X = \text{co}(\text{dent}(B_X))$. Thus, $B_X$ is the closed absolutely convex hull of $F$ and $X$ is an almost-CL-space.

$(i) \Rightarrow (iv)$. For $x^* \in B_{X^*}$, we have

\[ x^* \in \text{ex}(B_{X^*}) \iff |x^*(x)| = 1 \text{ for each } x \in \text{dent}(B_X). \]

One implication is [12, Lemma 1]; the other one arises from the fact that $\text{dent}(B_X)$ is norming for $X^*$.

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It follows from (11) that $K = \text{ex}(B_{X^*})$, with the $w^*$ topology, is a Hausdorff compact space. Let $J$ be the canonical injection from $X$ into $C(K)$. Now, since $J^*\delta_s = s \in \text{ex}(B_{X^*})$, we deduce from (ii) that $|x**(J^*\delta_s)| = 1$ for all $s \in K$ and $x** \in \text{ex}(B_{X^{**}})$.

Let us give some remarks on the above theorem.

Condition (ii) of the theorem is called E.P.I.P. in [10] and appeared in [11, §4]. In [10, Corollary 3.6], A. Lima proved $(ii) \Rightarrow (iii)$ for arbitrary real Banach spaces. Nevertheless, we do not know if Lima’s result is valid in the complex case without the RNP assumption.

Observe that, when proving $(iv) \Rightarrow (i)$, we do not need the RNP and that the result still holds if we change $C(K)$ to $C_b(\Omega)$, the Banach space of all bounded continuous functions from the topological Hausdorff space $\Omega$ into the scalar field. Therefore, we have the following by-product of the proof of Theorem 1.

**Corollary 2.** Let $\Omega$ be a Hausdorff topological space and let $X$ be a Banach space. If there exists a linear isometry $J : X \to C_b(\Omega)$ such that

$$|x**(J^*\delta_s)| = 1 \quad (s \in \Omega, \ x** \in \text{ex}(B_{X^{**}})),$$

then $n(X) = 1$.

The above corollary improves one result recently given by D. Werner [16, Corollary 2.2]. Some definitions are required. Following [16], a Banach space $X$ is said to be **nicely embedded** in $C_b(\Omega)$ if there exists a linear isometry $J : X \to C_b(\Omega)$ such that for all $s \in \Omega$ the following properties are satisfied:

1. $\|J^*\delta_s\| = 1$.
2. $\text{span}(J^*\delta_s)$ is an $L$-summand in $X^*$.

It is clear that nicely embedded spaces fulfill the conditions in Corollary 2, so they have numerical index 1. This is precisely Corollary 2.2 of [16] (see also the introduction of [14]). But the converse result is false even in the finite-dimensional setting. As a matter of fact, in the case when $X$ is the 3-dimensional $L$-space, $n(X) = 1$ and $X^*$ does not have any non-trivial $L$-summand.

In the real case, Corollary 2 can be written in a more suitable form by using notation similar to Werner’s result. Following A. Lima [9, 10], a closed subspace $Y$ of a Banach space $X$ is said to be a **semi $L$-summand** if for every $x \in X$ there exists a unique $y \in Y$ such that $\|x - y\| = d(x, Y)$, and moreover this $y$ satisfies $\|x\| = \|y\| + \|x - y\|$. A Banach space $X$ is said to be **semi-nicely embedded** in $C_b(\Omega)$ if there exists a linear isometry $J : X \to C_b(\Omega)$ such that for all $s \in \Omega$ the following properties are satisfied:

1. $\|J^*\delta_s\| = 1$.
2. $\text{span}(J^*\delta_s)$ is a semi $L$-summand in $X^*$.

Given a real Banach space $X$ and a point $x \in S_X$, [10, Theorem 3.1] says that $\text{span}(x)$ is a semi-$L$-summand if and only if $|x^*(x)| = 1$ for all $x^* \in \text{ex}(B_{X^*})$. Therefore, condition $(iv)$ in Theorem 1 is equivalent to the property of being semi-nicely embedded in some $C(K)$. Therefore, for real spaces, Theorem 1 reads as follows.

**Corollary 3.** Let $X$ be a real Banach space having the RNP. Then $n(X) = 1$ if and only if $X$ is semi-nicely embedded in some $C(K)$. 
It is now easy to give examples of Banach spaces semi-nicely embedded in some $C(K)$ which are not nicely embedded in any $C_0(\Omega)$. For instance, since the real space $l_1$ has the RNP and $n(l_1) = 1$, it is semi-nicely embedded in $C(\Delta)$, where $\Delta$ is the Cantor set. But $l_\infty$ does not have any non-trivial $L$-summand (see [2, Theorem 6.15], for instance).

References


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