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A NOTE ON THE ISOPERIMETRIC INEQUALITY

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Abstract. We show that the sharp integral form on the isoperimetric inequality holds for those orientation-preserving mappings $f \in W_{loc}^{\frac{n^2}{n+1}}(\Omega, \mathbb{R}^n)$ whose Jacobians obey the rule of integration by parts.

1. Introduction

The familiar geometric form of the isoperimetric inequality reads as

$$n^{n-1}\omega_{n-1}|U|^{n-1} \leq |\partial U|^n, \tag{1}$$

where $|U|$ stands for the volume of a domain $U \subset \mathbb{R}^n$ and $|\partial U|$ is its $(n-1)$-dimensional surface area. Now, if $f : B_r \to U$ is a diffeomorphism of a ball $B_r = B(x_0, r) \subset \mathbb{R}^n$ onto U, then $|U| = \left|\int_{B_r} J(x,f)\,dx\right|$ and $|\partial U| \leq \int_{\partial B_r} |D^\sharp f(x)|\,dx$. Here $D^\sharp f(x)$ stands for the cofactor matrix of the differential matrix $Df(x)$. In this way, we obtain what is known as the integral form of the isoperimetric inequality, namely

$$\left|\int_{B_r} J(x,f)\,dx\right| \leq I(n)\left(\int_{\partial B_r} |D^\sharp f(x)|\,dx\right)^{\frac{n}{n-1}} \tag{2}$$

with $I(n) = (n\sqrt[n-1]{\omega_{n-1}})^{-1}$. Above, we used the operator norm of the cofactor matrix, defined by $|D^\sharp f(x)| = \sup\{|D^\sharp f(x)h| : |h| = 1\}$.

Reshetnyak proved in [14] the sharp Hölder-continuity for a mapping of bounded distortion by extending certain ideas of Morrey's [10]. This required him to prove the isoperimetric inequality (2) for a mapping in the Sobolev class $W^{1,n}$ [15] (see also [2, Theorem 4.5.9 (31)]). Reshetnyak's proof is based on integration by parts as are the related proofs given in [11], [12] by Müller et al. One can check using a standard approximation argument that it suffices to prove the isoperimetric inequality (2) for all smooth mappings. The sharp constant $I(n)$ in inequality (2) plays a very crucial role in Reshetnyak's argument (also see [6, Chapter 7.7]). The Sobolev regularity

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$W^{1,n}$ cannot be substantially relaxed. Indeed, the mapping

$$f(x) = \frac{x}{|x|} \log\left(\frac{e}{|x|}\right) \tag{3}$$

belongs to $\bigcap_{p<n} W^{1,p}(B(0,1),\mathbb{R}^n)$ but (2) fails for all $0 < r < 1$.

For example in non-linear elasticity (see [1], [16] and [12]) it is natural to assume that the Jacobians of the mappings in consideration are positive a.e., because a deformation of an elastic body should be orientation preserving. Recently, a generalization of mappings of bounded distortion, the theory of mappings of finite distortion, with subexponentially distortion has emerged, partially motivated by non-linear elasticity. We refer the interested reader to the monograph [6] by Iwaniec and Martin. The assumptions of this theory imply that $f \in W^{1,1}_{loc}(\Omega,\mathbb{R}^n)$, $J(x,f) \geq 0$ a.e.,

$$|Df|^n \in L^P_{loc}(\Omega) \tag{4}$$

where

(5) the function $t \to P(t^{\frac{n}{n+1}})$ is increasing for large values of t,

$$\int_1^\infty \frac{P(t)}{t^2}\, dt = \infty \tag{6}$$

and P is an Orlicz-function (see [6, Chapter 4.12]). One can improve example (3) and find, for each given function P for which the integral (6) converges, a radial stretching f so that (4) holds and (2) fails ([9]). We proved in [5] that, under the above assumptions, the isoperimetric inequality holds, with some constant, depending only on the dimension n. In this paper, we will give a simple limiting argument to show that, under the above assumptions, the isoperimetric inequality (2) holds with the sharp constant $I(n)$. Actually this is a simple case of our more general theorem.

Let $f \in W^{1,\frac{n^2}{n+1}}_{loc}(\Omega,\mathbb{R}^n)$. We say that the Jacobian $J(\cdot,f)$ of f obeys the rule of integration by parts if the equation

$$\int_\Omega \varphi(x)J(x,f)\,dx = -\int_\Omega f_i(x)J(x,f_1,...,f_{i-1},\varphi,f_{i+1},...,f_n)\,dx \tag{7}$$

is valid for every test function $\varphi \in C_0^\infty(\Omega)$ and each index $i = 1,...,n$. Under the assumption $f \in W^{1,\frac{n^2}{n+1}}_{loc}(\Omega,\mathbb{R}^n)$, different choices of indices i yield the same value of the integral; see [3]. It is important to note that the right-hand side is well defined for mappings lying in the Sobolev space $W^{1,\frac{n^2}{n+1}}_{loc}(\Omega,\mathbb{R}^n)$ and so equation (7) implies, when the Jacobian does not change the sign, that

$$J(\cdot,f) \in L^1_{loc}(\Omega). \tag{8}$$

As an example, the Jacobian of an orientation-preserving mapping (i.e. $J(\cdot,f) \geq 0$ a.e.) in the class $W^{1,1}_{loc}(\Omega,\mathbb{R}^n)$ so that (4)-(6) hold, obeys the rule of integration by parts ([4], [9], [3] and [6, Theorem 7.2.1]; see also the fundamental paper [7] by Iwaniec and Sbordone).

Theorem 1.1. *Suppose that the Jacobian of* $f \in W^{1,\frac{n^2}{n+1}}_{loc}(\Omega,\mathbb{R}^n)$ *is non-negative a.e. and the mapping* f *obeys the rule* (7) *of integration by parts. Then* f *satisfies* *the isoperimetric inequality* (2) *for every* $x_0 \in \Omega$ *and almost every radius* $r \in (0, dist(x_0, \partial\Omega))$.

The question of the sharp constant is motivated by the study of sharp modulus of continuity properties for mappings of finite distortion; see the forthcoming papers [8] and [13].

2. Proof of Theorem 1.1

Let $B_R = B(x_0, R) \subset \Omega$ be a ball such that $\overline{B}_R \subset \Omega$. We approximate f in $W^{1,\frac{n^2}{n+1}}(B_R, \mathbb{R}^n)$ by mappings $f^i \in C^\infty(B_R, \mathbb{R}^n)$. Since the functions $|D^\sharp f^i|$ converge to $|D^\sharp f|$ in $L^1(B_R)$ (observe that the cofactor matrix is made up of $n-1$ subdeterminants of the differential matrix and $\frac{n^2}{n+1} \geq n-1$), we find by Fubini's theorem that $|D^\sharp f^i|$ converges to $|D^\sharp f|$ in $L^1(\partial B_r)$ for almost every radius $r \in (0, R)$. Fix $r \in (0, R)$ so that the functions $|D^\sharp f^i|$ converge to $|D^\sharp f|$ in $L^1(\partial B_r)$. Pick $0 < \epsilon < \frac{r}{2}$. We take a convolution approximation u_t^ϵ to the characteristic function $\chi_{B_{r-\epsilon}}$ of the ball $B_{r-\epsilon}$ by using the standard mollifiers Φ_t (see [6, Formula (4.6)]) where t is chosen to be so small that $u_t^\epsilon \in C_0^\infty(B_r)$. Then $0 \leq u_t^\epsilon \leq 1$ and so

$$\int_{B_r} u_t^\epsilon(x) J(x, f^i)\, dx \leq \int_{B_r} J(x, f^i)\, dx \leq I(n) \left(\int_{\partial B_r} |D^\sharp f^i(x)|\, dx \right)^{\frac{n}{n-1}}. \tag{9}$$

Applying Stokes' theorem for the smooth mapping f^i we find that

$$\int_{B_r} u_t^\epsilon(x) J(x, f^i)\, dx = -\int_{B_r} f_1^i(x) J(x, u_t^\epsilon, f_2^i, ..., f_n^i)\, dx. \tag{10}$$

The telescoping decomposition of the Jacobian (cf. [6, Chapter 8]) leads to the equation

$$\begin{aligned} &\int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2, ..., f_n)\, dx - \int_{B_r} f_1^i(x) J(x, u_t^\epsilon, f_2^i, ..., f_n^i)\, dx \\ = &\int_{B_r} (f_1(x) - f_1^i(x)) J(x, u_t^\epsilon, f_2, ..., f_n)\, dx \\ &+ \sum_{k=2}^n \int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2^i, ..., f_{k-1}^i, f_k - f_k^i, f_{k+1}, ..., f_n)\, dx. \end{aligned} \tag{11}$$

Combining Hadamard's inequality with Hölder's inequality we find that

$$\begin{aligned} &\left| \int_{B_r} f_1(x) J(x, u_t^\epsilon, f_2, ..., f_n)\, dx - \int_{B_r} f_1^i(x) J(x, u_t^\epsilon, f_2^i, ..., f_n^i)\, dx \right| \\ \leq &\int_{B_r} |f_1 - f_1^i| |\nabla u_t^\epsilon| |Df|^{n-1} + \sum_{k=2}^n \int_{B_r} |f_1| |\nabla u_t^\epsilon| |Df^i|^{k-2} |Df - Df^i| |Df|^{n-k} \\ \leq &|\nabla u_t^\epsilon|_{L^\infty(B_r)} \left(\int_{B_r} |f_1 - f_1^i|^{n^2} \right)^{\frac{1}{n^2}} \left(\int_{B_r} |Df|^{\frac{n^2}{n+1}} \right)^{\frac{n^2-1}{n^2}} \\ &+ C(n) |\nabla u_t^\epsilon|_{L^\infty(B_r)} \left(\int_{B_r} |f_1|^{n^2} \right)^{\frac{1}{n^2}} \left(\int_{B_r} (|Df^i| + |Df|)^{\frac{n^2}{n+1}} \right)^{\frac{n^2-n-2}{n^2}} \\ &\left(\int_{B_r} |Df - Df^i|^{\frac{n^2}{n+1}} \right)^{\frac{n+1}{n^2}}. \end{aligned} \tag{12}$$

By the Sobolev-Poincaré inequality we see that the right-hand side of inequality (12) tends to zero as i goes to infinity. Combining this with inequality (9) and equation (10) we find that

$$-\int_{B_r} f_1(x)J(x,u_t^\epsilon,f_2,...,f_n)\,dx \le I(n)\left(\int_{\partial B_r}|D^\sharp f(x)|\,dx\right)^{\frac{n}{n-1}}. \tag{13}$$

Applying the assumptions $u_t^\epsilon \in C_0^\infty(B_r)$ and (7) we conclude that

$$\int_{B_r} u_t^\epsilon(x)J(x,f)\,dx \le I(n)\left(\int_{\partial B_r}|D^\sharp f(x)|\,dx\right)^{\frac{n}{n-1}}. \tag{14}$$

Since $u_t^\epsilon(x)J(x,f) \le \chi_{B_r}(x)J(x,f)$ and $J(\cdot,f) \in L^1_{loc}(\Omega)$ by (8), we can use the dominated convergence theorem. First letting $t \to 0$ and then $\epsilon \to 0$, the claim follows.

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