A NOTE ON THE ISOPERIMETRIC INEQUALITY

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ABSTRACT. We show that the sharp integral form on the isoperimetric inequality holds for those orientation-preserving mappings $f \in W^{n+1}_{loc}(\Omega, \mathbb{R}^n)$ whose Jacobians obey the rule of integration by parts.

1. Introduction

The familiar geometric form of the isoperimetric inequality reads as

$$n^{n-1} \omega_{n-1} |U|^{n-1} \leq |\partial U|^{n},$$

where $|U|$ stands for the volume of a domain $U \subseteq \mathbb{R}^n$ and $|\partial U|$ is its $(n-1)$-dimensional surface area. Now, if $f : B_r \to U$ is a diffeomorphism of a ball $B_r = B(x_0, r) \subset \mathbb{R}^n$ onto $U$, then $|U| = \left| \int_{B_r} J(x, f) \, dx \right|$ and $|\partial U| \leq \int_{\partial B_r} |D^2 f(x)| \, dx$. Here $D^2 f(x)$ stands for the cofactor matrix of the differential matrix $D f(x)$. In this way, we obtain what is known as the integral form of the isoperimetric inequality, namely

$$\left| \int_{B_r} J(x, f) \, dx \right| \leq I(n) \left( \int_{\partial B_r} |D^2 f(x)| \, dx \right)^{\frac{n}{n-1}}$$

with $I(n) = \left( n^{-\sqrt{\omega_{n-1}}} \right)^{-1}$. Above, we used the operator norm of the cofactor matrix, defined by $|D^2 f(x)| = \sup\{|D^2 f(x)h| : |h| = 1\}$.

Reshetnyak proved in [14] the sharp Hölder-continuity for a mapping of bounded distortion by extending certain ideas of Morrey’s [10]. This required him to prove the isoperimetric inequality for a mapping in the Sobolev class $W^{1,n}$ [15] (see also [2, Theorem 4.5.9 (31)]). Reshetnyak’s proof is based on integration by parts as are the related proofs given in [11], [12] by Müller et al. One can check using a standard approximation argument that it suffices to prove the isoperimetric inequality for all smooth mappings. The sharp constant $I(n)$ in inequality plays a very crucial role in Reshetnyak’s argument (also see [6, Chapter 7.7]). The Sobolev regularity

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$W^{1,n}$ cannot be substantially relaxed. Indeed, the mapping
\begin{equation}
 f(x) = \frac{x}{|x|} \log \left( \frac{e}{|x|} \right)
\end{equation}
belongs to $\bigcap_{p<n} W^{1,p}(B(0,1),\mathbb{R}^n)$ but (2) fails for all $0 < r < 1$.

For example in non-linear elasticity (see [1], [16] and [12]) it is natural to assume that the Jacobians of the mappings in consideration are positive a.e., because a deformation of an elastic body should be orientation preserving. Recently, a generalization of mappings of bounded distortion, the theory of mappings of finite distortion, with subexponentially distortion has emerged, partially motivated by non-linear elasticity. We refer the interested reader to the monograph [6] by Iwaniec and Martin. The assumptions of this theory imply that $f \in W^{1,1}_{loc}(\Omega,\mathbb{R}^n)$, $J(x,f) \geq 0$ a.e.

\begin{equation}
 |Df|^n \in L^p_{loc}(\Omega)
\end{equation}
where
\begin{equation}
 the\ function\ t \to P(t^{\frac{n}{n-1}})\ is\ increasing\ for\ large\ values\ of\ t,
\end{equation}
\begin{equation}
 \int_1^\infty \frac{P(t)}{t^2} \, dt = \infty
\end{equation}
and $P$ is an Orlicz-function (see [6, Chapter 4.12]). One can improve example (3) and find, for each given function $P$ for which the integral (6) converges, a radial stretching $f$ so that (4) holds and (2) fails (9). We proved in [5] that, under the above assumptions, the isoperimetric inequality holds, with some constant, depending only on the dimension $n$. In this paper, we will give a simple limiting argument to show that, under the above assumptions, the isoperimetric inequality (2) holds with the sharp constant $I(n)$. Actually this is a simple case of our more general theorem.

Let $f \in W^{1,\frac{n^2}{n-1}}_{loc}(\Omega,\mathbb{R}^n)$. We say that the Jacobian $J(\cdot,f)$ of $f$ obeys the rule of integration by parts if the equation
\begin{equation}
 \int_\Omega \phi(x) J(x,f) \, dx = - \int_\Omega f_i(x) J(x,f_1,\ldots,f_{i-1},\phi,f_{i+1},\ldots,f_n) \, dx
\end{equation}
is valid for every test function $\phi \in C^\infty_0(\Omega)$ and each index $i = 1,\ldots,n$. Under the assumption $f \in W^{1,\frac{n^2}{n-1}}_{loc}(\Omega,\mathbb{R}^n)$, different choices of indices $i$ yield the same value of the integral; see [3]. It is important to note that the right-hand side is well defined for mappings lying in the Sobolev space $W^{1,\frac{n^2}{n-1}}_{loc}(\Omega,\mathbb{R}^n)$ and so equation (7) implies, when the Jacobian does not change the sign, that
\begin{equation}
 J(\cdot,f) \in L^1_{loc}(\Omega).
\end{equation}
As an example, the Jacobian of an orientation-preserving mapping (i.e. $J(\cdot,f) \geq 0$ a.e.) in the class $W^{1,1}_{loc}(\Omega,\mathbb{R}^n)$ so that (4)-(6) hold, obeys the rule of integration by parts (1, 9, 3 and 5, Theorem 7.2.1); see also the fundamental paper [7] by Iwaniec and Sbordone).

**Theorem 1.1.** Suppose that the Jacobian of $f \in W^{1,\frac{n^2}{n-1}}_{loc}(\Omega,\mathbb{R}^n)$ is non-negative a.e. and the mapping $f$ obeys the rule (7) of integration by parts. Then $f$ satisfies
the isoperimetric inequality \( \mathcal{L} \) for every \( x_0 \in \Omega \) and almost every radius \( r \in (0, \text{dist}(x_0, \partial \Omega)) \).

The question of the sharp constant is motivated by the study of sharp modulus of continuity properties for mappings of finite distortion; see the forthcoming papers \[8\] and \[13\].

2. Proof of Theorem 1.1

Let \( B_R = B(x_0, R) \subset \Omega \) be a ball such that \( \overline{B}_R \subset \Omega \). We approximate \( f \) in \( W^{1, \frac{n}{n+1}}(B_R, \mathbb{R}^n) \) by mappings \( f^i \in C^\infty(B_R, \mathbb{R}^n) \). Since the functions \( |D^i f^i| \) converge to \( |Df| \) in \( L^1(B_R) \) (observe that the cofactor matrix is made up of \( n-1 \) subdeterminants of the differential matrix and \( \frac{n^2}{n+1} \geq n-1 \)), we find by Fubini’s theorem that \( |D^i f^i| \) converges to \( |Df| \) in \( L^1(\partial B_r) \) for almost every radius \( r \in (0, R) \). Fix \( r \in (0, R) \) so that the functions \( |D^i f^i| \) converge to \( |Df| \) in \( L^1(\partial B_r) \). Pick \( 0 < \epsilon < \frac{r}{2} \). We take a convolution approximation \( u^*_i \) to the characteristic function \( \chi_{B_{r-\epsilon}} \) of the ball \( B_{r-\epsilon} \) by using the standard mollifiers \( \Phi_i \) (see \[6\] Formula (4.6)) where \( t \) is chosen to be so small that \( u^*_i \in C^\infty_0(B_r) \). Then \( 0 \leq u^*_i \leq 1 \) and so

\[
\int_{B_r} u^*_i(x) J(x, f^i) \, dx \leq \int_{B_r} J(x, f^i) \, dx \leq I(n) \left( \int_{\partial B_r} |D^i f^i(x)| \, dx \right)^{ \frac{n}{n+1} }.
\]

Applying Stokes’ theorem for the smooth mapping \( f^i \) we find that

\[
\int_{B_r} u^*_i(x) J(x, f^i) \, dx = -\int_{B_r} f^i_1(x) J(x, u^*_i, f^i_2, \dotsc, f^i_n) \, dx.
\]

The telescoping decomposition of the Jacobian (cf. \[8\] Chapter 8) leads to the equation

\[
\int_{B_r} f_1(x) J(x, u^*_i, f^i_2, \dotsc, f^i_n) \, dx - \int_{B_r} f^i_1(x) J(x, u^*_i, f^i_2, \dotsc, f^i_n) \, dx
\]

\[
= \int_{B_r} (f_1(x) - f^i_1(x)) J(x, u^*_i, f^i_2, \dotsc, f^i_n) \, dx
\]

\[
+ \sum_{k=2}^n \int_{B_r} f_k(x) J(x, u^*_i, f^i_2, \dotsc, f^i_{k-1}, f_k - f^i_k, f_{k+1}, \dotsc, f^i_n) \, dx.
\]

Combining Hadamard’s inequality with Hölder’s inequality we find that

\[
\left| \int_{B_r} f_1(x) J(x, u^*_i, f^i_2, \dotsc, f^i_n) \, dx - \int_{B_r} f^i_1(x) J(x, u^*_i, f^i_2, \dotsc, f^i_n) \, dx \right|
\]

\[
\leq \int_{B_r} |f_1 - f^i_1| |\nabla u^*_i| |Df|^{n-1} + \sum_{k=2}^n \int_{B_r} |f_k - f^i_k| |\nabla u^*_i| |Df|^{k-2} |Df| - |Df^i| |Df|^{n-k}
\]

\[
\leq |\nabla u^*_i|_{L^\infty(B_r)} \left( \int_{B_r} |f_1 - f^i_1|^{n^2} \right)^{\frac{1}{n^2}} \left( \int_{B_r} |Df|^{\frac{n^2}{n+1}} \right)^{\frac{n^2}{n+1}}
\]

\[
+ C(n) |\nabla u^*_i|_{L^\infty(B_r)} \left( \int_{B_r} |f_1|^{n^2} \right)^{\frac{1}{n^2}} \left( \int_{B_r} (|Df| + |Df^i|)^{n^2} \right)^{\frac{n^2}{n+1}}
\]

\[
\left( \int_{B_r} |Df - Df^i|^{\frac{n^2}{n+1}} \right)^{\frac{n}{n+1}}.
\]
By the Sobolev-Poincaré inequality we see that the right-hand side of inequality (12) tends to zero as $i$ goes to infinity. Combining this with inequality (9) and equation (10) we find that

\[\int_{B_{r}} f_{1}(x) J(x, u_{\epsilon}, f_{2}, \ldots, f_{n}) \, dx \leq I(n) \left( \int_{\partial B_{r}} |D^{i}f(x)| \, dx \right)^{\frac{n}{n-i}}.\]

Applying the assumptions $u_{\epsilon} \in C^\infty_0(B_{r})$ and (7) we conclude that

\[\int_{B_{r}} u_{\epsilon}^{i}(x) J(x, f) \, dx \leq I(n) \left( \int_{\partial B_{r}} |D^{i}f(x)| \, dx \right)^{\frac{n}{n-i}}.\]

Since $u_{\epsilon}^{i}(x) J(x, f) \leq \chi_{B_{r}}(x) J(x, f)$ and $J(\cdot, f) \in L^{1}_{\text{loc}}(\Omega)$ by (3), we can use the dominated convergence theorem. First letting $t \to 0$ and then $\epsilon \to 0$, the claim follows.

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**References**


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