AN EXAMPLE OF A C-MINIMAL GROUP WHICH IS NOT ABELIAN-BY-FINITE

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Abstract. In 1996 D. Macpherson and C. Steinhorn introduced C-minimality as an analogue, for valued fields and some groups with a definable chain of normal subgroups with trivial intersection, of the notion of o-minimality. One of the open questions of that paper was the existence of a non abelian-by-finite C-minimal group. We give here the first example of such a group.

1. Introduction

The notion of o-minimality has undergone a very important development in recent years and has found many applications, for example in the study of expansions of real closed fields by analytic functions. Recall that o-minimal structures are totally ordered structures in which the parameter-definable subsets are finite unions of intervals with endpoints in the structure. More recently D. Macpherson and C. Steinhorn introduced C-minimality in [5] as a variant of the notion of o-minimality. In a C-minimal structure, a ternary relation, with some specific properties, the C-relation plays the role analogous to the order in an o-minimal structure: any parameter-definable subset is quantifier-free definable with formulas using just the C-relation and equality. Such relations arise naturally in valued groups and fields. Less developed than o-minimality for the moment, this notion has already led to some promising results (see [5] and [1]). It applies to expansions of algebraically closed valued fields ([4]), and may be expected to have a development in some ways analogous to o-minimality (see [1]). Some of the tools of stability can be developed in this context ([2], [3]). Notwithstanding, some basic questions remain: while, as in the o-minimal case, C-minimal fields are characterized, they are exactly the algebraically closed valued fields, C-minimal groups are far less understood than the o-minimal: we do not know which groups can be endowed with a C-minimal structure. There are many examples of abelian C-minimal groups (see [5], [7]) and it is easy to construct non-abelian C-minimal groups by adding a finite non-abelian group to an abelian C-minimal group as a direct summand. However, up to now, there have been no examples of non-abelian-by-finite C-minimal groups. In this paper we give such an example, the first one as far as I know, answering a
question of D. Macpherson. While $C$-minimality is proved in general using a quantifier elimination result, our group is obtained as a reduct of some ring interpretable in an algebraically closed valued field, and we do not even know its theory. Note that a natural question that arises when studying algebraically closed valued fields is to determine which groups are interpretable in such a structure; our group will appear naturally in that context.

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The following definitions can be found in $[5]$ (when we say “definable” we always mean “parameter-definable”):

- A $C$-structure is a structure $(M, C)$ where $C(x; y, z)$ is a ternary relation satisfying the following axioms:
  - $C_1$: $\forall xyz \left( C(x; y, z) \rightarrow C(x; z, y) \right)$;
  - $C_2$: $\forall xyz \left( C(x; y, z) \rightarrow \neg C(y; x, z) \right)$;
  - $C_3$: $\forall x y z w \left( C(x; y, z) \rightarrow (C(w; y, z) \vee C(x; w, z)) \right)$;
  - $C_4$: $\forall x y z \left( x \neq y \rightarrow (y \neq z \wedge C(x; y, z)) \right)$.

- An expansion $\tilde{M} = (M, C, \ldots)$ of a $C$-structure $(M, C)$ is $C$-minimal if for every elementary extension $M' = (M', C, \ldots)$ of $\tilde{M}$, any definable subset of $M'$ is quantifier-free definable in $(M', C)$, that is, by a quantifier-free formula of the language containing only the $C$-relation and the equality.

- A $C$-group is a structure $G = (G, C; \cdot, -1, 1)$, where $(G, \cdot, -1, 1)$ is a group, $C$ is a $C$-relation and $G$ satisfies:
  $\forall xyzuv \left( C(x; y, z) \leftrightarrow C(uxv; uyv, uzv) \right)$.

A $C$-field is a structure $F = (F, C; +, -, 0, 1)$, where $(F, +, -, 0, 1)$ is a field, and $C$ is a $C$-relation for which both the additive group and the multiplicative group of $F$ are $C$-groups.

Let $F = (F, +, -, 0, 1)$ be a field. From any non-trivial (Krull) valuation $v$ from $F$ to an ordered abelian group, we can define a $C$-relation on $F$ by setting

$$C(x; y, z) \text{ iff } v(z - y) > v(z - x)$$

and this makes $(F, C) = (F, +, -, 0, 1, C)$ into a $C$-field. Conversely, any $C$-field can be made into a valued field such that the $C$-relation and the valuation satisfy the relation above. It was shown in $[5]$ and $[1]$ that the $C$-minimal $C$-fields correspond to the algebraically closed valued fields. With the induced $C$-relation, the additive group and the multiplicative group of a $C$-minimal $C$-field $F$ are $C$-minimal groups.

Let $(\mathbb{F}, C) = (F, +, -, 0, 1, C)$ be an algebraically closed $C$-field and $v$ the corresponding valuation. We use the following notations (for basics on Krull valuations see $[6]$): $\Gamma$ is the valuation group of $(\mathbb{F}, v)$, $A_v = \{ x \in F \mid v(x) \geq 0 \}$ the valuation ring and $M_v = \{ x \in F \mid v(x) > 0 \}$ its maximal ideal. For any $\gamma \in \Gamma$, $A_\gamma = \{ x \in F \mid v(x) \geq \gamma \}$ and $M_\gamma = \{ x \in F \mid v(x) > \gamma \}$. We also write $A_\infty = \{ 0 \}$ where $\infty$ is the valuation of 0 ($\infty$ does not belong to the group $\Gamma$ and is greater than any element of $\Gamma$). The $C$-field $(\mathbb{F}, C)$ being $C$-minimal, we can easily describe its definable subsets (see $[5]$ for details): any definable subset of any structure elementarily equivalent to $(\mathbb{F}, C)$ is a disjoint union of “truncated cones”. A truncated cone in $F$ can be described as a set

$$D = (a_0 + D_0) \setminus ((a_1 + D_1) \cup \ldots \cup (a_n + D_n))$$
where \(a_0, \ldots, a_n\) are elements of \(F\) and \(D_0, \ldots, D_n\) are equal either to \(F\) or to some \(A_\gamma\), or to some \(M_\gamma\), where \(\gamma \in \Gamma \cup \{\infty\}\). We may assume that \(a_1 + D_1, \ldots, a_n + D_n\) are disjoint subsets of \(a_0 + D_0\). We allow the case where \(n = 0\) and \(D = a_0 + D_0\).

Remember how these subsets are definable from the \(C\)-relation: if \(v(u) = \gamma\), then \(a + A_\gamma = \{x \in F \mid \neg C(x; a + u, a)\}\) and \(a + M_\gamma = \{x \in F \mid C(a + u; x, a)\}\).

For any strictly positive \(\gamma\), the ring \(V_\gamma = A_v/A_\gamma\) can be endowed with the \(C\)-relation induced by \(C\): for any \(x, y, z \in A_v\), \(C'(x + A_\gamma; y + A_\gamma, z + A_\gamma)\) holds if and only if \(C(x; y, z)\) holds and \(z - x \not\in A_\gamma\). Note that the last axiom for \(C\)-relations holds because the interval \([0, \gamma)\) in \(\Gamma\) has no last element since \(\Gamma\) is divisible. On the other hand, since \(V_\gamma\) is not a domain, the compatibility of the \(C\)-relation with the product is no longer true. We will call the structure \((V_\gamma, C')\) a \(C\)-ring, and denote by \(s_\gamma\) the canonical morphism from \(A_v\) to \(V_\gamma\). Although \(C\)-minimality is not preserved in general by interpretations, we have

**Lemma 2.1.** For any strictly positive \(\gamma\) the \(C\)-ring \((V_\gamma, C')\) is \(C\)-minimal.

**Proof.** Every definable subset of \(V_\gamma\) is the image by \(s_\gamma\) of a definable subset of \(A_v\) which is, by \(C\)-minimality of \((\bar{F}, C)\), a disjoint union of truncated cones included in \(A_v\). Obviously, the parameters used to define these truncated cones can be taken from \(A_v\). It is easy to see that the image by \(s_\gamma\) of a truncated cone of \(A_v\) is a truncated cone of \(V_\gamma\). We conclude that every definable subset of \((V_\gamma, C')\) is a disjoint union of truncated cones.

To prove that \((V_\gamma, C')\) is \(C\)-minimal we need to verify that every definable subset of every structure elementarily equivalent to \((V_\gamma, C')\) is a disjoint union of truncated cones. But every such structure \(M\) is an elementary substructure of an ultrapower \(\mathbb{N}\#\) of \((V_\gamma, C')\), and such an ultrapower is interpretable by the same means in an algebraically closed \(C\)-field. Thus we can apply the preceding argument to \(N\#\), and every formula \(\phi(\bar{F}, x)\) with parameters in \(M\) is equivalent in \(N\#\) to a formula \(\psi(\bar{g}, x)\) (with parameters in \(N\)) where \(\psi(\bar{g}, x)\) is a quantifier-free formula of the language containing only the \(C\)-relation and the equality. As \(M\) is an elementary substructure of \(\mathbb{N}\#\), we can find \(\bar{f} \in \mathbb{N}\) such that \(\phi(\bar{f}, x)\) is equivalent in \(M\) to \(\psi(\bar{f}, x)\). \(\square\)

From now on we assume that \(\bar{F}\) has characteristic \(p > 0\).

We define a new operation on \(A_v\): let \(T\) be an element of \(M_v \setminus \{0\}\), for any \(a, b \in A_v\),

\[
a * b = a + b + Ta^p b.
\]

This operation has the following properties (easy to verify and left to the reader):

for \(a, b, c \in A_v\),

(i) for every positive \(\gamma\), \(A_\gamma\) and \(M_\gamma\) are stable under \(*\).

(ii) \((a * b) * c = (a + b + Ta^p b) + c + T(a + b + Ta^p b)c = a + b + c + T(a^p b + a^p b + b^p c + T^{p+1} a^p b^p c)\) and

\[
a * (b * c) = a + b + c + T b^p c + T^a b^p (b + c + T^b c) = a + b + c + T^a b^p a + T b^p c + T^2 a b^p c.
\]

(iii) \(a * 0 = 0 * a = a\).

(iv) \(a * (-a + Ta^p a) = T^2 a^{2p+1}\) and \((-a + Ta^p a) * a = T^{p+1} a^{2p+1}\).

(v) \((a + b + Ta^p b) * a) = b = a + T(a^p b - b^p a) + T^2 d, \) with \(d \in A_v\).

(vi) \(((a + Ta^p a) * (b + Ta^p b)) * a) * b = T(a^p b - a^p b) + T^2 d, \) with \(d \in A_v\).

(vii) \(v(a * c - b * c) = v(c * a - c * b) = v(a - b)\).
Let $\gamma$ be the valuation of $T$. From the properties above we deduce that $*$ induces on $V_{2\gamma}$, a group law. By (iv), if $a \in A_v$, the inverse of $\overline{a} = a + A_{2\gamma}$ in $V_{2\gamma}$ is the element $\overline{a}^{-1} = -a + Ta^p a + A_{2\gamma}$. By (vii) this law is compatible with the $C$-relation defined in $V_{2\gamma}$: for every $a, b, c, d \in V_{2\gamma}$, $V_{2\gamma} = C'(a \ast d; b \ast d, c \ast d)$ if and only if $V_{2\gamma} = C'(a; b, c)$. Let $G = (V_{2\gamma}, \ast, ^{-1}, 0, C')$ be the $C$-group just defined. Clearly, any $C$-structure that is a reduct of a $C$-minimal structure is again $C$-minimal. As $G$ is a reduct of $(V_{2\gamma}, C')$, it is a $C$-minimal group.

Consider an element $a \in A_v$ and a strictly positive $\gamma \in \Gamma$. Define $Z(a, \gamma) = \{ x \in A_v \mid v(a^p x - x^p a) \geq \gamma \}$. Its image by $s_{2\gamma}$ is the centralizer in $G$ of $a + A_{2\gamma}$.

**Lemma 2.2.**

(i) if $v(a) \geq \gamma$, then $Z(a, \gamma) = A_v$,

(ii) if $\frac{v(a)}{p+1} \leq v(a) < \gamma$, then $Z(a, \gamma) = A_{-v(a)}$,

(iii) if $0 \leq v(a) < \frac{v(a)}{p+1}$, then $Z(a, \gamma) = \bigcup_{n \in \mathbb{F}_p} (n a + A_{-p v(a)})$.

**Proof.** (i) is obvious. Write $x = ta$ with $v(t) \geq v(a)$, then $x$ belongs to $Z(a, \gamma)$ if and only if $v(t - t') \geq \gamma - (p+1)v(a)$. If $\gamma - (p+1)v(a) \leq 0$ and $v(a) < \gamma$, this means that $v(t) \geq \gamma - (p+1)v(a) \geq \gamma - v(a)$. Thus prove (iii): if $\gamma - (p+1)v(a) > 0$, then $t$ can be written $t = n + t'$ where $n \in \mathbb{F}_p$, the field with $p$ elements, and $t' \in M_0$. Thus $v(tP - t) = v(tP - t') = v(t')$ so $x$ belongs to $Z(a, \gamma)$ if and only if $v(t') \geq \gamma - (p+1)v(a)$.

If $\alpha \in [0, 2\gamma)$, where $\gamma = v(T)$, we call $G_\alpha$, the image of $A_v$ by $s_{2\gamma}$. Clearly $G_\alpha$ is a subgroup of $G$. We conclude by:

**Theorem 2.3.** The group $G$ is a $C$-minimal group that is not abelian-by-finite. Moreover $G$ is a nilpotent group of class 2 and of exponent $p$ if $p$ is odd and 4 if $p = 2$.

**Proof.** By the preceding lemma, the set of elements of $V_{2\gamma}$ whose centralizer is of finite index in $G$ is equal to $G_\gamma$. Since $G_\gamma$ is not of finite index in $G$, the group $G$ is not abelian-by-finite. It is easy to see that its center is $G_2$, and its derived subgroup is also $G_2$. Therefore $G$ is a nilpotent group of class 2. Finally, computing by induction the $n^{th}$ power of $a \in A_v$, we find the formula $a \ast a \ast \ldots \ast a = na + T(\frac{n(n-1)}{2} a^{p+1})$ modulo $A_2$.

The valuation $v$ induces a map $v_{2\gamma}$ from $V_{2\gamma}$ to the ordered set $[0, 2\gamma) \cup \{ \infty \}$ defined by $v_{2\gamma}(a + A_{2\gamma}) = v(a)$ if $v(a) < 2\gamma$, and $v_{2\gamma}(A_{2\gamma}) = \infty$. This map is what we called in [?] a group valuation, and the $C$-group $G$ belongs to the class of what we called valued $C$-groups: the $C$-relation in $G$ can be defined from the valuation $v_{2\gamma}$ by

$$C'(x; y, z) \iff v_{2\gamma}(zy^{-1}) > v_{2\gamma}(zx^{-1}).$$

In [?] we prove that every $C$-minimal valued $C$-group is nilpotent-by-finite and that every connected (i.e. without proper definable subgroups of finite index) $C$-minimal valued $C$-group of finite exponent is nilpotent. The $C$-group $G$ defined above is nilpotent of class 2 and we do not have examples of $C$-minimal valued groups of nilpotent class greater than 2.
References


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