ON NON-MEASURABILITY OF $\ell_\infty/c_0$ IN ITS SECOND DUAL

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Abstract. We show that $\ell_\infty/c_0 = C([N^*])$ with the weak topology is not an intersection of $\aleph_1$ Borel sets in its Čech-Stone extension (and hence in any compactification). Assuming (CH), this implies that $(C([N^*]), \text{weak})$ has no continuous injection onto a Borel set in a compact space, or onto a Lindelöf space. Under (CH), this answers a question of Arhangel’skii.

1. Introduction

All of our spaces are completely regular. We shall identify $\ell_\infty$ with $C(\beta\mathbb{N})$, the space of continuous functions on the Čech-Stone compactification of the natural numbers $\mathbb{N}$, and $\ell_\infty/c_0$ with the function space $C([N^*])$. Talagrand [Ta] demonstrated that $\ell_\infty$ and $\ell_\infty/c_0$, canonically embedded in their second duals, are not Borel with respect to the weak* topology; cf. Edgar [Ed]. One checks that $\ell_\infty$ is an intersection of $2^{\aleph_0}$ Borel sets in $(\ell_\infty^{**}, \text{weak}^*)$; cf. Remark 3.2. However, assuming the Continuum Hypothesis (CH), the analogous fact is not true for $\ell_\infty/c_0$. This follows readily from the next result.

Theorem 1.1. The space $(\ell_\infty/c_0, \text{weak})$ is not an intersection of $\aleph_1$ Borel sets in its Čech-Stone extension.

Indeed, the universal properties of the Čech-Stone extension imply that $(\ell_\infty/c_0, \text{weak})$ is not an intersection of $\aleph_1$ Borel sets in any of its compactifications, and hence also in its $\sigma$-compact extension $(\ell_\infty^{**}, \text{weak}^*)$.

The following corollary answers under (CH) (in a rather strong form) a question posed by A.V. Arhangel’skii [Ar1] Problem 34, [Ar2] Problem 5) (cf. Remark 3.3). This question was also mentioned in [CS] where, as a partial answer, it was shown that $C_p([N^*])$ had no one-to-one continuous map onto a compact space, a result obtained independently in [AP].

Corollary 1.2. Assuming (CH), no continuous image of $(C([N^*]), \text{pointwise})$ under a map with $\sigma$-compact fibers is an intersection of $2^{\aleph_0}$ Borel sets in its Čech-Stone compactification.
2. Proof of Theorem 1.1

We shall prove the assertion in a more general setting. Let us recall that a compact space $K$ is an $F$-space if each continuous $c : U \to [0, 1]$, defined on an open $\sigma$-compact set $U$ in $K$, extends continuously over $K$. A compact set $L \subseteq K$ is a $P$-set if any countable intersection of neighborhoods of $L$ is a neighborhood of $L$. The space $\beta N$, and all of its compact subspaces, are $F$-spaces, and $\aleph_1^n$ contains compact non-open $P$-sets. From results of Jayne, Namioka and Rogers [JNR] it follows that for any infinite compact $F$-space $K$, $(C(K), \text{weak})$ is not Borel in its Čech-Stone extension.

**Proposition 2.1.** Let $K$ be a compact $F$-space containing a compact non-open $P$-set. Then the function space $(C(K), \text{weak})$ is not an intersection of $\aleph_1$ Borel sets in its Čech-Stone compactification.

**Proof.** By inductively choosing complements of neighborhoods of a compact non-open $P$-set in $K$, one can define a strictly increasing sequence
\[
(1) \quad W_1 \subseteq W_2 \subseteq \cdots \subseteq W_\xi \subseteq \cdots, \quad \xi < \omega_1, \quad \overline{W_\xi} \subseteq W_{\xi+1}, \quad W_\xi \neq W_{\xi+1}
\]
of open sets in $K$. Let
\[
(2) \quad H = \{ f \in C(K) : 0 \leq f \leq 1, \ f | (K \setminus \bigcup_{\xi < \omega_1} W_\xi) \equiv 0 \}.
\]
We let $H_w$ denote the set $H$ equipped with the topology inherited from $(C(K), \text{weak})$. We also consider a topology $\tau$ on $H$ generated by basic neighborhoods
\[
(3) \quad N(f, \xi) = \{ g \in H : f | W_\xi = g | W_\xi \}.
\]
We shall write $H_\tau$ for $(H, \tau)$. The identity map
\[
(4) \quad i : H_\tau \to H_w
\]
is continuous; cf. [DJP], [Ha], [JNR]. Since $K$ is an $F$-space, one easily checks that for any sequence $N(f_1, \xi_1) \supseteq N(f_2, \xi_2) \supseteq \cdots, \xi_1 < \xi_2 < \cdots$, there is $f \in H$ such that, for $\eta = \sup \xi_i$,
\[
(5) \quad \bigcap_i N(f_i, \xi_i) \supseteq N(f, \eta).
\]
This implies that $H_\tau$ is a Baire space and provides the essential property for the proof of the next lemma. \hfill $\square$

**Lemma 2.2.** The Čech-Stone compactification $\beta H_\tau$ cannot be covered by $\aleph_1$ meager sets.

**Proof of Lemma 2.2.** Assume $\beta H_\tau = \bigcup_{\alpha < \omega_1} A_\alpha$, where every $A_\alpha$ is closed and nowhere dense. By (5) one can inductively define $g_\alpha \in H$, $\phi(\alpha) < \omega_1$, with $\phi(\alpha) < \phi(\beta)$, for $\alpha < \beta$, such that $N(g_\alpha, \phi(\alpha)) \supseteq N(g_\beta, \phi(\beta))$ and $\overline{N(g_\alpha, \phi(\alpha))} \cap A_\alpha = \emptyset$ (the closure taken in $\beta H_\tau$). But then, $\bigcap_{\alpha < \omega_1} \overline{N(g_\alpha, \phi(\alpha))}$ is simultaneously non-empty, and disjoint from $\bigcup_{\alpha < \omega_1} A_\alpha$, a contradiction. That concludes the proof of the lemma.

Following an idea of De we shall show that $H_w$ is not an intersection of $\aleph_1$ Borel sets in $\beta H_w$. Since $H_w$ is closed in $(C(K), \text{weak})$, this will also show that the assertion of the proposition is true.

Aiming for a contradiction to Lemma 2.2, assume that
\[
(6) \quad H_w = \bigcap_{\alpha < \omega_1} E_\alpha, \quad \text{where every } E_\alpha \subseteq \beta H_w, \alpha < \omega_1, \text{ is Borel},
\]
and let
\[(7) \ i_\beta : \beta H_\tau \to \beta H_w\]
be the continuous extension of the identity map given in (4).

Let
\[(8) \ S_\alpha = (i^{\beta})^{-1}(E_\alpha).\]
Then \(S_\alpha\) is a Borel set in \(\beta H_\tau\) containing the Baire space \(H_\tau\), and therefore, \(S_\alpha\) being open modulo meager sets in \(\beta H_\tau\),
\[(9) \ \beta H_\tau \setminus M_\alpha \subseteq S_\alpha \text{ for } M_\alpha \text{ meager in } \beta H_\tau.\]
From (6), (8) and (9) we obtain
\[(10) \ \beta H_\tau \setminus \bigcup_{\alpha < \omega_1} M_\alpha \subseteq (i^{\beta})^{-1}(H_w).\]
Now let \(C_\alpha\) be the closure in \(\beta H_w\) of the set \(\{ f \in H : f(K \setminus W_\alpha) \equiv 0 \}\). If there exists \(h \in H_w \setminus \bigcup_{\alpha < \omega_1} C_\alpha\), then there is \(a > 0\) such that \(h^{-1}([a, 1]) \cap (K \setminus W_\alpha) \neq \emptyset\), for all \(\alpha < \omega_1\). But, \(K\) compact gives \(h^{-1}([a, 1]) \setminus \bigcup_{\alpha < \omega_1} W_\alpha \neq \emptyset\), a contradiction.
Hence,
\[(11) \ H_w \subseteq \bigcup_{\alpha < \omega_1} C_\alpha \text{ and } C_\alpha \text{ contains no } N(f, \xi).\]
Let
\[(12) \ L_\alpha = (i^{\beta})^{-1}(C_\alpha).\]
The second part of (11) shows that \(C_\alpha \cap H\) has empty interior in \(H_\tau\), and therefore
\[(13) \ L_\alpha \text{ is nowhere dense in } \beta H_\tau.\]
Then (10), (11) and (12) yield
\[(14) \ \beta H_\tau = \bigcup_{\alpha < \omega_1} (M_\alpha \cup L_\alpha),\]
with \(M_\alpha, L_\alpha\) meager in \(\beta H_\tau\). This, however, is impossible by Lemma 2.2. \(\square\)

3. Proof of Corollary 1.2 and remarks

Corollary 1.2 is an immediate consequence of the following observation.

**Lemma 3.1.** Let \(u : X \to Y\) be a continuous surjection with \(\sigma\)-compact fibers. If \(|Y| = 2^{\aleph_0}\) and \(Y\) is an intersection of \(2^{\aleph_0}\) Borel sets in \(\beta Y\), then \(X\) is an intersection of \(2^{\aleph_0}\) Borel sets in \(\beta X\).

**Proof.** Let \(u^\beta : \beta X \to \beta Y\) be the continuous extension and let
\[(1) \ B(Y) = (u^\beta)^{-1}(Y),\]
\[(2) \ B(y) = \beta X \setminus ((u^\beta)^{-1}(y) \setminus X).\]
Then \(B(Y)\) is an intersection of \(2^{\aleph_0}\) Borel sets in \(\beta X\). Since every \(u^{-1}(y) = (u^\beta)^{-1}(y)\) is \(\sigma\)-compact and hence \(F_\tau\), the sets \(B(y) = \beta X \setminus ((u^\beta)^{-1}(y) \setminus u^{-1}(y))\) are also Borel sets in \(\beta X\). Now, one readily checks that
\[X = B(Y) \cap \bigcap \{ B(y) : y \in Y \};\]
hence \(X\) is an intersection of \(2^{\aleph_0}\) Borel sets in \(\beta X\). \(\square\)
Remark 3.2. Similar arguments show that $\ell_\infty = C(\beta N)$ is an intersection of $2^{\aleph_0}$ Borel sets in $(C(\beta N)^*, \text{weak}^*)$. Indeed, let $A$ be the $\sigma$-compact space of all bounded sequences of reals with the pointwise topology. If $\delta_n \in C(\beta N)^*$ is the functional identified with the probability measure supported by $\{n\}$, the map $u : C(\beta N)^* \to A$, defined by $u(\phi)(n) = (\delta_n, \phi)$, is a surjection which is continuous with respect to the weak* topology. (Here, $(\cdot, \cdot)$ represents the duality map on $C(\beta N)^* \times C(\beta N)^*$.) For $y \in A$, let $y^0$ be the continuous extension over $\beta N$, and let

$$B(y) = C(\beta N)^* \setminus (u^{-1}(y) \setminus \{y^0\}).$$

Then, $B(y)$ is Borel in $(C(\beta N)^*, \text{weak}^*)$ and $C(\beta N) = \bigcap \{B(y) : y \in A\}$.

Remark 3.3. Corollary 1.2 also shows that, assuming (CH), $(C(\aleph^*)$, pointwise) cannot be mapped onto a Lindelöf space by any continuous function with $\sigma$-compact fibers (this answers the second part of a question of Arhangel’skii in [Ar1 Problem 34]). Assume on the contrary that a Lindelöf space $K$ is an image of $C(\aleph^*)$ under such a map. Since $(C(\aleph^*), \text{pointwise})$ is of cardinality $2^{\aleph_0}$, so is $X$. Therefore, $X$ has a continuous injection into the Tychonoff cube $K$ of weight $2^{\aleph_0}$ and in effect, one can assume that the Lindelöf space $X$ is a subspace of $K$. But then $X$ is an intersection of $2^{\aleph_0}$ $\sigma$-compact subsets of $K$. Indeed, $X$ being Lindelöf, each point in $K \setminus X$ is contained in a compact $G_\delta$-set in $K \setminus X$, and there are $2^{\aleph_0}$ compact $G_\delta$-sets in $K$. We arrive at a contradiction with Corollary 1.2.

Remark 3.4. Let $\kappa$ be a regular uncountable cardinal, let $\kappa_d$ be the set $\kappa$ with the discrete topology, and let $K_\kappa$ be obtained from $\beta \kappa_d$ by identifying to a point the set of uniform ultrafilters $\beta \kappa_d \setminus \bigcup \{A : A \subseteq \kappa, |A| < \kappa\}$. Then, $K_\kappa$ is an $F$-space with a non-isolated $P$-point. The reasoning in the proof of Proposition 2.1 shows that $(C(K_\kappa), \text{weak})$ is not an intersection of $\kappa$ Borel sets in its Čech-Stone compactification. Assuming that $\kappa$ is strongly inaccessible, one shows also, as in the proof of Corollary 1.2, that any continuous image of $(C(K_\kappa), \text{pointwise})$ under a map with $\sigma$-compact fibers, has Lindelöf number $\kappa$.

References

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