

ASYMPTOTIC LIMIT FOR CONDENSATE SOLUTIONS IN THE ABELIAN CHERN-SIMONS HIGGS MODEL II

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ABSTRACT. In this paper we show that the maximal condensate solutions $(\phi^\epsilon, A^\epsilon)$ in the Abelian Chern-Simons Higgs model converge to (ϕ_*, A_*) in higher norms, where ϕ_* is a harmonic map.

1. INTRODUCTION

In this paper we continue our study discussed in [4] on asymptotic behaviors of the maximal solutions of the following equations on Ω :

$$(1.1) \quad D_1\phi + iD_2\phi = 0,$$

$$(1.2) \quad F_A + \frac{2}{\epsilon^2}|\phi|^2(|\phi|^2 - 1) = 0.$$

Here $\epsilon > 0$, $\phi : \Omega \rightarrow \mathbb{C}$ is the complex Higgs field, $A : \Omega \rightarrow \mathbb{R}^2$ is the coupled gauge potential, $D_A\phi = \nabla\phi - iA\phi$ is the covariant derivative, and $F_A = \mathbf{curl}A$ is the magnetic field. The domain Ω is a basic lattice cell in \mathbb{R}^2 generated by two independent vectors \mathbf{a}^1 and \mathbf{a}^2 , namely,

$$\Omega = \{x \in \mathbb{R}^2 \mid x = s_1\mathbf{a}^1 + s_2\mathbf{a}^2, \ 0 < s_1, s_2 < 1\}.$$

The equations (1.1) and (1.2) arise from the self-dual Abelian Chern-Simons Higgs model proposed by Hong-Kim-Pac [5] and Jackiw-Weinberg [6]. It is easy to see that (1.1) and (1.2) are invariant under the gauge transformation

$$(\phi, A) \rightarrow (e^{i\chi}\phi, A + \nabla\chi),$$

where $\chi : \Omega \rightarrow \mathbb{R}$. In view of gauge invariance we impose the following 't Hooft boundary condition on Ω :

$$(1.3) \quad \begin{aligned} \exp(i\xi_j(x + \mathbf{a}^j))\phi(x + \mathbf{a}^j) &= \exp(i\xi_j(x))\phi(x), \\ (A_k + \partial_k\xi_j)(x + \mathbf{a}^j) &= (A_k + \partial_k\xi_j)(x), \quad k = 1, 2, \\ x \in \Gamma^1 \cup \Gamma^2 - \Gamma^j, \quad &j = 1, 2, \end{aligned}$$

where ξ_1 and ξ_2 are real-valued smooth functions defined in a neighborhood of $\Gamma^2 \cup \{\mathbf{a}^1 + \Gamma^2\}$, $\Gamma^1 \cup \{\mathbf{a}^2 + \Gamma^1\}$, respectively. Here

$$\Gamma^j = \{x \in \mathbf{R}^2 \mid x = s\mathbf{a}^j, \ 0 < s < 1\}, \quad j = 1, 2.$$

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The solutions of (1.1) and (1.2) on Ω under the above boundary conditions are called condensate solutions.

Using the classical Jaffe-Taubes arguments [7], one can show that the equation (1.1) implies that ϕ is holomorphic up to a nonvanishing multiple factor and has exactly N zeros allowing multiplicities. Thus in light of gauge invariance we may assume that ϕ takes the form

$$(1.4) \quad \phi(x) = \exp\left(\frac{1}{2}u(x) + i \sum_{j=1}^k n_j \arg(x - p_j)\right),$$

where the points p_1, \dots, p_k , called the vortex points, are the distinct zeros of ϕ with multiplicities n_1, \dots, n_k , respectively. Then the equation (1.2) can be reduced to

$$(1.5) \quad \Delta u = \frac{4}{\epsilon^2} e^u (e^u - 1) + 4\pi \sum_{j=1}^k n_j \delta_{p_j} \quad \text{on } \Omega,$$

u : doubly periodic.

Here δ_p denotes the Dirac measure concentrated on the point p . Conversely, once we find a solution u of (1.5), we may recover A from (1.1) by the formula

$$(1.6) \quad A_1 + iA_2 = -2i\bar{\partial} \ln \phi,$$

where $\bar{\partial} = (\partial_1 + i\partial_2)/2$.

The first result about the existence of solutions of (1.1) and (1.2) was given by Caffarelli-Yang [2]. They showed that there is a critical value $\epsilon_c < \sqrt{|\Omega|/4\pi N}$ so that for $0 < \epsilon < \epsilon_c$ the equations $(1.1)_\epsilon$ and $(1.2)_\epsilon$ with the boundary condition (1.3) admits a maximal solution $(\phi^\epsilon, A^\epsilon)$ of the form (1.4) and (1.6). The solution $(\phi^\epsilon, A^\epsilon)$ is maximal in the sense that $|\phi^\epsilon|$ has the largest possible value among all the solutions to $(1.1)_\epsilon$ and $(1.2)_\epsilon$ with the same zeros and multiplicities. In [8] it was proved by Tarantello that there exists another solution, and some asymptotic behaviors of solutions as $\epsilon \rightarrow 0$ were studied. See also [3] and references therein for recent progress for related topics.

In [4] we showed that

Theorem 1.1. *Let $(\phi^\epsilon, A^\epsilon)$ be the maximal solution of (1.1) and (1.2) corresponding to ϵ . Let $\Omega' = \Omega \setminus \{p_1, \dots, p_k\}$. Then for each $\alpha \in (0, 1)$, we have*

$$(\phi^\epsilon, A^\epsilon) \rightarrow (\phi_*, A_*) \quad \text{in } C_{loc}^{1,\alpha}(\Omega', \mathbb{C}) \times C_{loc}^{0,\alpha}(\Omega', \mathbb{R}^2)$$

as $\epsilon \rightarrow 0$, where (ϕ_*, A_*) belongs to $W_{loc}^{2,p}(\Omega', \mathbb{C}) \times W_{loc}^{1,p}(\Omega', \mathbb{R}^2)$ for all $p > 1$ and satisfies

$$(1.7) \quad \begin{aligned} \Delta \phi_* + \phi_* |\nabla \phi_*|^2 &= 0, \\ |\phi_*| &= 1, \\ \text{deg}(\phi_*, p_j) &= n_j, \\ A_* &= -i\overline{\phi_*} \nabla \phi_* \end{aligned}$$

on Ω' . In fact,

$$(1.8) \quad \phi_*(z) = \prod_{j=1}^k \frac{(z - p_j)^{n_j}}{|z - p_j|^{n_j}}$$

on Ω' .

The purpose of the present paper is to obtain the speed of the convergence of $(\phi^\epsilon, A^\epsilon)$ to (ϕ_*, A_*) in higher norms. In fact, we establish

Theorem 1.2. *Let $(\phi^\epsilon, A^\epsilon)$ be the maximal solution of (1.1) and (1.2) corresponding to ϵ . Let $\Omega' = \Omega \setminus \{p_1, \dots, p_k\}$. Then for each positive integer s , we have*

$$(\phi^\epsilon, A^\epsilon) \rightarrow (\phi_*, A_*) \quad \text{in } C_{loc}^s(\Omega', \mathbb{C}) \times C_{loc}^{s-1}(\Omega', \mathbb{R}^2).$$

Furthermore for each $K \subset\subset \Omega'$, as $\epsilon \rightarrow 0$,

$$(1.9) \quad \|\phi^\epsilon - \phi_*\|_{C^s(K)} \leq C_{K,s}\epsilon^2$$

and

$$(1.10) \quad \|A^\epsilon - A_*\|_{C^{s-1}(K)} \leq C_{K,s}\epsilon^2.$$

In the next section we give a proof of Theorem 1.2.

2. PROOF OF THEOREM 1.2

We notice that the equation (1.5) can be regarded as an exponential nonlinearity version of the following Ginzburg-Landau equations:

$$(2.1) \quad -\Delta v = \frac{1}{\epsilon^2}v(1 - |v|^2) \quad \text{in } \omega,$$

where $v : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{C}$. The main strategy of the proof of Theorem 1.2 is to apply the same arguments used in [1] to describe the asymptotic behavior of solutions of (2.1).

Let us denote

$$\Theta(x) = 2 \sum_{j=1}^k n_j \arg(x - p_j).$$

Then it comes from (1.4), (1.6), (1.7), and (1.8) that

$$\begin{aligned} \phi^\epsilon &= \exp(u_\epsilon/2 + i\Theta/2), \\ 2A^\epsilon &= \mathbf{curl}u^\epsilon + \nabla\Theta, \\ \phi_* &= \exp(i\Theta/2), \\ 2A_* &= \nabla\Theta, \end{aligned}$$

where $\mathbf{curl}u = (\partial_2 u, -\partial_1 u)$.

In order to prove Theorem 1.2, we need some lemmas.

Lemma 2.1 ([1]). *Let U be a bounded domain in \mathbb{R}^n . Suppose that*

$$\Delta v = f \quad \text{in } U.$$

Then for each $K \subset\subset U$,

$$\|\nabla v\|_{L^\infty(K)}^2 \leq C_K \|v\|_{L^\infty(U)} \left(\|f\|_{L^\infty(U)} + \|v\|_{L^\infty(U)} \right).$$

Lemma 2.2 ([3]). *For every compact subset K of $\Omega \setminus \{p_1, \dots, p_k\}$,*

$$(2.2) \quad 0 \leq 1 - |\phi^\epsilon(x)|^2 \leq C_K \epsilon^2, \quad \forall x \in K,$$

as $\epsilon \rightarrow 0$.

Lemma 2.3. *For each $K \subset\subset \Omega'$ and nonnegative integer s , we have*

$$(2.3) \quad \|1 - e^{u_\epsilon}\|_{C^s(K)} \leq C_{K,s}\epsilon^2$$

as $\epsilon \rightarrow 0$.

Proof. The proof is given by induction. The case $s = 0$ follows from (2.2) and we suppose that (2.3) holds up to $s \geq 0$.

Let us choose $x_0 \in \Omega'$ and $R < \inf\{|x_0 - p_j|/5 : j = 1, \dots, k\}$. Set

$$w_\epsilon = \frac{1 - e^{u_\epsilon}}{\epsilon^2}.$$

Then it comes from (1.5) that

$$(2.4) \quad \Delta u_\epsilon = 4\epsilon^2 w_\epsilon^2 - 4w_\epsilon \quad \text{on } B_{5R}(x_0),$$

where $B_{5R}(x_0)$ is a ball of radius $5R$ centered at x_0 . Since $\|w_\epsilon\|_{C^s(B_{5R})} \leq C$ by induction assumption, it follows from the elliptic estimates that $\|u_\epsilon\|_{W^{s+2,p}(B_{4R})} \leq C$ for all $p > 1$. In particular,

$$(2.5) \quad \|u_\epsilon\|_{C^{s+1}(B_{4R})} \leq C.$$

A short computation yields

$$(2.6) \quad \Delta w_\epsilon = \frac{4}{\epsilon^2} e^{2u_\epsilon} w_\epsilon - \frac{1}{\epsilon^2} e^{u_\epsilon} |\nabla u_\epsilon|^2 \equiv f_\epsilon \quad \text{on } B_{4R}.$$

Then by Lemma 2.1

$$\begin{aligned} & \|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{3R})}^2 \\ & \leq C \|\partial^s w_\epsilon\|_{L^\infty(B_{4R})} \left(\|\partial^s w_\epsilon\|_{L^\infty(B_{4R})} + \|\partial^s f_\epsilon\|_{L^\infty(B_{4R})} \right) \\ & \leq C \left(1 + \frac{1}{\epsilon^2} \|w_\epsilon\|_{C^s(B_{4R})} \cdot \|u_\epsilon\|_{C^{s+1}(B_{4R})} \right) \\ & \leq C\epsilon^{-2}. \end{aligned}$$

Hence we have

$$(2.7) \quad \|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{3R})} \leq C\epsilon^{-1}.$$

Differentiating (2.4) $(s+1)$ times and applying Lemma 2.1, we deduce from (2.5) and (2.7) that

$$\begin{aligned} & \|\partial^{s+2} u_\epsilon\|_{L^\infty(B_{2R})}^2 \\ & \leq C \|\partial^{s+1} u_\epsilon\|_{L^\infty(B_{3R})} \left(\|\partial^{s+1} u_\epsilon\|_{L^\infty(B_{3R})} + \|\Delta \partial^{s+1} u_\epsilon\|_{L^\infty(B_{3R})} \right) \\ & \leq C \left(1 + \|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{3R})} \right) \\ & \leq C\epsilon^{-1}. \end{aligned}$$

Thus

$$(2.8) \quad \|\partial^{s+2} u_\epsilon\|_{L^\infty(B_{2R})} \leq C\epsilon^{-1/2}.$$

Next differentiating (2.6) $(s+1)$ times and applying Lemma 2.1 again, we find by (2.7) and (2.8) that

$$\begin{aligned} & \|\partial^{s+2} w_\epsilon\|_{L^\infty(B_R)}^2 \\ & \leq C \|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{2R})} \left(\|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{2R})} + \|\partial^{s+1} f_\epsilon\|_{L^\infty(B_{2R})} \right) \\ & \leq \frac{C}{\epsilon^2} \|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{2R})} \left(\|\partial^{s+1} w_\epsilon\|_{L^\infty(B_{2R})} + \|\partial^{s+2} u_\epsilon\|_{L^\infty(B_{2R})} \right) \\ & \leq C\epsilon^{-4}. \end{aligned}$$

Therefore

$$(2.9) \quad \|\partial^{s+2}w_\epsilon\|_{L^\infty(B_R)} \leq C\epsilon^{-2}.$$

Now let us rewrite (2.6) as

$$(2.10) \quad -\epsilon^2\Delta w_\epsilon + 4w_\epsilon = 8\epsilon^2w_\epsilon^2 - 4\epsilon^4w_\epsilon^3 - \epsilon^2\nabla u_\epsilon \cdot \nabla w_\epsilon \equiv h_\epsilon \quad \text{on } B_R.$$

Then

$$-\epsilon^2\Delta\partial^{s+1}w_\epsilon + 4\partial^{s+1}w_\epsilon = \partial^{s+1}h_\epsilon \quad \text{on } B_R.$$

It is seen from (2.5), (2.7), (2.8), and (2.9) that

$$\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)} \leq C.$$

Set

$$v = \partial^{s+1}w_\epsilon - \frac{\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)}}{4}.$$

From (2.7) we are led to

$$(2.11) \quad \begin{aligned} -\epsilon^2\Delta v + 4v &\leq 0 && \text{in } B_R, \\ v &\leq \frac{C}{\epsilon} - \frac{\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)}}{4} && \text{on } \partial B_R. \end{aligned}$$

On the other hand it is easy to check that for $\epsilon < R$, the function

$$V(r) = \frac{C}{\epsilon} \exp\left(\frac{1}{2\epsilon R}(r^2 - R^2)\right), \quad r = |x - x_0|$$

is a supersolution of (2.11). This implies that

$$\partial^{s+1}w_\epsilon(r) \leq \frac{\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)}}{4} + \frac{C}{\epsilon} \exp\left(-\frac{3R}{8\epsilon}\right) \quad \text{on } B_{R/2}.$$

Similarly,

$$\partial^{s+1}w_\epsilon(r) \geq -\frac{\|\partial^{s+1}h_\epsilon\|_{L^\infty(B_R)}}{4} - \frac{C}{\epsilon} \exp\left(-\frac{3R}{8\epsilon}\right) \quad \text{on } B_{R/2}.$$

As a consequence we conclude that

$$\|\partial^{s+1}w_\epsilon\|_{L^\infty(B_{R/2})} \leq C,$$

and the proof is completed. \square

Corollary 2.4. *For each $K \subset\subset \Omega'$ and nonnegative integer s , we have*

$$(2.12) \quad \|\nabla u_\epsilon\|_{C^s(K)} \leq C_{K,s}\epsilon^2$$

as $\epsilon \rightarrow 0$.

Proof. By (2.2) we may assume that $e^{u_\epsilon} \geq 1/2$ on K as $\epsilon \rightarrow 0$. Then (2.12) immediately follows from (2.3). \square

We are now in a position to prove Theorem 1.2. Given $K \subset\subset \Omega'$, we may suppose by (2.2) that $e^{u_\epsilon} \geq 1/2$ on K . Then we observe from (2.3) that

$$(2.13) \quad \|1 - e^{u_\epsilon/2}\|_{C^s(K)} \leq C_{K,s}\epsilon^2.$$

In the sequel (1.9) and (1.10) is verified from (2.12) and (2.13) by the formula

$$\begin{aligned} \phi_* - \phi^\epsilon &= (1 - e^{u_\epsilon/2})e^{i\Theta/2}, \\ A_* - A^\epsilon &= \frac{1}{2}\mathbf{curl}u_\epsilon. \end{aligned}$$

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