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AN EXAMPLE OF A C-MINIMAL GROUP WHICH IS NOT ABELIAN-BY-FINITE

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Abstract. In 1996 D. Macpherson and C. Steinhorn introduced C-minimality as an analogue, for valued fields and some groups with a definable chain of normal subgroups with trivial intersection, of the notion of o-minimality. One of the open questions of that paper was the existence of a non abelian-by-finite C-minimal group. We give here the first example of such a group.

1. Introduction

The notion of o-minimality has undergone a very important development in recent years and has found many applications, for example in the study of expansions of real closed fields by analytic functions. Recall that *o-minimal* structures are totally ordered structures in which the parameter-definable subsets are finite unions of intervals with endpoints in the structure. More recently D. Macpherson and C. Steinhorn introduced *C-minimality* in [5] as a variant of the notion of o-minimality. In a C-minimal structure, a ternary relation, with some specific properties, the C-relation plays the role analogous to the order in an o-minimal structure: any parameter-definable subset is quantifier-free definable with formulae using just the C-relation and equality. Such relations arise naturally in valued groups and fields. Less developed than o-minimality for the moment, this notion has already led to some promising results (see [5] and [1]). It applies to expansions of algebraically closed valued fields ([4]), and may be expected to have a development in some ways analogous to o-minimality (see [1]). Some of the tools of stability can be developed in this context ([2], [3]). Notwithstanding, some basic questions remain: while, as in the o-minimal case, C-minimal fields are characterized, they are exactly the algebraically closed valued fields, C-minimal groups are far less understood than the o-minimal: we do not know which groups can be endowed with a C-minimal structure. There are many examples of abelian C-minimal groups (see [5], [7]) and it is easy to construct non-abelian C-minimal groups by adding a finite non-abelian group to an abelian C-minimal group as a direct summand. However, up to now, there have been no examples of non-abelian-by-finite C-minimal groups. In this paper we give such an example, the first one as far as I know, answering a

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question of D. Macpherson. While C-minimality is proved in general using a quantifier elimination result, our group is obtained as a reduct of some ring interpretable in an algebraically closed valued field, and we do not even know its theory. Note that a natural question that arises when studying algebraically closed valued fields is to determine which groups are interpretable in such a structure; our group will appear naturally in that context.

2.

The following definitions can be found in [5] (when we say "definable" we always mean "parameter-definable"):

- A *C-structure* is a structure (M, C) where $C(x; y, z)$ is a ternary relation satisfying the following axioms:
  - $\mathcal{C}_1$: $\forall xyz\ (C(x;y,z) \longrightarrow C(x;z,y))$;
  - $\mathcal{C}_2$: $\forall xyz\ (C(x;y,z) \longrightarrow \neg C(y;x,z))$;
  - $\mathcal{C}_3$: $\forall xyzw\ [C(x;y,z) \longrightarrow (C(w;y,z) \vee C(x;w,z))]$;
  - $\mathcal{C}_4$: $\forall xy \exists z\ [x \neq y \longrightarrow (y \neq z \wedge C(x;y,z))]$.
- An expansion $\mathbb{M} = (M, C, ...)$ of a C-structure (M, C) is *C-minimal* if for every elementary extension $\mathbb{M}' = (M', C, ...)$ of $\mathbb{M}$, any definable subset of M' is quantifier-free definable in (M', C), that is, by a quantifier-free formula of the language containing only the C-relation and the equality.
- A *C-group* is a structure $\mathbb{G} = (G, C, \cdot, ^{-1}, 1)$, where $(G, \cdot, ^{-1}, 1)$ is a group, C is a C-relation and $\mathbb{G}$ satisfies:

  $$\forall xyzuv\ (C(x;y,z) \longleftrightarrow C(uxv; uyv, uzv)).$$

  A *C-field* is a structure $\mathbb{F} = (F, C, +, -, \cdot, 0, 1)$, where $(F, +, -, \cdot, 0, 1)$ is a field, and C is a C-relation for which both the additive group and the multiplicative group of $\mathbb{F}$ are C-groups.

Let $\mathbb{F} = (F, +, -, \cdot, 0, 1)$ be a field. From any non-trivial (Krull) valuation v from F to an ordered abelian group, we can define a C-relation on F by setting

$$C(x; y, z) \text{ iff } v(z-y) > v(z-x)$$

and this makes $(\mathbb{F}, C) = (F, +, -, \cdot, 0, 1, C)$ into a C-field. Conversely, any C-field can be made into a valued field such that the C-relation and the valuation satisfy the relation above. It was shown in [5] and [1] that the C-minimal C-fields correspond to the algebraically closed valued fields. With the induced C-relation, the additive group and the multiplicative group of a C-minimal C-field $\mathbb{F}$ are C-minimal groups.

Let $(\mathbb{F}, C) = (F, +, -, \cdot, 0, 1, C)$ be an algebraically closed C-field and v the corresponding valuation. We use the following notations (for basics on Krull valuations see [6]): Γ is the valuation group of (F, v), $A_v = \{x \in F \mid v(x) \geqslant 0\}$ the valuation ring and $M_v = \{x \in F \mid v(x) > 0\}$ its maximal ideal. For any $\gamma \in \Gamma$, $A_\gamma = \{x \in F \mid v(x) \geqslant \gamma\}$ and $M_\gamma = \{x \in F \mid v(x) > \gamma\}$. We also write $A_\infty = \{0\}$ where ∞ is the valuation of 0 (∞ does not belong to the group Γ and is greater than any element of Γ). The C-field $(\mathbb{F}, C)$ being C-minimal, we can easily describe its definable subsets (see [5] for details): any definable subset of any structure elementarily equivalent to $(\mathbb{F}, C)$ is a disjoint union of "truncated cones". A truncated cone in F can be described as a set

$$D = (a_0 + D_0) \setminus ((a_1 + D_1) \cup ... \cup (a_n + D_n))$$

where $a_0, ..., a_n$ are elements of F and $D_0, ..., D_n$ are equal either to F or to some A_γ, or to some M_γ, where $\gamma \in \Gamma \cup \{\infty\}$. We may assume that $a_1 + D_1, ..., a_n + D_n$ are disjoint subsets of $a_0 + D_0$. We allow the case where $n = 0$ and $D = a_0 + D_0$. Remember how these subsets are definable from the C-relation: if $v(u) = \gamma$, then $a + A_\gamma = \{x \in F \mid \neg C(x; a + u, a)\}$ and $a + M_\gamma = \{x \in F \mid C(a + u; x, a)\}$.

For any strictly positive γ, the ring $V_\gamma = A_v/A_\gamma$ can be endowed with the C-relation induced by C: for any $x, y, z \in A_v$, $C'(x + A_\gamma; y + A_\gamma, z + A_\gamma)$ holds if and only if $C(x; y, z)$ holds and $z - x \notin A_\gamma$. Note that the last axiom for C-relations holds because the interval $[0, \gamma)$ in Γ has no last element since Γ is divisible. On the other hand, since $\mathbb{V}_\gamma$ is not a domain, the compatibility of the C-relation with the product is no longer true. We will call the structure $(\mathbb{V}_\gamma, C') = (V_\gamma, +, -, \cdot, 0, 1, C')$ a C-ring, and denote by s_γ the canonical morphism from A_v to V_γ. Although C-minimality is not preserved in general by interpretations, we have

Lemma 2.1. *For any strictly positive γ the C-ring $(\mathbb{V}_\gamma, C')$ is C-minimal.*

Proof. Every definable subset of V_γ is the image by s_γ of a definable subset of A_v which is, by C-minimality of $(\mathbb{F}, C)$, a disjoint union of truncated cones included in A_v. Obviously, the parameters used to define these truncated cones can be taken from A_v. It is easy to see that the image by s_γ of a truncated cone of A_v is a truncated cone of V_γ. We conclude that every definable subset of $(\mathbb{V}_\gamma, C')$ is a disjoint union of truncated cones.

To prove that $(\mathbb{V}_\gamma, C')$ is C-minimal we need to verify that every definable subset of every structure elementarily equivalent to $(\mathbb{V}_\gamma, C')$ is a disjoint union of truncated cones. But every such structure $\mathbb{M}$ is an elementary substructure of an ultrapower $\mathbb{N}^\#$ of $(\mathbb{V}_\gamma, C')$, and such an ultrapower is interpretable by the same means in an algebraically closed C-field. Thus we can apply the preceding argument to $N^\#$, and every formula $\phi(\overline{a}, x)$ with parameters in M is equivalent in $N^\#$ to a formula $\psi(\overline{b}, x)$ (with parameters in N) where $\psi(\overline{y}, x)$ is a quantifier-free formula of the language containing only the C-relation and the equality. As $\mathbb{M}$ is an elementary substructure $\mathbb{N}^\#$, we can find $\overline{c} \in M$ such that $\phi(\overline{a}, x)$ is equivalent in M to $\psi(\overline{c}, x)$. □

From now on we assume that $\mathbb{F}$ has characteristic $p > 0$.

We define a new operation on A_v: let T be an element of $M_v \setminus \{0\}$, for any $a, b \in A_v$,

$$a \, * \, b \; = \; a \; + \; b \; + \; T a^p b.$$

This operation has the following properties (easy to verify and left to the reader): for $a, b, c \in A_v$,

(i) for every positive γ, A_γ and M_γ are stable under $*$.
(ii) $(a * b) * c = (a + b + Ta^p b) + c + T(a + b + Ta^p b)^p c = a + b + c$
$+ T(a^p b + a^p c + b^p c) + T^{p+1} a^{p^2} b^p c$ and
$a * (b * c) = a + b + c + Tb^p c + Ta^p(b + c + Tb^p c) = a + b + c$
$+ T(a^p b + a^p c + b^p c) + T^2 a^p b^p c$.
(iii) $a * 0 = 0 * a = a$.
(iv) $a * (-a + Ta^p a) = T^2 a^{2p+1}$ and $(-a + Ta^p a) * a = T^{p+1} a^{p^2+p+1}$.
(v) $((-b + Tb^p b) * a) * b = a + T(a^p b - b^p a) + T^2 d$, with $d \in A_v$.
(vi) $(((-a + Ta^p a) * (-b + Tb^p b)) * a) * b = T(a^p b - b^p a) + T^2 d$, with $d \in A_v$.
(vii) $v(a * c - b * c) = v(c * a - c * b) = v(a - b)$.

Let γ be the valuation of T. From the properties above we deduce that $*$ induces on $V_{2\gamma}$ a group law. By (iv), if $a \in A_v$, the inverse of $\overline{a} = a + A_{2\gamma}$ in $V_{2\gamma}$ is the element $\overline{a}^{-1} = -a + Ta^p a + A_{2\gamma}$. By (vii) this law is compatible with the C-relation defined in $V_{2\gamma}$: for every $a, b, c, d \in V_{2\gamma}$, $\mathbb{V}_{2\gamma} \models C'(a * d; b * d, c * d)$ if and only if $\mathbb{V}_{2\gamma} \models C'(d * a; d * b, d * c)$ if and only if $\mathbb{V}_{2\gamma} \models C'(a; b, c)$. Let $\mathbb{G} = (V_{2\gamma}, *, ^{-1}, 0, C')$ be the C-group just defined. Clearly, any C-structure that is a reduct of a C-minimal structure is again C-minimal. As $\mathbb{G}$ is a reduct of $(\mathbb{V}_{2\gamma}, C')$, it is a C-minimal group.

Consider an element $a \in A_v$ and a strictly positive $\gamma \in \Gamma$. Define $Z_{(a,\gamma)} = \{x \in A_v \mid v(a^p x - x^p a) \geqslant \gamma\}$. Its image by $s_{2\gamma}$ is the centralizer in $\mathbb{G}$ of $a + A_{2\gamma}$.

Lemma 2.2. (i) *if* $v(a) \geqslant \gamma$, *then* $Z_{(a,\gamma)} = A_v$,
(ii) *if* $\frac{\gamma}{p+1} \leqslant v(a) < \gamma$, *then* $Z_{(a,\gamma)} = A_{\frac{\gamma - v(a)}{p}}$,
(iii) *if* $0 \leqslant v(a) < \frac{\gamma}{p+1}$, *then* $Z_{(a,\gamma)} = \bigcup_{n \in \mathbb{F}_p} (na + A_{\gamma - pv(a)})$.

Proof. (i) is obvious. Write $x = ta$ with $v(t) \geqslant -v(a)$. Then x belongs to $Z_{(a,\gamma)}$ if and only if $v(t - t^p) \geqslant \gamma - (p+1)v(a)$. If $\gamma - (p+1)v(a) \leqslant 0$ and $v(a) < \gamma$, this means that $pv(t) \geqslant \gamma - (p+1)v(a)$ and $pv(x) \geqslant \gamma - v(a)$ and proves (ii). We now prove (iii): if $\gamma - (p+1)v(a) > 0$, then t can be written $t = n + t'$ where $n \in \mathbb{F}_p$, the field with p elements, and $t' \in M_v$. Thus $v(t^p - t) = v(t'^p - t') = v(t')$ so x belongs to $Z_{(a,\gamma)}$ if and only if $v(t') \geqslant \gamma - (p+1)v(a)$. □

If $\alpha \in [0, 2\gamma)$, where $\gamma = v(T)$, we call G_α the image of A_α by $s_{2\gamma}$. Clearly G_α is a subgroup of $\mathbb{G}$. We conclude by:

Theorem 2.3. *The group* $\mathbb{G}$ *is a* C*-minimal group that is not abelian-by-finite. Moreover* $\mathbb{G}$ *is a nilpotent group of class* 2 *and of exponent* p *if* p *is odd and* 4 *if* $p = 2$.

Proof. By the preceding lemma, the set of elements of $V_{2\gamma}$ whose centralizer is of finite index in $\mathbb{G}$ is equal to G_γ. Since G_γ is not of finite index in $\mathbb{G}$, the group $\mathbb{G}$ is not abelian-by-finite. It is easy to see that its center is G_γ and its derived subgroup is also G_γ. Therefore $\mathbb{G}$ is a nilpotent group of class 2. Finally, computing by induction the n^{th} power of $a \in A_v$, we find the formula $a * a * ... * a = na + T(\frac{n(n-1)}{2} a^{p+1})$ modulo $A_{2\gamma}$. □

The valuation v induces a map $v_{2\gamma}$ from $V_{2\gamma}$ to the ordered set $[0, 2\gamma) \cup \{\infty\}$ defined by $v_{2\gamma}(a + A_{2\gamma}) = v(a)$ if $v(a) < 2\gamma$, and $v_{2\gamma}(A_{2\gamma}) = \infty$. This map is what we called in [7] a group valuation, and the C-group $\mathbb{G}$ belongs to the class of what we called valued C-groups: the C-relation in $\mathbb{G}$ can be defined from the valuation $v_{2\gamma}$ by

$$C'(x; y, z) \text{ iff } v_{2\gamma}(zy^{-1}) > v_{2\gamma}(zx^{-1}).$$

In [8] we prove that every C-minimal valued C-group is nilpotent-by-finite and that every connected (i.e. without proper definable subgroups of finite index) C-minimal valued C-group of finite exponent is nilpotent. The C-group $\mathbb{G}$ defined above is nilpotent of class 2 and we do not have examples of C-minimal valued groups of nilpotent class greater than 2.

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