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# ON NON-MEASURABILITY OF $\ell_\infty/c_0$ IN ITS SECOND DUAL

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Abstract. We show that $\ell_\infty/c_0 = C(\mathbb{N}^*)$ with the weak topology is not an intersection of $\aleph_1$ Borel sets in its Čech-Stone extension (and hence in any compactification). Assuming (CH), this implies that $(C(\mathbb{N}^*), \text{weak})$ has no continuous injection onto a Borel set in a compact space, or onto a Lindelöf space. Under (CH), this answers a question of Arhangel'skiĭ.

## 1. Introduction

All of our spaces are completely regular. We shall identify $\ell_\infty$ with $C(\beta\mathbb{N})$, the space of continuous functions on the Čech-Stone compactification of the natural numbers $\mathbb{N}$, and $\ell_\infty/c_0$ with the function space $C(\mathbb{N}^*)$. Talagrand [Ta] demonstrated that $\ell_\infty$ and $\ell_\infty/c_0$, canonically embedded in their second duals, are not Borel with respect to the weak$^*$ topology; cf. Edgar [Ed]. One checks that $\ell_\infty$ is an intersection of $2^{\aleph_0}$ Borel sets in $({\ell_\infty}^{**}, \text{weak}^*)$; cf. Remark 3.2. However, assuming the Continuum Hypothesis (CH), the analogous fact is not true for $\ell_\infty/c_0$. This follows readily from the next result.

**Theorem 1.1.** *The space* $(\ell_\infty/c_0, \text{weak})$ *is not an intersection of* $\aleph_1$ *Borel sets in its Čech-Stone extension.*

Indeed, the universal properties of the Čech-Stone extension imply that $(\ell_\infty/c_0, \text{weak})$ is not an intersection of $\aleph_1$ Borel sets in any of its compactifications, and hence also in its $\sigma$-compact extension $({\ell_\infty}^{**}, \text{weak}^*)$.

The following corollary answers under (CH) (in a rather strong form) a question posed by A.V. Arhangel'skiĭ [Ar1, Problem 34], [Ar2, Problem 5] (cf. Remark 3.3). This question was also mentioned in [C-S] where, as a partial answer, it was shown that $C_p(\mathbb{N}^*)$ had no one-to-one continuous map onto a compact space, a result obtained independently in [AP].

**Corollary 1.2.** *Assuming (CH), no continuous image of* $(C(\mathbb{N}^*), \text{pointwise})$ *under a map with* $\sigma$*-compact fibers is an intersection of* $2^{\aleph_0}$ *Borel sets in its Čech-Stone compactification.*

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## 2. Proof of Theorem 1.1

We shall prove the assertion in a more general setting. Let us recall that a compact space $K$ is an *$F$-space* if each continuous $c : U \to [0,1]$, defined on an open $\sigma$-compact set $U$ in $K$, extends continuously over $K$. A compact set $L \subseteq K$ is a $P$-set if any countable intersection of neighborhoods of $L$ is a neighborhood of $L$. The space $\beta\mathbb{N}$, and all of its compact subspaces, are $F$-spaces, and $\mathbb{N}^*$ contains compact non-open $P$-sets. From results of Jayne, Namioka and Rogers [JNR] it follows that for any infinite compact $F$-space $K$, $(C(K), \text{weak})$ is not Borel in its Čech-Stone extension.

**Proposition 2.1.** *Let $K$ be a compact $F$-space containing a compact non-open $P$-set. Then the function space* $(C(K), \text{weak})$ *is not an intersection of $\aleph_1$ Borel sets in its Čech-Stone compactification.*

*Proof.* By inductively choosing complements of neighborhoods of a compact non-open $P$-set in $K$, one can define a strictly increasing sequence

(1) $W_1 \subseteq W_2 \subseteq \ldots \subseteq W_\xi \subseteq \ldots,\ \xi < \omega_1,\ \overline{W_\xi} \subseteq W_{\xi+1},\ W_\xi \neq W_{\xi+1}$

of open sets in $K$. Let

(2) $H = \{f \in C(K) : 0 \leq f \leq 1,\ \ f|(K \setminus \bigcup_{\xi<\omega_1} W_\xi) \equiv 0\}$.

We let $H_w$ denote the set $H$ equipped with the topology inherited from $(C(K),$ weak). We also consider a topology $\tau$ on $H$ generated by basic neighborhoods

(3) $N(f,\xi) = \{g \in H : f|W_\xi = g|W_\xi\}$.

We shall write $H_\tau$ for $(H, \tau)$. The identity map

(4) $i : H_\tau \to H_w$

is continuous; cf. [DJP], [Ha], [JNR]. Since $K$ is an $F$-space, one easily checks that for any sequence $N(f_1,\xi_1) \supseteq N(f_2,\xi_2) \supseteq \ldots,\ \xi_1 < \xi_2 < \ldots$, there is $f \in H$ such that, for $\eta = \sup\limits_i \xi_i$,

(5) $\bigcap\limits_i N(f_i,\xi_i) \supseteq N(f,\eta)$.

This implies that $H_\tau$ is a Baire space and provides the essential property for the proof of the next lemma. □

**Lemma 2.2.** *The Čech-Stone compactification $\beta H_\tau$ cannot be covered by $\aleph_1$ meager sets.*

*Proof of Lemma* 2.2. Assume $\beta H_\tau = \bigcup\limits_{\alpha<\omega_1} A_\alpha$, where every $A_\alpha$ is closed and nowhere dense. By (5) one can inductively define $g_\alpha \in H$, $\phi(\alpha) < \omega_1$, with $\phi(\alpha) < \phi(\beta)$, for $\alpha < \beta$, such that $N(g_\alpha, \phi(\alpha)) \supseteq N(g_\beta, \phi(\beta))$ and $\overline{N(g_\alpha, \phi(\alpha))}^\beta \cap A_\alpha = \emptyset$ (the closure taken in $\beta H_\tau$). But then, $\bigcap\limits_{\alpha<\omega_1} \overline{N(g_\alpha, \phi(\alpha))}^\beta$ is simultaneously non-empty, and disjoint from $\bigcup\limits_{\alpha<\omega_1} A_\alpha$, a contradiction. That concludes the proof of the lemma.

Following an idea of [De] we shall show that $H_w$ is not an intersection of $\aleph_1$ Borel sets in $\beta H_w$. Since $H_w$ is closed in $(C(K), \text{weak})$, this will also show that the assertion of the proposition is true.

Aiming for a contradiction to Lemma 2.2, assume that

(6) $H_w = \bigcap\limits_{\alpha<\omega_1} E_\alpha$, where every $E_\alpha \subseteq \beta H_w$, $\alpha < \omega_1$, is Borel,

and let

(7) $i^\beta : \beta H_\tau \to \beta H_w$

be the continuous extension of the identity map given in (4).

Let

(8) $S_\alpha = (i^\beta)^{-1}(E_\alpha)$.

Then $S_\alpha$ is a Borel set in $\beta H_\tau$ containing the Baire space $H_\tau$, and therefore, $S_\alpha$ being open modulo meager sets in $\beta H_\tau$,

(9) $\beta H_\tau \setminus M_\alpha \subseteq S_\alpha$ for $M_\alpha$ meager in $\beta H_\tau$.

From (6), (8) and (9) we obtain

(10) $\beta H_\tau \setminus \bigcup_{\alpha<\omega_1} M_\alpha \subseteq (i^\beta)^{-1}(H_w)$.

Now let $C_\alpha$ be the closure in $\beta H_w$ of the set $\{f \in H : f|(K \setminus W_\alpha) \equiv 0\}$. If there exists $h \in H_w \setminus \bigcup_{\alpha<\omega_1} C_\alpha$, then there is $a > 0$ such that $h^{-1}([a,1]) \cap (K \setminus W_\alpha) \neq \emptyset$, for all $\alpha < \omega_1$. But, $K$ compact gives $h^{-1}([a,1]) \setminus \bigcup_{\alpha<\omega_1} W_\alpha \neq \emptyset$, a contradiction. Hence,

(11) $H_w \subseteq \bigcup_{\alpha<\omega_1} C_\alpha$ and $C_\alpha$ contains no $N(f,\xi)$.

Let

(12) $L_\alpha = (i^\beta)^{-1}(C_\alpha)$.

The second part of (11) shows that $C_\alpha \cap H$ has empty interior in $H_\tau$, and therefore

(13) $L_\alpha$ is nowhere dense in $\beta H_\tau$.

Then (10), (11) and (12) yield

(14) $\beta H_\tau = \bigcup_{\alpha<\omega_1} (M_\alpha \cup L_\alpha)$,

with $M_\alpha, L_\alpha$ meager in $\beta H_\tau$. This, however, is impossible by Lemma 2.2. □

## 3. Proof of Corollary 1.2 and remarks

Corollary 1.2 is an immediate consequence of the following observation.

**Lemma 3.1.** *Let $u : X \to Y$ be a continuous surjection with $\sigma$-compact fibers. If $|Y| = 2^{\aleph_0}$ and $Y$ is is an intersection of $2^{\aleph_0}$ Borel sets in $\beta Y$, then $X$ is an intersection of $2^{\aleph_0}$ Borel sets in $\beta X$.*

*Proof.* Let $u^\beta : \beta X \to \beta Y$ be the continuous extension and let

(1) $B(Y) = (u^\beta)^{-1}(Y)$,

(2) $B(y) = \beta X \setminus ((u^\beta)^{-1}(y) \setminus X)$.

Then $B(Y)$ is an intersection of $2^{\aleph_0}$ Borel sets in $\beta X$. Since every $u^{-1}(y) = (u^\beta)^{-1}(y) \cap X$ is $\sigma$-compact and hence $F_\sigma$, the sets $B(y) = \beta X \setminus ((u^\beta)^{-1}(y) \setminus u^{-1}(y))$ are also Borel sets in $\beta X$. Now, one readily checks that

$$X = B(Y) \cap \bigcap \{B(y) : y \in Y\};$$

hence $X$ is an intersection of $2^{\aleph_0}$ Borel sets in $\beta X$. □

*Remark* 3.2. Similar arguments show that $\ell_\infty = C(\beta\mathbb{N})$ is an intersection of $2^{\aleph_0}$ Borel sets in $(C(\beta\mathbb{N})^{**}, \text{weak}^*)$. Indeed, let $\Lambda$ be the $\sigma$-compact space of all bounded sequences of reals with the pointwise topology. If $\delta_n \in C(\beta\mathbb{N})^*$ is the functional identified with the probability measure supported by $\{n\}$, the map $u : C(\beta\mathbb{N})^{**} \to \Lambda$, defined by $u(\phi)(n) = \langle \delta_n, \phi\rangle$, is a surjection which is continuous with respect to the weak$^*$-topology. (Here, $\langle\, ,\rangle$ represents the duality map on $C(\beta\mathbb{N})^* \times C(\beta\mathbb{N})^{**}$.) For $y \in \Lambda$, let $y^\beta$ be the continuous extension over $\beta\mathbb{N}$, and let

$$B(y) = C(\beta\mathbb{N})^{**} \setminus (u^{-1}(y) \setminus \{y^\beta\}).$$

Then, $B(y)$ is Borel in $(C(\beta\mathbb{N})^{**}, \text{weak}^*)$ and $C(\beta\mathbb{N}) = \bigcap\{B(y) : y \in \Lambda\}$.

*Remark* 3.3. Corollary 1.2 also shows that, assuming (CH), $(C(\mathbb{N}^*), \text{pointwise})$ cannot be mapped onto a Lindelöf space by any continuous function with $\sigma$-compact fibers (this answers the second part of a question of Arhangel'skiĭ in [Ar1, Problem 34]). Assume on the contrary that a Lindelöf space $X$ is an image of $C(\mathbb{N}^*)$ under such a map. Since $(C(\mathbb{N}^*), \text{pointwise})$ is of cardinality $2^{\aleph_0}$, so is $X$. Therefore, $X$ has a continuous injection into the Tychonoff cube $K$ of weight $2^{\aleph_0}$ and in effect, one can assume that the Lindelöf space $X$ is a subspace of $K$. But then $X$ is an intersection of $2^{\aleph_0}$ $\sigma$-compact subsets of $K$. Indeed, $X$ being Lindelöf, each point in $K \setminus X$ is contained in a compact $G_\delta$-set in $K \setminus X$, and there are $2^{\aleph_0}$ compact $G_\delta$-sets in $K$. We arrive at a contradiction with Corollary 1.2.

*Remark* 3.4. Let $\kappa$ be a regular uncountable cardinal, let $\kappa_d$ be the set $\kappa$ with the discrete topology, and let $K_\kappa$ be obtained from $\beta\kappa_d$ by identifying to a point the set of uniform ultrafilters $\beta\kappa_d \setminus \bigcup\{\overline{A} : A \subseteq \kappa, |A| < \kappa\}$. Then, $K_\kappa$ is an $F$-space with a non-isolated $P$-point. The reasoning in the proof of Proposition 2.1 shows that $(C(K_\kappa), \text{weak})$ is not an intersection of $\kappa$ Borel sets in its Čech-Stone compactification. Assuming that $\kappa$ is strongly inaccessible, one shows also, as in the proof of Corollary 1.2, that any continuous image of $(C(K_\kappa), \text{pointwise})$ under a map with $\sigma$-compact fibers, has Lindelöf number $\kappa$.

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