

ZEROES OF COMPLETE POLYNOMIAL VECTOR FIELDS

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Dedicated to my father

ABSTRACT. We prove that a complete polynomial vector field on \mathbb{C}^2 has at most one zero, and analyze the possible cases of those with exactly one which is not of Poincaré-Dulac type. We also obtain the possible nonzero first jet singularities of the foliation \mathcal{F}_X at infinity and the nongenericity of completeness. Connections with the Jacobian Conjecture are established.

INTRODUCTION AND RESULTS

Let $X = P(z_1, z_2)\frac{\partial}{\partial z_1} + Q(z_1, z_2)\frac{\partial}{\partial z_2}$ be a polynomial vector field on \mathbb{C}^2 of degree $m = \max\{\deg P, \deg Q\} \geq 2$ with isolated zeroes. It is known, [9], that X extends as a rational vector field in $\mathbb{C}\mathbb{P}^2$ having a pole along the line at infinity, L_∞ . Removing the pole, we obtain a foliation \mathcal{F}_X of degree d , where $d = m$ if L_∞ is invariant and $d = m - 1$ if it is not. We denote by $Sing(\mathcal{F}_X)$ the singular set of \mathcal{F}_X .

Recall that a holomorphic vector field X in a complex manifold M is said to be complete if, for every $p \in M$, the differential equation defined by X can be solved for every complex time t .

In this paper we study complete polynomial vector fields X on \mathbb{C}^2 through some properties of the leaves of \mathcal{F}_X . In section 1, we analyze the trajectories of X at infinity and we give in Theorem 1.1 the possible nonzero first jet of \mathcal{F}_X at its singular points in L_∞ , thus proving Corollary 1.1: foliations induced by complete polynomial vector fields of degree m give a nowhere dense set in the space of degree m foliations, $\mathcal{F}(m, 2)$, providing a polynomial version of Buzzard-Fornaess's result, [5]. We also apply our results to the problem of exploding orbits of complex polynomial Hamiltonians, obtaining a simple geometric proof of Fornæss and Grellier's result, [8], in that case.

In section 2, we further study the isolated zeroes of X . A natural question (posed in [1] and [15]) is if there exist complete holomorphic vector fields on \mathbb{C}^2 with more than one isolated zero. The answer, given in Theorem 2.1, is no for polynomial ones. Our result relies on the study of proper orbits due to Brunella in [4]. We also classify the complete polynomial vector fields with rational first integral and, using

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Andersen's result in [2], those with one zero p which is not of Poincaré-Dulac type, when it is nondicritical and at least two of the separatrices through it are algebraic at infinity. When p is dicritical with no rational first integral, or nondicritical with just one separatrix algebraic at infinity, the induced foliation \mathcal{F}_X is, as in Brunella's result [4], P -complete where P can be written in a simple form due to [17] and [16] (Proposition 2.1 and Theorem 2.2).

In section 3, we state the Jacobian Conjecture in terms of completeness of certain vector fields, and characterize the complete commutative bases of \mathbb{C} -derivations of the polynomial ring.

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1. SEPARATRICES AT INFINITY AND COMPLETENESS

A germ Σ of an analytic irreducible curve is said to be a trajectory of X at $p \in L_\infty$ if $p \in \Sigma$ and $\Sigma \setminus \{p\}$ is invariant by X . In this case one can extend $\Sigma \setminus \{p\}$ by analytic continuation to obtain the complex orbit L of X . If $\gamma : \mathbb{D} \rightarrow \Sigma$ is the (minimal) Puiseux's parametrization of a neighborhood U_p of p in Σ , $\mathcal{L} = L \cup \{p\}$ can be endowed with an abstract Riemann surface structure as follows: for any $q \in L$, by the existence of local solutions for X , we can take the parametrization γ_q of an open neighborhood $U_q \subset L$, and define the local chart as $z_q = \gamma_q^{-1} : U_q \rightarrow \mathbb{C}$. Otherwise, γ defines the local chart around p in \mathcal{L} as $z_p = \gamma^{-1} : \Sigma \rightarrow \mathbb{D}$.

Lemma 1.1. *Let X be a polynomial vector field in \mathbb{C}^2 , and let Σ be a trajectory of X at $p \in L_\infty$. Then, if X is complete on $\mathcal{L} \setminus \{p\}$, it extends to p as a zero of order 1 or 2.*

Proof. As $\mathcal{L} \setminus \{p\}$ is uniformized by \mathbb{C} , and it is contained in the Stein manifold \mathbb{C}^2 , then $\mathcal{L} \setminus \{p\}$ is (analytically) isomorphic to \mathbb{C} or \mathbb{C}^* . If $\mathcal{L} \setminus \{p\} \simeq \mathbb{C}$, it follows that $\mathcal{L} \simeq \mathbb{CP}^1$ and X extends to p as zero of order 2, by Riemann-Roch. On the other hand, if $\mathcal{L} \setminus \{p\} \simeq \mathbb{C}^*$, then X extends to p as zero of order 1. We refer to [10] for the study of complete vector fields on Riemann surfaces. \square

Corollary 1.1. *If X is a complete polynomial vector field on \mathbb{C}^2 , then L_∞ is invariant by \mathcal{F}_X .*

Remark 1.1. If $\mathcal{L} \setminus \{p\} \simeq \mathbb{C}$, by Chow's Theorem $\mathcal{L} \setminus \{p\}$ is contained in a rational curve.

Remark 1.2. Lemma 1.1 is valid for polynomial vector fields on \mathbb{C}^n , $n \geq 2$.

Let $p \in \text{Sing}(\mathcal{F}_X) \cap L_\infty$, and let $\Sigma \neq L_\infty$ be a separatrix of \mathcal{F}_X through p , parametrized by $\gamma : \mathbb{D} \rightarrow \Sigma$. Without loss of generality assume that $p = (0 : 1 : 0)$. Then if $\gamma(t) = (y_1(t), y_2(t))$, with $(y_1, y_2) = (\varphi_1 \circ \varphi_0^{-1})(z_1, z_2) = (\frac{1}{z_1}, \frac{z_2}{z_1})$ the usual change of charts in \mathbb{CP}^2 , we denote by σ the order of $y_1(t)$ at $t = 0$, which is the order of contact of Σ with L_∞ at p . Since $\Sigma \setminus \{p\}$ is invariant by \mathcal{F}_X , γ^*X is a holomorphic vector field on \mathbb{D}^* whose order at 0 is called the multiplicity of \mathcal{F}_X with respect to Σ . We will denote it by $\text{ind}_p(\mathcal{F}_X, \Sigma)$. From now on, if no other conditions are explicitly given, X will be a complete polynomial vector field on \mathbb{C}^2 of degree $m \geq 2$ with isolated zeroes.

Lemma 1.2. $\text{ind}_p(\mathcal{F}_X, \Sigma) - \sigma(m - 1) = 1$ or 2.

Proof. We obtain the extension of $X|_{\mathcal{L}\setminus\{p\}}$ to p as $\gamma^*(\varphi_1 \circ \varphi_0^{-1})_*X = f(t)\frac{\partial}{\partial t}$. Thus $(y_1(t))^{m-1}f(t)$ equals

$$(1) \quad - \sum_{i=0}^m \frac{(y_1(t))^{m+1-i}}{y_1'(t)} \cdot P_i(1, y_2(t)), \text{ or } \sum_{i=0}^m \frac{(y_1(t))^{m-i}}{y_2'(t)} \cdot G_i(y_2(t)),$$

where P_i and Q_i denote the homogeneous components of degree i of P and Q respectively, and $G_i(y_2) = Q_i(1, y_2) - y_2P_i(1, y_2)$. As L_∞ is invariant by Corollary 1.1, $y_1^{m-1}(\varphi_1 \circ \varphi_0^{-1})_*X$ represents \mathcal{F}_X in U_1 . Thus $ord_0f(t) = ind_p(\mathcal{F}_X, \Sigma) - \sigma(m - 1)$, and the result follows from Lemma 1.1. \square

1.1. Foliations with nonzero first jet singularities at infinity. We say that \mathcal{F}_X has nonzero first jet at a singularity p if the linear part at p of a vector field Y which represents \mathcal{F}_X in a neighbourhood of p is not zero. Let λ and μ be the eigenvalues of DY_p and suppose that λ and μ are not both zero. Then, we say that p is a saddle-node point if $\lambda\mu = 0$. If $\lambda/\mu \in \mathbb{Q}^+$, the singularity is either dicritical or of Poincaré-Dulac type: after a local analytic change of coordinates Y is given by $x\frac{\partial}{\partial x} + (ny + x^n)\frac{\partial}{\partial y}$, with $n \in \mathbb{N}^+$ [3]. We will suppose that $p = (0, \alpha) \in Sing(\mathcal{F}_X) \cap L_\infty$. Let us rewrite the jacobian $D(y_1^{m-1}(\varphi_1 \circ \varphi_0^{-1})_*X)_p$ as

$$(2) \quad J_p = \begin{pmatrix} -P_m(1, \alpha) & 0 \\ G_{m-1}(\alpha) & G'_m(\alpha) \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ \nu & \mu \end{pmatrix}.$$

Theorem 1.1. *Let $p \in Sing(\mathcal{F}_X) \cap L_\infty$ be a point at which \mathcal{F}_X has nonzero first jet. Let us suppose that λ and μ are not both zero. Then,*

- (i) *either p is a saddle-node point and L_∞ defines the strong direction, that is, $\lambda = 0, \mu \neq 0$;*
- (ii) *or p is of Poincaré-Dulac type.*

Proof. We study the following cases:

1) If $detJ_p = 0$, then $\lambda = 0$. To see this, we use Corollary 1.1, and observe that if $\lambda \neq 0$, L_∞ is a smooth separatrix tangent to the weak direction $\mu = 0$ and there is just one more smooth separatrix Σ , tangent to the strong direction. Σ is transversal to L_∞ at p , so $ind_p(\mathcal{F}_X, \Sigma) = 1 < m$, contradicting Lemma 1.2. Then (i) holds, and there is at most one more separatrix $\Sigma \neq L_\infty$, [11, pp. 521–522].

2) If $detJ_p \neq 0$, then $\lambda/\mu \in \mathbb{Q}^+$, as otherwise there are exactly two transversal smooth separatrices through p [11, pp. 518–521], and we get a contradiction as before. Moreover, p is nondicritical. If not, take a separatrix $\Sigma \neq L_\infty$, then $ind_p(\mathcal{F}_X, \Sigma) = 1 < 1 + \sigma(m - 1)$, again a contradiction by Lemma 1.2. Thus p is of Poincaré-Dulac type, [3]. \square

Remark 1.3. Note that if $\Sigma \neq L_\infty$ is a separatrix through p , then p is a saddle-node point and L_∞ defines the strong direction. Example ([7]): $X = z_1\frac{\partial}{\partial z_1} - z_2(1+z_1)\frac{\partial}{\partial z_2}$, with $p = (0 : 1 : 0) \in L_\infty$.

Corollary 1.2. *For each $m \geq 2$, the set of degree m foliations defined by complete polynomial vector fields is a nowhere dense set in $\mathcal{F}(m, 2)$.*

Application: exploding orbits of polynomial Hamiltonians. Given $H \in \mathbb{C}[z_1, z_2]_m$, the space of polynomials of degree $\leq m$, we get a polynomial Hamiltonian, X_H . The (complex) orbit of a point $p \in \mathbb{C}^2$ is said to explode if it is unbounded on some $\mathbb{D}^* \subset \mathbb{C}$.

Proposition 1.1. *The existence of a dense set of points in \mathbb{C}^2 whose complex orbit explodes is a generic property in $\mathbb{C}[z_1, z_2]_m$, $m \geq 3$.*

Proof. Consider the Zariski open of $\mathbb{C}[z_1, z_2]_m$ defined by $W_m = \{H \mid H_m = 0 \text{ defines } m \text{ distinct points in } \mathbb{C}\mathbb{P}^1\}$. For any $H \in W_m$ and $p = (0, \alpha) \in \text{Sing}\mathcal{F}_{X_H} \cap L_\infty$, as $\partial H_m / \partial z_2(1, \alpha) \neq 0$, (2) is not 0. Since \mathcal{F}_{X_H} is given by the pencil defined by H , and L_∞ is invariant, p can be taken to be dicritical. For each separatrix $\Sigma \neq L_\infty$ through p , $X|_{\mathcal{L} \setminus \{p\}}$ extends to p as a pole of order $k \geq 1$, Theorem 1.1. Thus the norm of X is unbounded on \mathbb{D}^* and L explodes. \square

2. ON THE NUMBER OF ZEROES OF A COMPLETE POLYNOMIAL VECTOR FIELD

Proposition 2.1. *Suppose that X has a rational first integral. Then, there exists a polynomial automorphism $\varphi \in \text{Aut}[\mathbb{C}^2]$ such that*

- (i) *If X is not singular, $\varphi^*X = \frac{\partial}{\partial z_1}$.*
- (ii) *If X is singular, $\varphi^*X = mz_1 \frac{\partial}{\partial z_1} + nz_2 \frac{\partial}{\partial z_2}$ where $m, n \in \mathbb{Z}^*$.*

Proof. Let $H = F/G$ be a rational first integral of X . By Stein’s factorization, we may assume that the generic fiber of H is connected, i.e., H is a primitive rational first integral. Since X is complete there exists a subset $E \subset \mathbb{C}^2$ of zero transverse logarithmic capacity, which is invariant by the flow of X , and such that the orbits of X on $\mathbb{C}^2 \setminus E$ are all isomorphic either to \mathbb{C} or to \mathbb{C}^* (see [18], [19]). We say that the generic orbit of X is \mathbb{C} or \mathbb{C}^* , and also that H is of type \mathbb{C} or \mathbb{C}^* .

- Assume that the generic orbit of X is \mathbb{C} . Suppose that $\{H = 0\} \simeq \mathbb{C}$, so that according to Abhyankar-Moh-Suzuki’s Theorem [16], there exists $\varphi \in \text{Aut}[\mathbb{C}^2]$ with $H \circ \varphi(z_1, z_2) = z_2$. Therefore $\varphi^*X = \frac{\partial}{\partial z_1}$.

- If the generic orbit of X is \mathbb{C}^* , following an improvement of a theorem of Saito [17], after a polynomial automorphism Φ , we have that $H \circ \Phi(z_1, z_2) = h \circ Q(z_1, z_2)$, where h is a rational function of degree one and either $Q = (z_1^m(z_1^l z_2 + p(z_1))^n)$, $m, n \in \mathbb{Z}^*$, $l \in \mathbb{N}^+$, $p(z_1)$ is a polynomial of degree $\leq l - 1$ with $p(0) \neq 0$, or $Q(z_1, z_2) = z_1^m z_2^n$. In the first case, removing the one-dimensional singular locus of dQ , $i_{\Phi^*X}(dz_1 \wedge dz_2)$ equals

$$\frac{\lambda(z_1^{-2m})^a ((z_1^l z_2 + p(z_1))^{-2n})^b}{z_1^{m-1} (z_1^l z_2 + p(z_1))^{n-1} h'(Q)} dQ, \text{ where } \begin{cases} a = 0 & \text{if } m > 0, \\ a = 1 & \text{if } m < 0, \\ b = 0 & \text{if } n > 0, \\ b = 1 & \text{if } n < 0, \end{cases}$$

and $\lambda \in \mathbb{C}^*$. Thus Φ^*X equals

$$Az_1^{l+1} \frac{\partial}{\partial z_1} + (Bz_1^l z_2 + Cp(z_1) + Dz_1 p'(z_1)) \frac{\partial}{\partial z_2}, \quad A \in \mathbb{C}^* \text{ and } B, C, D \in \mathbb{C}.$$

Let us consider the trajectory $L = \{z_1^l z_2 + p(z_1) = 0\}$, and let Σ be the branch of \bar{L} at $(0 : 1 : 0)$, parametrized by $\gamma(t) = (t, -\tilde{p}(1, t))$, where $\tilde{p}(x, z)$ is the homogenization of p . Then, $\text{ind}_p(\mathcal{F}_{\Phi^*X}, \Sigma) = 1 < 1 + l$, and by Lemma 1.2 $X|_L$ is not complete.

If $Q = z_1^m z_2^n$, taking $\varphi = \frac{1}{\lambda} \Phi$, then $\varphi^*X = mz_1 \frac{\partial}{\partial z_1} + nz_2 \frac{\partial}{\partial z_2}$. \square

Proposition 2.2. *Let p be a nondicritical zero of X (polynomial but not necessarily complete). If Γ is an irreducible algebraic invariant curve through p such that $X|_{\Gamma \setminus \{p\}}$ is complete, then there exists $\Phi \in \text{Aut}[\mathbb{C}^2]$ such that $\Phi(\Gamma)$ is a line.*

Proof. As $X_{|\Gamma \setminus \{p\}}$ is complete and \mathbb{C}^2 is Stein, $\Gamma \setminus \{p\} \simeq \mathbb{C}^*$. Consider the unique branch of Γ at p and its parametrization $\gamma : \mathbb{D} \rightarrow \Gamma$. The extension of γ^*X to 0 has a zero of order 1. Then DX_p is not zero, and we denote by λ and μ its eigenvalues.

- If $\lambda = \mu = 0$, after a linear change of coordinates

$$DX_p = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Suppose that Γ is singular at p . There exists $\psi \in \text{Aut}[\mathbb{C}^2]$ such that $\psi(\Gamma) = \{z_1^k - az_2^l = 0, (k, l) = 1, a \in \mathbb{C}^*\}$, [21]. Then $\gamma(t) = (\varepsilon t^l, t^k)$, with $\varepsilon^k = a$, and $D(\psi_*X)_p = D\psi_p \cdot DX_p \cdot D\psi_p^{-1}$, so we have that $\gamma^*(\psi_*X) = \Delta(t) \frac{\partial}{\partial t}$, where $\Delta(t)$ equals

$$(3) \quad \frac{\alpha b(d\varepsilon t^l - bt^k) + P(\varepsilon t^l, t^k)}{\varepsilon l t^{l-1}} = \frac{\alpha d(d\varepsilon t^l - bt^k) + Q(\varepsilon t^l, t^k)}{k t^{k-1}},$$

where $a, b, c, d \in \mathbb{C}$, $\alpha = (ad - bc)^{-1} \in \mathbb{C}^*$, and $P, Q \in \mathbb{C}[z_1, z_2]$ have order ≥ 2 at p . If $bd \neq 0$ (the case $bd = 0$ is similar), as $\gamma^*(\psi_*X)$ has a zero of order 1, the orders of the numerators in (3) are l and k , respectively. It should be $k > l$; otherwise, the term $-\alpha b^2 t^k$ is cancelled with one of the terms of $P(\varepsilon t^l, t^k)$, and thus $k = jl$ with $j \geq 2$. But $k > l$ implies that $\alpha d^2 \varepsilon t^l$ is cancelled with one of the terms of $Q(\varepsilon t^l, t^k)$, and hence $l = jk$ with $j \geq 2$, a contradiction.

- If $\lambda/\mu \in \mathbb{Q}^+$, as p is nondicritical, p is of Poincaré-Dulac type [3], and hence Γ is smooth at p .
- If $\lambda/\mu \notin \mathbb{Q}^+$, or $\lambda \neq 0$ and $\mu = 0$, according to [11, pp. 518–522] Γ is smooth at p .

By [16], there exists $\Phi \in \text{Aut}[\mathbb{C}^2]$ such that $\Phi(\Gamma)$ is a line. □

Let Σ be a separatrix through a zero p of X . Consider the orbit L defined extending $\Sigma \setminus \{p\}$. As \mathbb{C}^2 is Stein, $L \simeq \mathbb{C}^*$. Thus L has two planar isolated ends; one defined by $\Sigma \setminus \{p\}$ and the other by $L \setminus \Sigma$. If the end defined by $L \setminus \Sigma$ is algebraic (transcendental), one says that Σ is *algebraic (transcendental) at infinity* (see definitions in [4]).

Proposition 2.3. *Either L is defined by the (unique) local branch at p of an algebraic curve $\Gamma \subset \mathbb{C}^2$, such that $\Gamma \setminus \{p\} \simeq L$, or $L \setminus \Sigma$ defines a planar isolated end which is properly imbedded and transcendental.*

Proof. Take $x \in L$ and let $j : \mathbb{C} \rightarrow \mathbb{C}^2$ be the map $j(t) = \varphi(t, x)$, where φ is the flow of X . We know that its analytic closure $\overline{L} \subset \mathbb{C}^2$ is of pure dimension 1, [19]. Then L is properly embedded in \mathbb{C}^2 (j is proper). If $L \setminus \Sigma$ is not transcendental, then L defines a separatrix through the point $r = \lim(L \setminus \Sigma) \in \text{Sing}(\mathcal{F}_X) \cap L_\infty$. Therefore $\overline{L} \cup \{r\} \simeq \mathbb{C}\mathbb{P}^1$ is an algebraic curve by Chow's Theorem. □

Theorem 2.1. *X has at most one zero in \mathbb{C}^2 .*

Proof. Suppose that $p_1 \neq p_2$ are zeroes of X . By [6], there exists a separatrix Σ_i through p_i , $i = 1, 2$. First assume that each Σ_i is algebraic at infinity through a nondicritical p_i . Let $\Phi \in \text{Aut}[\mathbb{C}^2]$, given in Proposition 2.2, such that $\Phi(\Gamma_1)$ is a line $L_{\Phi(p_1)}$ through $\Phi(p_1)$. Let $\overline{C_2}$ be the closure of $C_2 := \Phi(\Gamma_2)$ in $\mathbb{C}\mathbb{P}^2$. Thus $\overline{L_{\Phi(p_1)}} \cap \overline{C_2} = \{r\} \in L_\infty$; otherwise if $\alpha : Z_2 \rightarrow \overline{C_2}$ is the resolution of $\overline{C_2}$, α^*X extends to Z_2 with at least three zeroes, which is a contradiction. Analogously, $L_\infty \cap \overline{C_2} = \{r\}$. As $\overline{L_{\Phi(p_1)}}$ and L_∞ just intersect $\overline{C_2}$ at r , $\overline{C_2}$ has to be a line as it cannot have two branches at r . Suppose that $L_{\Phi(p_1)} = \{z_1 = a\}$ and $L_{\Phi(p_2)} :=$

$C_2 = \{z_1 = b\}$. The orbit of $(z_1^0, z_2^0) \in \mathbb{C}^2$ with $a \neq z_1^0 \neq b$ is defined by the image of the entire map $\varphi_{(z_1^0, z_2^0)}(t) = \varphi(t, z_1^0, z_2^0) = (z_1(t), z_2(t))$, where φ is the flow of Φ_*X . Since $z_1(\mathbb{C}) \subset \mathbb{C} \setminus \{a, b\}$, by Picard's Theorem $z_1(t) \equiv k \in \mathbb{C}$, and thus $\varphi_{(z_1^0, z_2^0)}(\mathbb{C})$ is contained in a line parallel to both $L_{\Phi(p_1)}$ and $L_{\Phi(p_2)}$, and hence $\Phi_*X = \frac{\partial}{\partial z_2}$, a contradiction.

Observe that if p_i is dicritical, Σ_i can be taken to be transcendental at infinity. Otherwise Darboux's Theorem and Proposition 2.1 imply that X has at most one zero. Thus it only remains to analyze the case when Σ_i is transcendental at infinity. Now, we take from [4] the notion of P -completeness, that will be used in what follows. Let $P : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a nonconstant polynomial. \mathcal{F}_X is P -complete if there exists a finite set $Q \subset \mathbb{C}$ such that, for all $t \notin Q$, $P^{-1}(t)$ is transverse to \mathcal{F}_X and there is a neighbourhood U_t of t in \mathbb{C} such that $P|_{P^{-1}(U_t)}$ is a fibration and $\mathcal{F}_X|_{P^{-1}(U_t)}$ defines a local trivialization on it. Thus, if one Σ_i is transcendental at infinity, it follows from [4] that there is a nonconstant (primitive) polynomial $P : \mathbb{C}^2 \rightarrow \mathbb{C}$ of type \mathbb{C} or \mathbb{C}^* such that \mathcal{F}_X is P -complete. The set of points where \mathcal{F}_X is not transverse to P is an algebraic curve $S \subset P^{-1}(Q)$, so $p_i \in S$. If P is of type \mathbb{C} , since by [16] there is $\varphi \in \text{Aut}[\mathbb{C}^2]$ such that $P \circ \varphi(z_1, z_2) = z_1$, one sees as above that $p_i \in \{z_1 = \lambda\}$, for $i = 1, 2$, again a contradiction.

On the other hand, if P is of type \mathbb{C}^* by [17], as noted in the proof of Proposition 2.1, after a polynomial automorphism φ , $P \circ \varphi$ can be easily written and since $(P \circ \varphi)^{-1}(\lambda) \simeq \mathbb{C}^*$ for all $\lambda \neq 0$, and X is complete on each component of S , one has that $p_i \in (P \circ \varphi)^{-1}(0) = \{z_1 = 0\}$ or $\{z_1 z_2 = 0\}$, but in both cases one has two zeroes on an invariant line of X , a contradiction. \square

Theorem 2.2. *Let p be a zero of X which is not of Poincaré-Dulac type, and let \mathcal{S} be the set of separatrices through p that are algebraic at infinity. Up to polynomial automorphism,*

- (1) *when p is dicritical and all the separatrices through it belong to \mathcal{S} : $X = mz_1 \frac{\partial}{\partial z_1} + nz_2 \frac{\partial}{\partial z_2}$ where $m, n \in \mathbb{Z}^*$, and $mn < 0$;*
- (2) *when p is nondicritical, but $\#\mathcal{S} \geq 2$: $X = z_1(\lambda + qf(z_1^p z_2^q)) \frac{\partial}{\partial z_1} + z_2(\mu + pf(z_1^p z_2^q)) \frac{\partial}{\partial z_2}$, where $f \in \mathbb{C}[z]$, $p, q \in \mathbb{N}$ and $\lambda\mu \in \mathbb{C}^*$.*

If there is at least one separatrix $\Sigma \notin \mathcal{S}$, then $\#\mathcal{S} \geq 1$, and either

- (3) *$X = \lambda z_1 \frac{\partial}{\partial z_1} + (a(z_1) + b(z_1)z_2) \frac{\partial}{\partial z_2}$, with $a, b \in \mathbb{C}[z_1]$ and $b(0)\lambda \in \mathbb{C}^*$, or*
- (4) *\mathcal{F}_X is P -complete for a polynomial $P = (z_1^m(z_1^l z_2 + p(z_1)))^n$, where $m, n, l \in \mathbb{N}^+$, $p \in \mathbb{C}[z_1]$ of degree $\leq l - 1$ with $p(0) \neq 0$, or $P = z_1^m z_2^n$.*

Proof. By Theorem 2.1, p is the unique zero of X in \mathbb{C}^2 . Suppose that $p = (0, 0)$. Since the restriction of X to any open neighbourhood of p is semicomplete, [13], one has that $\lambda\mu \neq 0$, [14].

We can distinguish two cases. Suppose that p is dicritical. Then, if there is a rational first integral, one has by Proposition 2.1 that X can be written as in (1). If there is no rational first integral, there is a separatrix through p , Σ , which is transcendental at infinity, and according to [4], the foliation \mathcal{F}_X is P -complete, with P of type \mathbb{C} or \mathbb{C}^* . As noted in the proof of Theorem 2.1, if P is of type \mathbb{C} , then $P = z_1$ and it follows that X is as in (3), while if it is of type \mathbb{C}^* , then P can be written so that it reads as in (4).

Assume now that p is nondicritical. As p is not of Poincaré-Dulac type, then there are at least two separatrices through p , [3] and [11]. If at least two of them

are algebraic at infinity, according to Propositions 2.2 and 2.3 they are defined by the smooth algebraic curves $\Gamma_1 = \{P_1 = 0\}$ and $\Gamma_2 = \{P_2 = 0\}$. Consider the simply connected algebraic curve $\Gamma_1 \cup \Gamma_2 = \{P_1 P_2 = 0\}$. After a polynomial automorphism Φ , $\Phi(\Gamma_1 \cup \Gamma_2) = \{z_1^k z_2^l = 0\}$, [21]. Moreover, since $\Phi_* X$ is complete on $\mathbb{C}^2 \setminus (\{z_1 = 0\} \cup \{z_2 = 0\}) \simeq (\mathbb{C}^*)^2$, the classification of such vector fields, [2], shows that $\Phi_* X$ takes the form (2) of the statement. The nonexistence of two separatrices through p algebraic at infinity implies the existence of at least one Σ which is transcendental at infinity, and by [4]; X can be expressed as it is pointed out in (3) or (4).

3. COMPLETENESS AND THE JACOBIAN CONJECTURE

Jacobian Conjecture. *If $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map such that $\det(JF) \in \mathbb{C}^*$, then F is invertible, that is, F has an inverse which is also a polynomial map.*

In fact, from a theorem due to Bialynicki-Birula and Rosenlicht, [20], if F is injective it is surjective, and the inverse is a polynomial map. Thus the Jacobian Conjecture is equivalent to: “if $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map such that $\det(JF) \in \mathbb{C}^*$, then F is injective”. The conjecture is true for $n = 1$, and is an open problem for $n \geq 2$.

Following Nousiainen and Sweedler, [20], we can associate to $F = (F_1, \dots, F_n)$ n polynomial vector fields on \mathbb{C}^n , $\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n}$, defined by

$$\left(\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n} \right) := \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right) (JF)^{-1},$$

with the following properties:

- (1) They are \mathbb{C} -linearly independent on \mathbb{C}^n .
- (2) $\mathcal{L}_{\frac{\partial}{\partial F_i}} F_j = DF_j \left(\frac{\partial}{\partial F_i} \right) = \delta_{ij}$ and $\left[\frac{\partial}{\partial F_i}, \frac{\partial}{\partial F_j} \right] = 0$, with $1 \leq i, j \leq n$.

Therefore, we obtain for each $i = 1, \dots, n$ a nonsingular algebraic foliation by curves in \mathbb{C}^n defined by the vector field $\frac{\partial}{\partial F_i}$ whose leaves are given by the intersection of the level sets of $F_j, j \neq i$.

Theorem 3.1. *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map such that $\det(JF) \in \mathbb{C}^*$. Then F is injective if and only if the polynomial vector fields $\frac{\partial}{\partial F_i}, i = 1, \dots, n$, are complete.*

Proof. Suppose that F is injective. If $F_i(z_1, \dots, z_n) = w_i, i = 1, \dots, n$, as the vector fields $\frac{\partial}{\partial w_i}$ are complete, $\frac{\partial}{\partial F_i} = F^* \frac{\partial}{\partial w_i}$ are also complete.

Conversely, if $\frac{\partial}{\partial F_i}, i = 1, \dots, n$, are complete, and there are two different points $p, q \in \mathbb{C}^n$ such that $F(p) = F(q) = \alpha = (\alpha_1, \dots, \alpha_n)$, there are n leaves $L_i = \bigcap_{j \neq i} \{F_j = \alpha_j\}$, one of each foliation induced by $\frac{\partial}{\partial F_i}$, having at least two different points p and q of intersection.

But each leaf $L_i, i = 1, \dots, n$, is equipped with a holomorphic 1-form $DF_i|_{L_i}$ such that $DF_i|_{L_i} \left(\frac{\partial}{\partial F_i} \right) \equiv 1$. Following [13], since $\frac{\partial}{\partial F_i}$ is complete the 1-form $DF_i|_{L_i}$ is defined by the “différentielle du temps”, which is locally given by its flow.

Let us fix $i_0 \in \{1, \dots, n\}$, and an injective smooth path $c_{i_0} : [0, 1] \rightarrow L_{i_0}$ from p to q . The integral of $DF_{i_0}|_{L_{i_0}}$ along c_{i_0} has to be nonzero [13], but $\int_{c_{i_0}} DF_{i_0}|_{L_{i_0}} = F_{i_0}(q) - F_{i_0}(p) = 0$. Thus F is injective. □

After our work was completed, we noticed that Theorem 3.1 was proved by Meisters and Olech in [12] for the real case in another context. We denote by \mathcal{D} the $\mathbb{C}[z_1, \dots, z_n]$ -module of all \mathbb{C} -derivations of $\mathbb{C}[z_1, \dots, z_n]$. It is well known that \mathcal{D} is free and of rank n . A basis is said to be commutative when $[X_i, X_j] = 0$, $0 \leq i, j \leq n$. If each X_i is complete, we will say that it is complete.

Proposition 3.1. *A commutative basis (X_1, \dots, X_n) of \mathcal{D} is complete if and only if there exists a polynomial automorphism F of \mathbb{C}^n such that $F_*X_i = \frac{\partial}{\partial z_i}$, $i = 1, \dots, n$.*

Proof. Suppose first that (X_1, \dots, X_n) is a complete commutative basis. Then by a result of A. Nowicki [20], there exists a polynomial map $F = (F_1, \dots, F_n)$ with $\det(JF) \in \mathbb{C}^*$ such that $X_i = \frac{\partial}{\partial F_i}$. Thus by Theorem 3.1, F is a polynomial automorphism such that $X_i = (F^{-1})_* \frac{\partial}{\partial z_i}$.

Now, suppose that there is a polynomial automorphism F of \mathbb{C}^n such that $F_*X_i = \frac{\partial}{\partial z_i}$. Then, the X_i are complete and moreover $[X_i, X_j] = (F^{-1})_* [\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}] = 0$, thus proving the converse. \square

Corollary 3.1. *The Jacobian Conjecture holds if and only if every commutative basis of \mathcal{D} is complete.*

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