

AN APPLICATION OF BOCHNER'S TECHNIQUE TO THE DEFORMATIONS OF THE COMPLEX STRUCTURE OF $\mathbb{C}\mathbb{P}^n$

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ABSTRACT. We consider the deformation of the complex structure of $\mathbb{C}\mathbb{P}^n$. We show that a harmonic deformation on $\mathbb{C}\mathbb{P}^n$ equipped with the Fubini-Study metric is trivial if its supernorm is appropriately small.

INTRODUCTION

Let $\mathbb{C}\mathbb{P}^n$ be the n -dimensional complex projective space equipped with the Fubini-Study metric. Let T be its holomorphic tangent bundle, and let M be the underlying differentiable manifold. Suppose that φ is a smooth $(0, 1)$ -form on $\mathbb{C}\mathbb{P}^n$ with value in T . If the supernorm of φ is < 1 and φ satisfies the integrability condition

$$(1) \quad \bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi],$$

then such a φ determines a complex structure on M . We call φ a **deformation** and denote the deformed complex manifold by M_φ . A natural question to ask is whether M_φ is biholomorphic to $\mathbb{C}\mathbb{P}^n$. It is well known that M_φ is biholomorphic to $\mathbb{C}\mathbb{P}^n$ if φ and its derivatives are sufficiently small. However it seems unclear if M_φ is biholomorphic to $\mathbb{C}\mathbb{P}^n$ for $n > 2$ if we only assume that the supernorm of φ is appropriately small. If a deformation φ also satisfies the equation

$$(2) \quad \bar{\partial}^*\varphi = 0,$$

where $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$, then (1) and (2) become a strongly elliptic system if the supernorm of φ is appropriately small. Equation (2) was introduced by Kuranishi in [2] to study the versal deformation of compact complex manifolds. For convenience sake, we call a deformation φ a **harmonic deformation** if it also satisfies equation (2). One can ask if such a φ on $\mathbb{C}\mathbb{P}^n$ is zero so that M_φ is biholomorphic to $\mathbb{C}\mathbb{P}^n$. The arguments in [2] (see also [1]) only demonstrate that such a φ is 0 if we assume a priori that the Sobolev norm $\|\varphi\|_k$, say for $k \geq n + 2$, is sufficiently small. It follows from the Sobolev embedding theorem that the $C^{1,\alpha}$ norm of such a φ must be assumed to be small. In this note we show that we can remove the assumption that the Sobolev norm of φ is sufficiently small. Using Bochner's technique, we prove that $\varphi = 0$ if the supernorm of φ is appropriately small.

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1. PRELIMINARIES

In this section we fix some notation and list some well-known formulas needed in section 2.

Let M be an n -dimensional compact complex manifold with a Kähler metric g . Let ω be its Kähler form. Let T be the holomorphic tangent bundle of M , and let $A^{p,q}(T)$ denote the space of $C^\infty(p, q)$ -forms with values in T .

In general, $e(\psi)$ denotes the exterior multiplication by ψ , i.e.,

$$e(\psi).\varphi = \psi \wedge \varphi.$$

In particular, we write $L = e(\omega)$, i.e.,

$$L\varphi = \omega \wedge \varphi,$$

so that

$$L : A^{p,q}(T) \longrightarrow A^{p+1,q+1}(T).$$

We set

$$\Lambda = *^{-1} \circ L \circ * : A^{p,q}(T) \longrightarrow A^{p-1,q-1}(T)$$

where $*$ is the Hodge star operator and Λ is the adjoint of L . We note that both L and Λ are algebraic operators.

Let ∇ be the canonical connection on T . Then

$$\nabla = \nabla' + \bar{\partial}$$

where

$$\nabla' : A^{p,q}(T) \longrightarrow A^{p+1,q}(T)$$

and

$$\bar{\partial} : A^{p,q}(T) \longrightarrow A^{p,q+1}(T).$$

We define

- (a) $\bar{\square} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$,
- (b) $\square = \nabla'\nabla'^* + \nabla'^*\nabla'$,

on $A^{p,q}(T)$. We have the following well-known identity:

Lemma 1. $\bar{\square} = \square + [i\Theta(T), \Lambda]$ where $\Theta(T)$ is the curvature form of T .

We also need the following result:

Lemma 2. Let p be a point on M . Let $\{e_1, \dots, e_n\}$ be a normal frame field of $(1, 0)$ type around p . Suppose that $\varphi \in A^{0,1}(T)$. Then at p we have

- (a1) $\bar{\square}\varphi = -\nabla_{e_i}\nabla_{\bar{e}_i}\varphi + S(\varphi)$,
- (b1) $\square\varphi = -\nabla_{\bar{e}_i}\nabla_{e_i}\varphi$,

where

$$S(\varphi)(\bar{X}) = R(e_i, \bar{X})\varphi.\bar{e}_i$$

for $\bar{X} \in T_pM^{0,1}$ and

$$R(e_i, \bar{X}) = \nabla_{e_i}\nabla_{\bar{X}} - \nabla_{\bar{X}}\nabla_{e_i} - \nabla_{[e_i, \bar{X}]}$$

2. HARMONIC DEFORMATIONS ON $\mathbb{C}\mathbb{P}^n$

In this section we prove the following result:

Theorem 1. *Suppose that φ is a harmonic deformation on $\mathbb{C}\mathbb{P}^n$ equipped with the Fubini-Study metric. There exists a constant $\epsilon(n) > 0$ such that if $\|\varphi\|_\infty \leq \epsilon(n)$, then*

$$\varphi = 0.$$

Here $\|\varphi\|_\infty$ is the supernorm of φ .

Proof. If we can show that

$$\bar{\square}|\varphi|^2 = \square\langle\varphi, \varphi\rangle \geq c(\|\bar{\partial}\varphi\|^2 + \|\nabla'\varphi\|^2)$$

for some constant $c > 0$ when $\|\varphi\|_\infty$ is appropriately small, then $\bar{\partial}\varphi = 0$. This together with $\bar{\partial}^*\varphi = 0$ implies $\varphi = 0$.

Suppose that p is an arbitrary point on $\mathbb{C}\mathbb{P}^n$. Then we can find a coordinate (z^1, \dots, z^n) around p such that $p = (0, \dots, 0)$ and

$$\begin{aligned} \Phi &= \log(1 + |z|^2) \\ &= |z|^2 - |z|^4/2 + |z|^6/3 - \dots \end{aligned}$$

is the Kähler potential of the Fubini-Study metric. The metric at p is given by

$$g_{i\bar{j}}(0) = \frac{\partial^2\Phi(0)}{\partial z^i\partial\bar{z}^j} = \delta_{ij}$$

and the curvature is

$$R^i_{j\bar{a}\bar{b}}(0) = \frac{\partial^4(|z|^4/2)}{\partial z^a\partial\bar{z}^b\partial z^j\partial\bar{z}^i}.$$

The only nonzero $R^i_{j\bar{a}\bar{b}}$ are

$$\begin{aligned} R^a_{aa\bar{a}\bar{a}} &= 2, \\ R^i_{ia\bar{a}\bar{a}} &= 1, i \neq a, \\ R^i_{ji\bar{j}\bar{j}} &= 1, i \neq j. \end{aligned}$$

We can find a normal frame field $\{e_1, \dots, e_n\}$ of type $(1, 0)$ around p such that

$$e_i(0) = \partial/\partial z^i(0), i = 1, \dots, n.$$

Let $\{\theta^1, \dots, \theta^n\}$ be the dual frame of $\{e_1, \dots, e_n\}$. Then in a neighborhood of p , we have

$$\begin{aligned} L &= ie(\theta^k)e(\bar{\theta}^k), \\ \Lambda &= -i\tau(\bar{\theta}^k)\tau(\theta^k) \end{aligned}$$

where τ is the interior product operator. We observe

$$\begin{aligned} \Theta(T) &= \sum(R^i_{j\bar{a}\bar{b}}\theta^a \wedge \bar{\theta}^b)\theta^j \otimes e_i \\ &= \Theta_1 + \Theta_2 + \Theta_3 \end{aligned}$$

where

$$\begin{aligned}\Theta_1 &= 2 \sum (\theta^i \wedge \bar{\theta}^i) \theta^i \otimes e_i, \\ \Theta_2 &= \sum_{i \neq a} (\theta^a \wedge \bar{\theta}^a) \theta^i \otimes e_i, \\ \Theta_3 &= \sum_{i \neq j} (\theta^i \wedge \bar{\theta}^j) \theta^j \otimes e_i.\end{aligned}$$

At p we have

$$\begin{aligned}\bar{\square} \langle \varphi, \varphi \rangle &= e_i \bar{e}_i \langle \varphi, \varphi \rangle \\ &= e_i \langle \nabla_{\bar{e}_i} \varphi, \varphi \rangle + e_i \langle \varphi, \nabla_{e_i} \varphi \rangle \\ &= \langle \nabla_{e_i} \nabla_{\bar{e}_i} \varphi, \varphi \rangle + \langle \nabla_{\bar{e}_i} \varphi, \nabla_{e_i} \varphi \rangle \\ (3) \quad &+ \langle \nabla_{e_i} \varphi, \nabla_{e_i} \varphi \rangle + \langle \varphi, \nabla_{\bar{e}_i} \nabla_{e_i} \varphi \rangle \\ &= |\nabla_{e_i} \varphi|^2 + |\nabla_{\bar{e}_i} \varphi|^2 + \langle \nabla_{e_i} \nabla_{\bar{e}_i} \varphi, \varphi \rangle \\ &+ \langle \varphi, \nabla_{\bar{e}_i} \nabla_{e_i} \varphi \rangle.\end{aligned}$$

It follows from the hypothesis that

$$(4) \quad \bar{\square} \varphi = \frac{1}{2} \bar{\partial}^* [\varphi, \varphi].$$

Lemmas 1 and 2 and (4) yield

$$\begin{aligned}&\langle \nabla_{e_i} \nabla_{\bar{e}_i} \varphi, \varphi \rangle + \langle \varphi, \nabla_{\bar{e}_i} \nabla_{e_i} \varphi \rangle \\ &= -\langle \bar{\square} \varphi, \varphi \rangle + \langle \varphi, S(\varphi) \rangle - \langle \varphi, \square \varphi \rangle \\ (5) \quad &= -\langle \bar{\square} \varphi, \varphi \rangle - \langle \square \varphi, \varphi \rangle + \langle \varphi, S(\varphi) \rangle \\ &= -2\langle \bar{\square} \varphi, \varphi \rangle + \langle [i\Theta(T), \Lambda] \varphi, \varphi \rangle + \langle \varphi, S(\varphi) \rangle \\ &= -\langle \bar{\partial}^* [\varphi, \varphi], \varphi \rangle + \langle [i\Theta(T), \Lambda] \varphi, \varphi \rangle + \langle \varphi, S(\varphi) \rangle \\ &= -\langle [\varphi, \varphi], \bar{\partial} \varphi \rangle + \langle [i\Theta(T), \Lambda] \varphi, \varphi \rangle + \langle \varphi, S(\varphi) \rangle.\end{aligned}$$

$[i\Theta(T), \Lambda] \varphi$ can be written as:

$$\begin{aligned}&[i\Theta(T), \Lambda] \varphi \\ &= -i\Lambda \Theta(T) \varphi \\ (6) \quad &= -\sum_p \tau(\bar{\theta}^p) \tau(\theta^p) \Theta(T) \varphi \\ &= -\sum_{j=1}^3 \sum_p \tau(\bar{\theta}^p) \tau(\theta^p) \Theta_j \varphi.\end{aligned}$$

Let

$$\varphi = \varphi_k^l \bar{\theta}^k \otimes e_l.$$

We compute each term in (6) separately:

$$\begin{aligned}
 \varphi_1 &= - \sum_p \tau(\bar{\theta}^p)\tau(\theta^p)\Theta_1\varphi \\
 &= - \sum_p \tau(\bar{\theta}^p)\tau(\theta^p)\{2 \sum (\theta^i \wedge \bar{\theta}^i)\theta^i \otimes e_i(\varphi_k^l \bar{\theta}^k \otimes e_l)\} \\
 (7) \quad &= - \sum_{i \neq k} 2\delta_{il}\varphi_k^l \bar{\theta}^k \otimes e_i \\
 &= -2 \sum_{k \neq l} \varphi_k^l \bar{\theta}^k \otimes e_l,
 \end{aligned}$$

$$\begin{aligned}
 \varphi_2 &= - \sum_p \tau(\bar{\theta}^p)\tau(\theta^p)\Theta_2\varphi \\
 &= - \sum_p \tau(\bar{\theta}^p)\tau(\theta^p)\{\sum_{i \neq a} (\theta^a \wedge \bar{\theta}^a)\theta^i \otimes e_i(\varphi_k^l \bar{\theta}^k \otimes e_l)\} \\
 (8) \quad &= - \sum_{\substack{a \neq k \\ a \neq i}} \delta_{il}\varphi_k^l \bar{\theta}^k \otimes e_i \\
 &= - \sum_k (n-1)\varphi_k^k \bar{\theta}^k \otimes e_k - \sum_{k \neq l} (n-2)\varphi_k^l \bar{\theta}^k \otimes e_l,
 \end{aligned}$$

$$\begin{aligned}
 \varphi_3 &= - \sum_p \tau(\bar{\theta}^p)\tau(\theta^p)\Theta_3\varphi \\
 &= - \sum_p \tau(\bar{\theta}^p)\tau(\theta^p)\{\sum_{i \neq j} (\theta^i \wedge \bar{\theta}^j)\theta^j \otimes e_i(\varphi_k^l \bar{\theta}^k \otimes e_l)\} \\
 (9) \quad &= \sum_{j \neq k} \delta_{jl}\varphi_k^l \bar{\theta}^j \otimes e_k \\
 &= \sum_{k \neq l} \varphi_k^l \bar{\theta}^l \otimes e_k.
 \end{aligned}$$

Summarizing (7), (8), (9), we obtain

$$(10) \quad [i\Theta(T), \Lambda]\varphi = -(n-1) \sum_k \varphi_k^k \bar{\theta}^k \otimes e_k - n \sum_{k \neq l} \varphi_k^l \bar{\theta}^k \otimes e_l + \sum_{k \neq l} \varphi_k^l \bar{\theta}^l \otimes e_k.$$

Hence

$$\begin{aligned}
 (11) \quad &\langle [i\Theta(T), \Lambda]\varphi, \varphi \rangle \\
 &= -(n-1) \sum_k |\varphi_k^k|^2 - n \sum_{k \neq l} |\varphi_k^l|^2 + \sum_{k \neq l} \varphi_k^l \bar{\varphi}_l^k.
 \end{aligned}$$

Now let's evaluate $S(\varphi)$:

$$\begin{aligned}
 (12) \quad &S(\varphi)(\bar{e}_j) \\
 &= R(e_i, \bar{e}_j)\varphi \cdot \bar{e}_i \\
 &= \varphi_k^l \{ \langle R(e_i, \bar{e}_j)\bar{\theta}^k \otimes e_l, \bar{e}_i \rangle + \langle \bar{\theta}^k \otimes R(e_i, \bar{e}_j)e_l, \bar{e}_i \rangle \} \\
 &= \varphi_k^l \{ -\langle \bar{\theta}^k \otimes R(e_i, \bar{e}_j)\bar{e}_i, e_l \rangle + \delta_{ki}R(e_i, \bar{e}_j)e_l \} \\
 &= \varphi_k^l \{ \langle \bar{\theta}^k, R(e_j, \bar{e}_i)e_i \rangle e_l + \delta_{ki}R(e_i, \bar{e}_j)e_l \} \\
 &= \varphi_k^l \{ R_{ij\bar{i}}^k e_l + \delta_{ki}R_{l\bar{i}j}^p e_p \}.
 \end{aligned}$$

Let $f(j, k, l) = R_{ij\bar{i}}^k e_l + \delta_{ki} R_{li\bar{j}}^p e_p$. When $k = l$ and $j = k$, we have

$$\begin{aligned}
 f(j, k, l) &= R_{ik\bar{i}}^k e_k + \delta_{ki} R_{ki\bar{k}}^p e_p \\
 &= R_{kk\bar{k}}^k e_k + \sum_{i \neq k} R_{ik\bar{i}}^k e_k + R_{kk\bar{k}}^p e_p \\
 &= 2e_k + (n-1)e_k + 2e_k \\
 &= (n+3)e_k.
 \end{aligned}
 \tag{13}$$

When $k = l$ and $j \neq k$, we have

$$\begin{aligned}
 f(j, k, l) &= R_{ij\bar{i}}^k e_k + R_{kk\bar{j}}^p e_p \\
 &= 0.
 \end{aligned}
 \tag{14}$$

When $k \neq l$ and $j = k$, we have

$$\begin{aligned}
 f(j, k, l) &= R_{ik\bar{i}}^k e_l + \delta_{ki} R_{li\bar{k}}^p e_p \\
 &= R_{kk\bar{k}}^k e_l + \sum_{i \neq k} R_{ik\bar{i}}^k e_l + R_{lk\bar{k}}^p e_p \\
 &= 2e_l + (n-1)e_l + e_l \\
 &= (n+2)e_l.
 \end{aligned}
 \tag{15}$$

When $k \neq l$, $j \neq k$ and $j = l$, we have

$$\begin{aligned}
 f(j, k, l) &= R_{il\bar{i}}^k e_l + R_{lk\bar{l}}^p e_p \\
 &= 0 + e_k \\
 &= e_k.
 \end{aligned}
 \tag{16}$$

When $k \neq l$, $j \neq k$ and $j \neq l$, we have

$$\begin{aligned}
 f(j, k, l) &= 0 + R_{lk\bar{j}}^p e_p \\
 &= 0.
 \end{aligned}
 \tag{17}$$

Hence we obtain

$$S(\varphi)\bar{e}_j = \sum_j (n+3)\varphi_j^j e_j + \sum_{l \neq j} (n+2)\varphi_j^l e_l + \sum_{k \neq j} \varphi_k^j e_k.
 \tag{18}$$

Thus

$$\begin{aligned}
 S(\varphi) &= \sum_j (n+3)\varphi_j^j \bar{\theta}^j \otimes e_j + \sum_{l \neq j} (n+2)\varphi_j^l \bar{\theta}^j \otimes e_l \\
 &\quad + \sum_{k \neq j} \varphi_k^j \bar{\theta}^j \otimes e_k.
 \end{aligned}
 \tag{19}$$

Therefore

$$\langle \varphi, S(\varphi) \rangle = \sum_k (n+3)|\varphi_k^k|^2 + \sum_{k \neq l} (n+2)|\varphi_k^l|^2 + \sum_{k \neq l} \varphi_k^l \bar{\varphi}_l^k
 \tag{20}$$

and

$$\begin{aligned}
 & \langle [i\Theta(T), \Lambda]\varphi, \varphi \rangle + \langle \varphi, S(\varphi) \rangle \\
 &= 4 \sum_k |\varphi_k^k|^2 + 2 \sum_{k \neq l} |\varphi_k^l|^2 + 2 \sum_{k \neq l} \varphi_k^l \bar{\varphi}_l^k \\
 (21) \quad & \geq 4 \sum_k |\varphi_k^k|^2 + 2 \sum_{k \neq l} |\varphi_k^l|^2 - 2 \sum_{k \neq l} |\varphi_k^l|^2 \\
 &= 4 \sum_k |\varphi_k^k|^2 \\
 & \geq 0.
 \end{aligned}$$

We observe

$$\begin{aligned}
 (22) \quad & \|\nabla_{e_i} \varphi\|^2 + \|\nabla_{\bar{e}_i} \varphi\|^2 + \langle [\varphi, \varphi], \bar{\partial} \phi \rangle \\
 & \geq \|\nabla_{e_i} \varphi\|^2 + \|\nabla_{\bar{e}_i} \varphi\|^2 - \|[\varphi, \varphi]\| \|\bar{\partial} \phi\| \\
 & \geq c(\|\nabla_{e_i} \varphi\|^2 + \|\nabla_{\bar{e}_i} \varphi\|^2)
 \end{aligned}$$

for some constant $c > 0$ if $\|\varphi\|_\infty$ is appropriately small. Equations (3), (5), (21) and (22) yield

$$(23) \quad \bar{\square} \|\varphi\|^2 \geq c(\|\nabla_{e_i} \varphi\|^2 + \|\nabla_{\bar{e}_i} \varphi\|^2)$$

and this implies

$$(24) \quad \varphi = 0.$$

□

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