ZEROES OF COMPLETE POLYNOMIAL VECTOR FIELDS

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Dedicated to my father

Abstract. We prove that a complete polynomial vector field on \( \mathbb{C}^2 \) has at most one zero, and analyze the possible cases of those with exactly one which is not of Poincaré-Dulac type. We also obtain the possible nonzero first jet singularities of the foliation \( \mathcal{F}_X \) at infinity and the nongenericity of completeness. Connections with the Jacobian Conjecture are established.

Introduction and results

Let \( X = P(z_1, z_2)\frac{\partial}{\partial z_1} + Q(z_1, z_2)\frac{\partial}{\partial z_2} \) be a polynomial vector field on \( \mathbb{C}^2 \) of degree \( m = \max\{\deg P, \deg Q\} \geq 2 \) with isolated zeroes. It is known, [9], that \( X \) extends as a rational vector field in \( \mathbb{CP}^2 \) having a pole along the line at infinity, \( L_\infty \). Removing the pole, we obtain a foliation \( \mathcal{F}_X \) of degree \( d \), where \( d = m \) if \( L_\infty \) is invariant and \( d = m - 1 \) if it is not. We denote by \( \text{Sing}(\mathcal{F}_X) \) the singular set of \( \mathcal{F}_X \).

Recall that a holomorphic vector field \( X \) in a complex manifold \( M \) is said to be complete if, for every \( p \in M \), the differential equation defined by \( X \) can be solved for every complex time \( t \).

In this paper we study complete polynomial vector fields \( X \) on \( \mathbb{C}^2 \) through some properties of the leaves of \( \mathcal{F}_X \). In section 1, we analyze the trajectories of \( X \) at infinity and we give in Theorem 1.1 the possible nonzero first jet of \( \mathcal{F}_X \) at its singular points in \( L_\infty \), thus proving Corollary 1.1 foliations induced by complete polynomial vector fields of degree \( m \) give a nowhere dense set in the space of degree \( m \) foliations, \( \mathcal{F}(m, 2) \), providing a polynomial version of Buzzard-Fornaess’s result, [5]. We also apply our results to the problem of exploding orbits of complex polynomial Hamiltonians, obtaining a simple geometric proof of Fornaess and Grellier’s result, [8], in that case.

In section 2, we further study the isolated zeroes of \( X \). A natural question (posed in [1] and [15]) is if there exist complete holomorphic vector fields on \( \mathbb{C}^2 \) with more than one isolated zero. The answer, given in Theorem 2.1 is no for polynomial ones. Our result relies on the study of proper orbits due to Brunella in [4]. We also classify the complete polynomial vector fields with rational first integral and, using
Andersen’s result in [2], those with one zero $p$ which is not of Poincaré-Dulac type, when it is nondicritical and at least two of the separatrices through it are algebraic at infinity. When $p$ is dicritical with no rational first integral, or nondicritical with just one separatrix algebraic at infinity, the induced foliation $\mathcal{F}_X$ is, as in Brunella’s result [4], $P$-complete where $P$ can be written in a simple form due to [17] and [16] (Proposition 2.1 and Theorem 2.2).

In section 3, we state the Jacobian Conjecture in terms of completeness of certain vector fields, and characterize the complete commutative bases of $\mathbb{C}$-derivations of the polynomial ring.

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1. Separatrices at infinity and completeness

A germ $\Sigma$ of an analytic irreducible curve is said to be a trajectory of $X$ at $p \in L_\infty$ if $p \in \Sigma$ and $\Sigma \setminus \{p\}$ is invariant by $X$. In this case one can extend $\Sigma \setminus \{p\}$ by analytic continuation to obtain the complex orbit $L$ of $X$. If $\gamma : \mathbb{D} \to \Sigma$ is the (minimal) Puiseaux’s parametrization of a neighborhood $U_p$ of $p$ in $\Sigma$, $\mathcal{L} = L \cup \{p\}$ can be endowed with an abstract Riemann surface structure as follows: for any $q \in L$, by the existence of local solutions for $X$, we can take the parametrization $\gamma_q$ of an open neighborhood $U_q \subset L$, and define the local chart as $z_q = \gamma_q^{-1} : U_q \to \mathbb{C}$. Otherwise, $\gamma$ defines the local chart around $p$ in $\mathcal{L}$ as $\gamma^{-1} : \Sigma \to \mathbb{D}$.

**Lemma 1.1.** Let $X$ be a polynomial vector field in $\mathbb{C}^2$, and let $\Sigma$ be a trajectory of $X$ at $p \in L_\infty$. Then, if $X$ is complete on $\mathcal{L} \setminus \{p\}$, it extends to $p$ as a zero of order 1 or 2.

**Proof.** As $\mathcal{L} \setminus \{p\}$ is uniformized by $\mathbb{C}$, and it is contained in the Stein manifold $\mathbb{C}^2$, then $\mathcal{L} \setminus \{p\}$ is (analytically) isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$. If $\mathcal{L} \setminus \{p\} \simeq \mathbb{C}$, it follows that $\mathcal{L} \simeq \mathbb{CP}^1$ and $X$ extends to $p$ as zero of order 2, by Riemann-Roch. On the other hand, if $\mathcal{L} \setminus \{p\} \simeq \mathbb{C}^*$, then $X$ extends to $p$ as zero of order 1. We refer to [10] for the study of complete vector fields on Riemann surfaces.

**Corollary 1.1.** If $X$ is a complete polynomial vector field on $\mathbb{C}^2$, then $L_\infty$ is invariant by $\mathcal{F}_X$.

**Remark 1.1.** If $\mathcal{L} \setminus \{p\} \simeq \mathbb{C}$, by Chow’s Theorem $\mathcal{L} \setminus \{p\}$ is contained in a rational curve.

**Remark 1.2.** Lemma 1.1 is valid for polynomial vector fields on $\mathbb{C}^n$, $n \geq 2$.

Let $p \in \text{Sing}(\mathcal{F}_X) \cap L_\infty$, and let $\Sigma \neq L_\infty$ be a separatrix of $\mathcal{F}_X$ through $p$, parametrized by $\gamma : \mathbb{D} \to \Sigma$. Without loss of generality assume that $p = (0 : 1 : 0)$. Then if $\gamma(t) = (y_1(t), y_2(t))$, with $(y_1, y_2) = (\varphi_1 \circ \varphi_0^{-1})(z_1, z_2) = (\frac{1}{z_1}, \frac{z_2}{z_1})$ the usual change of charts in $\mathbb{CP}^2$, we denote by $\sigma$ the order of $y_1(t)$ at $t = 0$, which is the order of contact of $\Sigma$ with $L_\infty$ at $p$. Since $\Sigma \setminus \{p\}$ is invariant by $\mathcal{F}_X$, $\gamma^*X$ is a holomorphic vector field on $\mathbb{D}^*$ whose order at 0 is called the multiplicity of $\mathcal{F}_X$ with respect to $\Sigma$. We will denote it by $\text{ind}_p(\mathcal{F}_X, \Sigma)$. From now on, if no other conditions are explicitly given, $X$ will be a complete polynomial vector field on $\mathbb{C}^2$ of degree $m \geq 2$ with isolated zeroes.

**Lemma 1.2.** $\text{ind}_p(\mathcal{F}_X, \Sigma) - \sigma(m - 1) = 1$ or 2.
Proof. We obtain the extension of $X_{|\mathcal{L}\backslash \{p\}}$ to $p$ as $\gamma^*(\varphi_1 \circ \varphi_0^{-1})_*, X = f(t)\frac{\partial}{\partial t}$. Thus $(y_1(t))^{m-1} f(t)$ equals

\[ -\sum_{i=0}^{m} \frac{(y_1(t))^{m+1-i}}{y'_1(t)} \cdot P_i(1, y_2(t)), \quad \text{or} \quad \sum_{i=0}^{m} \frac{(y_1(t))^{m-i}}{y'_2(t)} \cdot G_i(1, y_2(t)), \]

where $P_i$ and $Q_i$ denote the homogeneous components of degree $i$ of $P$ and $Q$ respectively, and $G_i(y_2) = Q_i(1, y_2) - y_2 P_i(1, y_2)$. As $L_\infty$ is invariant by Corollary 1.1, $y_1^{m-1}(\varphi_1 \circ \varphi_0^{-1})_*, X$ represents $\mathcal{F}_X$ in $U_1$. Thus $ord_0 f(t) = ind_p(\mathcal{F}_X, \Sigma) - \sigma(m-1)$, and the result follows from Lemma 1.1.

\[ \square \]

1.1. Foliations with nonzero first jet singularities at infinity. We say that $\mathcal{F}_X$ has nonzero first jet at a singularity $p$ if the linear part at $p$ of a vector field $Y$ which represents $\mathcal{F}_X$ in a neighbourhood of $p$ is not zero. Let $\lambda$ and $\mu$ be the eigenvalues of $DY_p$ and suppose that $\lambda$ and $\mu$ are not both zero. Then, we say that $p$ is a saddle-node point if $\lambda \mu = 0$. If $\lambda / \mu \in \mathbb{Q}^+$, the singularity is either dicritical or of Poincaré-Dulac type: after a local analytic change of coordinates $Y$ is given by $x\frac{\partial}{\partial x} + (ny + x^n)\frac{\partial}{\partial y}$, with $n \in \mathbb{N}^+$. We will suppose that $p = (0, \alpha) \in Sing(\mathcal{F}_X) \cap L_\infty$. Let us rewrite the Jacobian $D(y_1^{m-1}(\varphi_1 \circ \varphi_0^{-1})_*, X)_p$ as

\[ (2) \quad J_p = \begin{pmatrix} -P_m(1, \alpha) & 0 \\ G_{m-1}(\alpha) & G'_m(\alpha) \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ \nu & \mu \end{pmatrix}. \]

Theorem 1.1. Let $p \in Sing(\mathcal{F}_X) \cap L_\infty$ be a point at which $\mathcal{F}_X$ has nonzero first jet. Let us suppose that $\lambda$ and $\mu$ are not both zero. Then,

(i) either $p$ is a saddle-node point and $L_\infty$ defines the strong direction, that is, $\lambda = 0, \mu \neq 0$;

(ii) or $p$ is of Poincaré-Dulac type.

Proof. We study the following cases:

1) If $det J_p = 0$, then $\lambda = 0$. To see this, we use Corollary 1.1, and observe that if $\lambda \neq 0$, $L_\infty$ is a smooth separatix tangent to the weak direction $\mu = 0$ and there is just one more smooth separatix $\Sigma$, tangent to the strong direction. $\Sigma$ is transversal to $L_\infty$ at $p$, so $ind_p(\mathcal{F}_X, \Sigma) = 1 < m$, contradicting Lemma 1.2. Then (i) holds, and there is at most one more separatix $\Sigma \neq L_\infty$, [11] pp. 521-522.

2) If $det J_p \neq 0$, then $\lambda / \mu \in \mathbb{Q}^+$, as otherwise there are exactly two transversal smooth separatrices through $p$ [11] pp. 518-521, and we get a contradiction as before. Moreover, $p$ is nondicritical. If not, take a separatix $\Sigma \neq L_\infty$, then $ind_p(\mathcal{F}_X, \Sigma) = 1 < 1 + \sigma(m-1)$, again a contradiction by Lemma 1.2. Thus $p$ is of Poincaré-Dulac type, [3].

Remark 1.3. Note that if $\Sigma \neq L_\infty$ is a separatix through $p$, then $p$ is a saddle-node point and $L_\infty$ defines the strong direction. Example (2): $X = z_1 \frac{\partial}{\partial z_1} - z_2(1 + z_1) \frac{\partial}{\partial z_2}$, with $p = (0 : 1 : 0) \in L_\infty$.

Corollary 1.2. For each $m \geq 2$, the set of degree $m$ foliations defined by complete polynomial vector fields is a nowhere dense set in $\mathcal{F}(m, 2)$.

Application: exploding orbits of polynomial Hamiltonians. Given $H \in \mathbb{C}[z_1, z_2]_m$, the space of polynomials of degree $\leq m$, we get a polynomial Hamiltonian, $X_H$. The (complex) orbit of a point $p \in \mathbb{C}^2$ is said to explode if it is unbounded on some $D^* \subset \mathbb{C}$. 

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Proposition 1.1. The existence of a dense set of points in \( \mathbb{C}^2 \) whose complex orbit explodes is a generic property in \( \mathbb{C}[z_1, z_2]_m \), \( m \geq 3 \).

Proof. Consider the Zariski open of \( \mathbb{C}[z_1, z_2]_m \) defined by \( W_m = \{ H \ | \ H_m = 0 \} \) defines \( m \) distinct points in \( \mathbb{C}P^1 \). For any \( H \in W_m \) and \( p = (0, \alpha) \in \text{Sing} \mathcal{F}_{X_H} \cap L_\infty \), as \( \partial H_m/\partial z_2(1, \alpha) \neq 0 \), \( \mathcal{L} \) is not 0. Since \( \mathcal{F}_{X_H} \) is given by the pencil defined by \( H \), and \( L_\infty \) is invariant, it can be taken to be dicritical. For each separatrix \( \Sigma \neq L_\infty \) through \( p \), \( X_{|C \setminus \{p\}} \) extends to \( p \) as a pole of order \( k \geq 1 \), Theorem 1.1.

Thus the norm of \( X \) is unbounded on \( \mathbb{D}_* \) and \( L \) explodes. \( \square \)

2. On the number of zeroes of a complete polynomial vector field

Proposition 2.1. Suppose that \( X \) has a rational first integral. Then, there exists a polynomial automorphism \( \varphi \in \text{Aut}[\mathbb{C}^2] \) such that

(i) If \( X \) is not singular, \( \varphi^*X = \frac{\partial}{\partial z_1} \); 

(ii) If \( X \) is singular, \( \varphi^*X = m z_1^{l} + n z_2 \partial / \partial z_2 \) where \( m, n \in \mathbb{Z}^* \).

Proof. Let \( H = F/G \) be a rational first integral of \( X \). By Stein’s factorization, we may assume that the generic fiber of \( H \) is connected, i.e., \( H \) is a primitive rational first integral. Since \( X \) is complete there exists a subset \( E \subset \mathbb{C}^2 \) of zero transverse logarithmic capacity, which is invariant by the flow of \( X \), and such that the orbits of \( X \) on \( \mathbb{C}^2 \setminus E \) are all isomorphic either to \( \mathbb{C} \) or to \( \mathbb{C}^* \) (see [15], [19]). We say that the generic orbit of \( X \) is \( \mathbb{C} \) or \( \mathbb{C}^* \), and also that \( H \) is of type \( \mathbb{C} \) or \( \mathbb{C}^* \).

- Assume that the generic orbit of \( X \) is \( \mathbb{C} \). Suppose that \( \{ H = 0 \} \simeq \mathbb{C} \), so that according to Abhyankar-Moh-Suzuki’s Theorem [16], there exists \( \varphi \in \text{Aut}[\mathbb{C}^2] \) with \( H \circ \varphi(z_1, z_2) = z_2 \). Therefore \( \varphi^*X = \frac{\partial}{\partial z_1} \).

- If the generic orbit of \( X \) is \( \mathbb{C}^* \), following an improvement of a theorem of Saito [17], after a polynomial automorphism \( \Phi \), we have that \( H \circ \Phi(z_1, z_2) = h \circ Q(z_1, z_2) \), where \( h \) is a rational function of degree one and either \( Q = (z_1^m(z_1^2 + p(z_1))^n) \), \( m, n \in \mathbb{Z}^* \), \( l \in \mathbb{N}^+ \), \( p(z_1) \) is a polynomial of degree \( \leq l - 1 \) with \( p(0) \neq 0 \), or \( Q(z_1, z_2) = z_1^m z_2^n \). In the first case, removing the one-dimensional singular locus of \( dQ, i_{\Phi, X}(dz_1 \wedge dz_2) \) equals

\[
\frac{\lambda(z_1^2 + p(z_1))^{-2m} h(Q)}{z_1^{m-1}(z_1^2 + p(z_1))^{n-1}} \quad \text{where} \quad a = 0 \quad \text{if} \quad m > 0, \\
b = 0 \quad \text{if} \quad n > 0, \\
1 = 1 \quad \text{if} \quad n < 0,
\]

and \( \lambda \in \mathbb{C}^* \). Thus \( \Phi^*X \) equals

\[
A z_1^{l+1} \frac{\partial}{\partial z_1} + (B z_1^l z_2 + C p(z_1) + D z_1 p'(z_1)) \frac{\partial}{\partial z_2} \quad \text{where} \quad A \in \mathbb{C}^* \text{ and } B, C, D \in \mathbb{C}.
\]

Let us consider the trajectory \( L = \{ z_1^l z_2 + p(z_1) = 0 \} \), and let \( \Sigma \) be the branch of \( L \) at \( (0 : 1 : 0) \), parametrized by \( \gamma(t) = (t, -\tilde{p}(1, t)) \), where \( \tilde{p}(x, z) \) is the homogenization of \( p \). Then, \( \text{ind}_{\tilde{p}}(\mathcal{F}_{\Phi, X}, \Sigma) = 1 < 1 + l \), and by Lemma 2.2 \( X_L \) is not complete.

If \( Q = z_1^m z_2^n \), taking \( \varphi = \frac{\partial}{\partial z_1} \Phi \), then \( \varphi^*X = m z_1 \frac{\partial}{\partial z_1} + n z_2 \frac{\partial}{\partial z_2}. \) \( \square \)

Proposition 2.2. Let \( p \) be a nondicritical zero of \( X \) (polynomial but not necessarily complete). If \( \Gamma \) is an irreducible algebraic invariant curve through \( p \) such that \( X_{|\Gamma \setminus \{p\}} \) is complete, then there exists \( \Phi \in \text{Aut}[\mathbb{C}^2] \) such that \( \Phi(\Gamma) \) is a line.
Proof. As $X_{\Gamma \setminus \{p\}}$ is complete and $\mathbb{C}^2$ is Stein, $\Gamma \setminus \{p\} \simeq \mathbb{C}^*$. Consider the unique branch of $\Gamma$ at $p$ and its parametrization $\gamma : \mathbb{D} \to \Gamma$. The extension of $\gamma^*X$ to $0$ has a zero of order $1$. Then $DX_p$ is not zero, and we denote by $\lambda$ and $\mu$ its eigenvalues.

- If $\lambda = \mu = 0$, after a linear change of coordinates

$$DX_p = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  

Suppose that $\Gamma$ is singular at $p$. There exists $\psi \in Aut(\mathbb{C}^2)$ such that $\psi(\Gamma) = \{z_1^k - a z_2 = 0, (k,l) = 1, a \in \mathbb{C}^*\}$. Then $\gamma(t) = (z^l, t^k)$, with $\varepsilon^k = a$, and $D(\psi, X)_p = D\psi_p \cdot DX_p \cdot D\psi_p^{-1}$, so we have that $\gamma^*(\psi, X) = \Delta(t)\frac{dz}{dt}$, where $\Delta(t)$ equals

$$ab d \varepsilon^l - b t^k + P(\varepsilon^l, t^k) \varepsilon^{l-1} \frac{dt}{\varepsilon t^k} = \frac{ab d \varepsilon^l - b t^k + Q(\varepsilon^l, t^k) \frac{dt}{\varepsilon t^k}}{kt^{l-1}},$$

where $a, b, c, d \in \mathbb{C}$, $a = (ad - bc)^{-1}$ is $\mathbb{C}^*$, and $P, Q \in \mathbb{C}[z_1, z_2]$ have order $\geq 2$ at $p$. If $bd \neq 0$ (the case $bd = 0$ is similar), as $\gamma^*(\psi, X)$ has a zero of order $1$, the orders of the numerators in (3) are $l$ and $k$, respectively. It should be $k > l$; otherwise, the term $-a b t^k$ is cancelled with one of the terms of $P(\varepsilon^l, t^k)$, and thus $k = j l$ with $j \geq 2$. But $k > l$ implies that $b d^2 \varepsilon^l$ is cancelled with one of the terms of $Q(\varepsilon^l, t^k)$, and hence $l = j k$ with $j \geq 2$, a contradiction.

- If $\lambda \mu \in \mathbb{Q}^+$, as $p$ is nondicritical, $p$ is of Poincaré-Dulac type [3], and hence $\Gamma$ is smooth at $p$.

- If $\lambda / \mu \notin \mathbb{Q}^+$, or $\lambda \neq 0$ and $\mu = 0$, according to [11] pp. 518–522 $\Gamma$ is smooth at $p$.

By [10], there exists $\Phi \in Aut(\mathbb{C}^2)$ such that $\Phi(\Gamma)$ is a line. 

Let $\Sigma$ be a separatrix through a zero $p$ of $X$. Consider the orbit $L$ defined extending $\Sigma \setminus \{p\}$. As $\mathbb{C}^2$ is Stein, $L \simeq \mathbb{C}^*$. Thus $L$ has two planar isolated ends; one defined by $\Sigma \setminus \{p\}$ and the other by $L \setminus \Sigma$. If the end defined by $L \setminus \Sigma$ is algebraic (transcendental), one says that $\Sigma$ is algebraic (transcendental) at infinity (see definitions in [4]).

Proposition 2.3. Either $L$ is defined by the (unique) local branch at $p$ of an algebraic curve $\Gamma \subset \mathbb{C}^2$, such that $\Gamma \setminus \{p\} \simeq L$, or $L \setminus \Sigma$ defines a planar isolated end which is properly imbedded and transcendental.

Proof. Take $x \in L$ and let $j : \mathbb{C} \to \mathbb{C}^2$ be the map $j(t) = \varphi(t, x)$, where $\varphi$ is the flow of $X$. We know that its analytic closure $\overline{L} \subset \mathbb{C}^2$ is of pure dimension $1$, [19]. Then $L$ is properly embedded in $\mathbb{C}^2$ ($j$ is proper). If $L \setminus \Sigma$ is not transcendental, then $L$ defines a separatrix through the point $r = \lim(L \setminus \Sigma) \in \operatorname{Sing}(\mathcal{F}_X) \cap L_\infty$. Therefore $\overline{L} \cup \{r\} \simeq \mathbb{CP}^1$ is an algebraic curve by Chow’s Theorem. 

Theorem 2.1. $X$ has at most one zero in $\mathbb{C}^2$.

Proof. Suppose that $p_1 \neq p_2$ are zeroes of $X$. By [6], there exists a separatrix $\Sigma_i$ through $p_i$, $i = 1, 2$. First assume that each $\Sigma_i$ is algebraic at infinity through a nondicritical $p_i$. Let $\Phi \in Aut(\mathbb{C}^2)$, given in Proposition [22], such that $\Phi(\Sigma_1)$ is a line $L_{\Phi(p_1)}$ through $\Phi(p_1)$. Let $\overline{C_2}$ be the closure of $C_2 := \Phi(\Sigma_2)$ in $\mathbb{CP}^2$. Thus $L_{\Phi(p_1)} \cap C_2 = \{r\} \subset L_\infty$; otherwise if $\alpha : \mathbb{P}^2 \to C_2$ is the resolution of $\overline{C_2}$, $\alpha^*X$ extends to $Z_2$ with at least three zeroes, which is a contradiction. Analogously, $L_\infty \cap \overline{C_2} = \{r\}$. As $L_{\Phi(p_1)}$ and $L_\infty$ just intersect $\overline{C_2}$ at $r$, $\overline{C_2}$ has to be a line as it cannot have two branches at $r$. Suppose that $L_{\Phi(p_1)} = \{z_1 = a\}$ and $L_{\Phi(p_2)} :=$
$C_2 = \{z_1 = b\}$. The orbit of $(z_1^0, z_2^0) \in \mathbb{C}^2$ with $a \neq z_1^0 \neq b$ is defined by the image of the entire map $\varphi(x, y) = \varphi(t, z_1^0, z_2^0) = (z_1(t), z_2(t))$, where $\varphi$ is the flow of $\Phi, X$. Since $z_1(\mathbb{C}) \subset \mathbb{C} \setminus \{a, b\}$, by Picard’s Theorem $z_1(t) \equiv k \in \mathbb{C}$, and thus $\varphi(z_1^0, z_2^0)(\mathbb{C})$ is contained in a line parallel to both $L_{\Phi(p_1)}$ and $L_{\Phi(p_2)}$, and hence $\Phi, X = \frac{\partial}{\partial z_1^0}$, a contradiction.

Observe that if $p_1$ is dicritical, $\Sigma_i$ can be taken to be transcendental at infinity. Otherwise Darboux’s Theorem and Proposition 2.1 imply that $X$ has at most one zero. Thus it only remains to analyze the case when $\Sigma_i$ is transcendental at infinity. Now, we take from [4] the notion of $P$-completeness, that will be used in what follows. Let $P : \mathbb{C}^2 \to \mathbb{C}$ be a nonconstant polynomial. $\mathcal{F}_X$ is $P$-complete if there exists a finite set $Q \subset \mathbb{C}$ such that, for all $t \not\in Q$, $P^{-1}(t)$ is transverse to $\mathcal{F}_X$ and there is a neighbourhood $U_t$ of $t$ in $\mathbb{C}$ such that $P_1^{-1}(U_t)$ is a fibration and $\mathcal{F}_X|_{P^{-1}(U_t)}$ defines a local trivialization on it. Thus, if one $\Sigma_i$ is transcendental at infinity, it follows from [4] that there is a nonconstant (primitive) polynomial $P : \mathbb{C}^2 \to \mathbb{C}$ of type $\mathbb{C}$ or $\mathbb{C}^*$ such that $\mathcal{F}_X$ is $P$-complete. The set of points where $\mathcal{F}_X$ is not transverse to $P$ is an algebraic curve $S \subset P^{-1}(Q)$, so $p_i \in S$. If $P$ is of type $\mathbb{C}$, since by [16] there is $\varphi \in \text{Aut}[\mathbb{C}^2]$ such that $P \circ \varphi(z_1, z_2) = z_1$, one sees as above that $p_i \in \{z_1 = \lambda\}$, for $i = 1, 2$, again a contradiction.

On the other hand, if $P$ is of type $\mathbb{C}^*$ by [17], as noted in the proof of Proposition 2.1 after a polynomial automorphism $\varphi$, $P \circ \varphi$ can be easily written and since $(P \circ \varphi)^{-1}(\lambda) \simeq \mathbb{C}^*$ for all $\lambda \neq 0$, and $X$ is complete on each component of $S$, one has that $p_i \in (P \circ \varphi)^{-1}(0)$. Therefore $S \cap (P \circ \varphi)^{-1}(0) = \{z_1 = 0\}$ or $\{z_1 z_2 = 0\}$, but in both cases one has two zeroes on an invariant line of $X$, a contradiction. □

Theorem 2.2. Let $p$ be a zero of $X$ which is not of Poincaré-Dulac type, and let $S$ be the set of separatrices through $p$ that are algebraic at infinity. Up to polynomial automorphism,

(1) when $p$ is dicritical and all the separatrices through it belong to $S$: $X = mz_1 \frac{\partial}{\partial z_1} + nz_2 \frac{\partial}{\partial z_2}$ where $m, n \in \mathbb{Z}^*$, and $mn < 0$;

(2) when $p$ is nondicritical, but $\sharp S \geq 2$: $X = z_1(\lambda + qf(z_1^0 z_2^0)) \frac{\partial}{\partial z_1} + z_2(\mu + pf(z_1^0 z_2^0)) \frac{\partial}{\partial z_2}$, where $f \in \mathbb{C}[z]$, $p, q \in \mathbb{N}$ and $\lambda \mu \in \mathbb{C}^*$.

If there is at least one separatrix $\Sigma \not\in S$, then $\sharp S \geq 1$, and either

(3) $X = \lambda z_1 \frac{\partial}{\partial z_1} + (a(z_1) + b(z_1)z_2) \frac{\partial}{\partial z_2}$, with $a, b \in \mathbb{C}[z_1]$ and $b(0) \lambda \in \mathbb{C}^*$, or

(4) $\mathcal{F}_X$ is $P$-complete for a polynomial $P = (z_1^m (z_1^2 + p(z_1))^n)$, where $m, n, l \in \mathbb{N}^+$, $p \in \mathbb{C}[z_1]$ of degree $\leq l - 1$ with $p(0) \neq 0$, or $P = z_1^m z_2^n$.

Proof. By Theorem 2.1 $p$ is the unique zero of $X$ in $\mathbb{C}^2$. Suppose that $p = (0, 0)$. Since the restriction of $X$ to any open neighbourhood of $p$ is semicomplete, [13], one has that $\lambda \mu \neq 0$, [14].

We can distinguish two cases. Suppose that $p$ is dicritical. Then, if there is a rational first integral, one has by Proposition 2.1 that $X$ can be written as in (1). If there is no rational first integral, there is a separatrix through $p$, $\Sigma$, which is transcendental at infinity, and according to [4], the foliation $\mathcal{F}_X$ is $P$-complete, with $P$ of type $\mathbb{C}$ or $\mathbb{C}^*$. As noted in the proof of Theorem 2.1 if $P$ is of type $\mathbb{C}$, then $P = z_1$ and it follows that $X$ is as in (3), while if it is of type $\mathbb{C}^*$, then $P$ can be written so that it reads as in (4).

Assume now that $p$ is nondicritical. As $p$ is not of Poincaré-Dulac type, there are at least two separatrices through $p$, [3] and [11]. If at least two of them
are algebraic at infinity, according to Propositions 2.2 and 2.3 they are defined by the smooth algebraic curves $\Gamma_1 = \{P_1 = 0\}$ and $\Gamma_2 = \{P_2 = 0\}$. Consider the simply connected algebraic curve $\Gamma_1 \cup \Gamma_2 = \{P_1P_2 = 0\}$. After a polynomial automorphism $\Phi$, $\Phi(\Gamma_1 \cup \Gamma_2) = \{z_1^2z_2^2 = 0\}$, [21]. Moreover, since $\Phi_*X$ is complete on $\mathbb{C}^2 \setminus \{(z_1 = 0) \cup \{z_2 = 0\}\} \simeq (\mathbb{C}^*)^2$, the classification of such vector fields, [2], shows that $\Phi_*X$ takes the form (2) of the statement. The nonexistence of two separatrices through $p$ algebraic at infinity implies the existence of at least one $\Sigma$ which is transcendental at infinity, and by [4]; $X$ can be expressed as it is pointed out in (3) or (4).

3. Completeness and the Jacobian Conjecture

Jacobian Conjecture. If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map such that $\det(JF) \in \mathbb{C}^*$, then $F$ is invertible, that is, $F$ has an inverse which is also a polynomial map.

In fact, from a theorem due to Bialynicki-Birula and Rosenlicht, [20], if $F$ is injective it is surjective, and the inverse is a polynomial map. Thus the Jacobian Conjecture is equivalent to: “if $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map such that $\det(JF) \in \mathbb{C}^*$, then $F$ is injective”. The conjecture is true for $n = 1$, and is an open problem for $n \geq 2$.

Following Nousiainen and Sweedler, [20], we can associate to $F = (F_1, \ldots, F_n)$ $n$ polynomial vector fields on $\mathbb{C}^n$, $\partial F_1, \ldots, \partial F_n$, defined by

$$\left( \frac{\partial}{\partial F_1}, \ldots, \frac{\partial}{\partial F_n} \right) := \left( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \right) (JF)^{-1},$$

with the following properties:

1. They are $\mathbb{C}$-linearly independent on $\mathbb{C}^n$.

2. $\mathcal{L} \frac{\partial}{\partial F_j} F_j = DF_j \left( \frac{\partial}{\partial F_i} \right) = \delta_{ij}$ and $\left[ \frac{\partial}{\partial F_i}, \frac{\partial}{\partial F_j} \right] = 0$, with $1 \leq i, j \leq n$.

Therefore, we obtain for each $i = 1, \ldots, n$ a nonsingular algebraic foliation by curves in $\mathbb{C}^n$ defined by the vector field $\frac{\partial}{\partial F_i}$ whose leaves are given by the intersection of the level sets of $F_j$, $j \neq i$.

**Theorem 3.1.** Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $\det(JF) \in \mathbb{C}^*$. Then $F$ is injective if and only if the polynomial vector fields $\frac{\partial}{\partial F_i}$, $i = 1, \ldots, n$, are complete.

**Proof.** Suppose that $F$ is injective. If $F_i(z_1, \ldots, z_n) = w_i$, $i = 1, \ldots, n$, as the vector fields $\frac{\partial}{\partial w_i}$ are complete, $\frac{\partial}{\partial F_i} = F^*\frac{\partial}{\partial w_i}$ are also complete.

Conversely, if $\frac{\partial}{\partial F_i}$, $i = 1, \ldots, n$, are complete, and there are two different points $p, q \in \mathbb{C}^n$ such that $F(p) = F(q) = \alpha = (\alpha_1, \ldots, \alpha_n)$, there are $n$ leaves $L_i = \bigcap_j \{F_j = \alpha_j\}$, one of each foliation induced by $\frac{\partial}{\partial F_i}$, having at least two different points $p$ and $q$ of intersection.

But each leaf $L_i$, $i = 1, \ldots, n$, is equipped with a holomorphic 1–form $DF_i|_{L_i}$ such that $DF_i|_{L_i}(\frac{\partial}{\partial F_i}) = 1$. Following [13], since $\frac{\partial}{\partial F_i}$ is complete the 1–form $DF_i|_{L_i}$ is defined by the “dérivée du temps”, which is locally given by its flow.

Let us fix $i_0 \in \{1, \ldots, n\}$, and an injective smooth path $c_{i_0} : [0, 1] \to L_{i_0}$ from $p$ to $q$. The integral of $DF_{i_0}|_{L_{i_0}}$ along $c_{i_0}$ has to be nonzero [13], but $\int_{c_{i_0}} DF_{i_0}|_{L_{i_0}} = F_{i_0}(q) - F_{i_0}(p) = 0$. Thus $F$ is injective.
After our work was completed, we noticed that Theorem 3.1 was proved by Meisters and Olech in [12] for the real case in another context. We denote by $\mathcal{D}$ the $\mathbb{C}[z_1, \ldots, z_n]$-module of all $\mathbb{C}$-derivations of $\mathbb{C}[z_1, \ldots, z_n]$. It is well known that $\mathcal{D}$ is free and of rank $n$. A basis is said to be commutative when $[X_i, X_j] = 0$, $0 \leq i, j \leq n$. If each $X_i$ is complete, we will say that it is complete.

**Proposition 3.1.** A commutative basis $(X_1, \ldots, X_n)$ of $\mathcal{D}$ is complete if and only if there exists a polynomial automorphism $F$ of $\mathbb{C}^n$ such that $F_i = \partial F_i / \partial z_i$, $i = 1, \ldots, n$.

**Proof.** Suppose first that $(X_1, \ldots, X_n)$ is a complete commutative basis. Then by a result of A. Nowicki [20], there exists a polynomial map $F = (F_1, \ldots, F_n)$ with $\det(JF) \in \mathbb{C}^*$ such that $X_i = \partial F_i / \partial z_i$. Thus by Theorem 3.1 $F$ is a polynomial automorphism such that $X_i = (F^{-1})_* \partial / \partial z_i$.

Now, suppose that there is a polynomial automorphism $F$ of $\mathbb{C}^n$ such that $F_i X_i = \partial / \partial z_i$. Then, the $X_i$ are complete and moreover $[X_i, X_j] = (F^{-1})_* \left[ \partial / \partial z_i, \partial / \partial z_j \right] = 0$, thus proving the converse. \hfill $\square$

**Corollary 3.1.** The Jacobian Conjecture holds if and only if every commutative basis of $\mathcal{D}$ is complete.

**References**


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