

ON ALTERNATING ANALOGUES OF TORNHEIM'S DOUBLE SERIES

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ABSTRACT. In this paper, we give some evaluation formulas for alternating analogues of Tornheim's double series. These can be regarded as alternating analogues of Mordell's formulas. This gives a partial answer to the problem posed by Subbarao-Sitaramachandrarao.

1. INTRODUCTION

Tornheim considered the double series $T(p, q, r)$ defined by

$$(1.1) \quad T(p, q, r) = \sum_{m, n=1}^{\infty} \frac{1}{m^p n^q (m+n)^r},$$

where p, q, r are nonnegative integers with $p+r > 1$, $q+r > 1$ and $p+q+r > 2$ (see [5]). He showed that $T(p, q, N-p-q)$ is a polynomial in $\{\zeta(j) \mid 2 \leq j \leq N\}$ with rational coefficients when N is odd and $N \geq 3$ (see also [2]).

In [3], Mordell gave an evaluation formula for $T(2k, 2k, 2k)$ for a positive integer k . Furthermore, in [4], Subbarao and Sitaramachandrarao generalized Mordell's formula, and considered alternating analogues of (1.1) defined by

$$(1.2) \quad R(p, q, r) = \sum_{m, n=1}^{\infty} \frac{(-1)^n}{m^p n^q (m+n)^r},$$

$$(1.3) \quad S(p, q, r) = \sum_{m, n=1}^{\infty} \frac{(-1)^{m+n}}{m^p n^q (m+n)^r}.$$

They posed the problem to evaluate $R(p, p, p)$ and $S(p, p, p)$ for any positive integer p . As a partial answer to their problem, we gave an evaluation formula for $S(p, p, p)$ for any positive *odd* integer p (see [6], Corollary 3).

The purpose of this paper is to give an evaluation formula for $R(p, p, p)$ for any *odd* positive integer p with $p \geq 3$ (see Theorem 3.6). In order to prove this formula,

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we make use of the same method as we introduced in [6]. Indeed, we consider *partial* Tornheim's double series

$$(1.4) \quad \mathfrak{T}_{b_1, b_2}(p, q, r) = \sum_{m, n=0}^{\infty} \frac{1}{(2m + b_1)^p (2n + b_2)^q (2m + 2n + b_1 + b_2)^r},$$

where $b_1, b_2 \in \{1, 2\}$. As a result, we can write $\mathfrak{T}_{1,1}(p, p, q)$ as a rational linear combination of products of Riemann's zeta values at positive integers, when p and q are odd positive integers with $q \geq 3$ (see Proposition 3.5).

More general results on *partial* Tornheim's double series defined by (1.4) will be given in [7].

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2. PRELIMINARIES

Let \mathbb{N} be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathbb{Z} the ring of rational integers, and \mathbb{R} the field of real numbers. Let $i = \sqrt{-1}$. Throughout this paper we fix $\delta \in \mathbb{R}$ with $\delta > 0$. For $u \in \mathbb{R}$ with $u \in [1, 1 + \delta]$ and $s \in \mathbb{R}$, we define $\rho(s; u) := \sum_{m \geq 0} (-u)^{-m} / (2m + 1)^s$. If $u > 1$, then $\rho(s; u)$ is convergent for any $s \in \mathbb{Z}$. Let $\rho(s) := \rho(s; 1)$. We define a set of numbers $\{\mathcal{E}_m(u)\}$ by

$$(2.1) \quad F(x; u) = \frac{2ue^x}{e^{2x} + u} = \sum_{m=0}^{\infty} \mathcal{E}_m(u) \frac{x^m}{m!}.$$

Note that $\{\mathcal{E}_m(1)\}$ are the Euler numbers (see, e.g., [1]). So we have $\mathcal{E}_{2j+1}(1) = 0$ ($j \in \mathbb{N}_0$). It follows from (2.1) that if $u \in [1, 1 + \delta]$, then

$$(2.2) \quad \liminf_{m \rightarrow \infty} \left(\frac{|\mathcal{E}_m(u)|}{m!} \right)^{-1/m} \geq \frac{\pi}{2}.$$

From the relation $F(x; u) = 2 \sum_{n \geq 0} (-u)^{-n} e^{(2n+1)x}$, we obtain the following.

Lemma 2.1. For $k \in \mathbb{N}_0$ and $u \in (1, 1 + \delta]$,

$$(2.3) \quad \rho(-k; u) = \frac{1}{2} \mathcal{E}_k(u).$$

For $r \in \mathbb{N}$, $p \in \mathbb{N}_0$, $u \in [1, 1 + \delta]$ and $\theta \in \mathbb{R}$, we define

$$(2.4) \quad \mathcal{X}_p(\theta, r; u) := \sum_{n=0}^{\infty} \frac{(-u)^{-n} \sin^{(p)}((2n+1)\theta)}{(2n+1)^r},$$

where we denote the l -th derivative of a function $f(\theta)$ by $f^{(l)}(\theta)$. Using the well-known relation

$$(2.5) \quad \sin^{(p)}(\theta) = \frac{i^{p-1}}{2} (e^{i\theta} + (-1)^{p-1} e^{-i\theta}) = i^{p-1} \sum_{n=0}^{\infty} \lambda_{p+1+n} \frac{(i\theta)^n}{n!},$$

where $\lambda_j := (1 + (-1)^j)/2$, we have

$$(2.6) \quad \mathcal{X}_p(\theta, r; u) = i^{p-1} \sum_{m=0}^{\infty} \rho(r-m; u) \lambda_{p+1+m} \frac{(i\theta)^m}{m!},$$

when $u \in (1, 1 + \delta]$. By (2.2) and (2.3), we see that (2.6) is uniformly convergent with respect to $u \in (1, 1 + \delta]$ when $\theta \in (-\pi/2, \pi/2)$. Furthermore we define

$$(2.7) \quad \mathcal{Y}_p(\theta, r; u) := \mathcal{X}_p(\theta, r; u) - i^{p-1} \sum_{j=0}^r \rho(r-j; u) \lambda_{p+1+j} \frac{(i\theta)^j}{j!},$$

for $r \in \mathbb{N}$, $p \in \mathbb{N}_0$, $u \in [1, 1 + \delta]$ and $\theta \in \mathbb{R}$. When $u \in (1, 1 + \delta]$,

$$\mathcal{Y}_p(\theta, r; u) = i^{p-1} \sum_{n=1}^{\infty} \rho(-n; u) \lambda_{p+1+n+r} \frac{(i\theta)^{n+r}}{(n+r)!}.$$

This is also uniformly convergent with respect to $u \in (1, 1 + \delta]$ when $\theta \in (-\pi/2, \pi/2)$. So it follows from Lemma 2.1 and the fact $\mathcal{E}_{2j+1}(1) = 0$ ($j \in \mathbb{N}_0$) that

$$(2.8) \quad \mathcal{Y}_r(\theta, r; u) \rightarrow 0 \quad (u \rightarrow 1; r \in \mathbb{N}, \theta \in (-\pi/2, \pi/2)).$$

Now we define

$$(2.9) \quad \mathfrak{S}(k, s; u) := \sum_{m,n=0}^{\infty} \frac{(-u)^{-m-n}}{\{(2m+1)(2n+1)\}^{2k+1} (2m+2n+2)^s},$$

$$(2.10) \quad \mathfrak{R}(k, s; u) := \sum_{m,n=0}^{\infty} \frac{(-u)^{-2m-n-1}}{\{(2m+1)(2m+2n+3)\}^{2k+1} (2n+2)^s},$$

for $k \in \mathbb{N}_0$, $s \in \mathbb{Z}$, $u \in [1, 1 + \delta]$. By an elementary calculation just the same as that in Lemma 3 of [6], we obtain the following.

Lemma 2.2. For $k \in \mathbb{N}_0$, $u \in (1, 1 + \delta]$ and $\theta \in \mathbb{R}$,

$$(2.11) \quad \sum_{m=0}^{\infty} \frac{(-u)^{-m} e^{i(2m+1)\theta}}{(2m+1)^{2k+1}} \cdot \sum_{n=0}^{\infty} \frac{(-u)^{-n} e^{i(2n+1)\theta}}{(2n+1)^{2k+1}} = \sum_{m=0}^{\infty} \mathfrak{S}(k, -m; u) \frac{(i\theta)^m}{m!},$$

$$(2.12) \quad \sum_{m=0}^{\infty} \frac{(-u)^{-m} e^{-i(2m+1)\theta}}{(2m+1)^{2k+1}} \cdot \sum_{n=0}^{\infty} \frac{(-u)^{-n} e^{i(2n+1)\theta}}{(2n+1)^{2k+1}} \\ = \sum_{m=0}^{\infty} \mathfrak{R}(k, -m; u) \{1 + (-1)^m\} \frac{(i\theta)^m}{m!} + \sum_{m=0}^{\infty} \frac{u^{-2m}}{(2m+1)^{4k+2}}.$$

For $n \in \mathbb{Z}$, $k \in \mathbb{N}_0$ and $u \in (1, 1 + \delta]$, we define

$$(2.13) \quad \beta_n(k; u) := \frac{1}{2} \{ \mathfrak{S}(k, -n; u) + (1 + (-1)^n) \mathfrak{R}(k, -n; u) \} \\ - \sum_{\nu=0}^k \binom{n}{2\nu} \rho(2k+1-2\nu; u) \rho(2k+1+2\nu-n; u).$$

In particular when $n \leq -1$, we define $\beta_n(k; 1)$ by (2.13) with $u = 1$. By combining (2.13) and Lemma 2.2, we obtain the following.

Lemma 2.3. For $k \in \mathbb{N}_0$, $u \in (1, 1 + \delta]$ and $\theta \in \mathbb{R}$,

$$(2.14) \quad \mathcal{Y}_{2k+1}(\theta, 2k+1; u) \sum_{n=0}^{\infty} \frac{(-u)^{-n} e^{i(2n+1)\theta}}{(2n+1)^{2k+1}} \\ = (-1)^k \sum_{n=0}^{\infty} \beta_n(k; u) \frac{(i\theta)^n}{n!} + \frac{(-1)^k}{2} \sum_{m=0}^{\infty} \frac{u^{-2m}}{(2m+1)^{4k+2}}.$$

Proof. By (2.5) and Lemma 2.2, we have

$$\begin{aligned} & \mathcal{X}_{2k+1}(\theta, 2k + 1; u) \sum_{n=0}^{\infty} \frac{(-u)^{-n} e^{i(2n+1)\theta}}{(2n + 1)^{2k+1}} \\ &= \frac{i^{2k}}{2} \sum_{n=0}^{\infty} \{ \mathfrak{S}(k, -n; u) + (1 + (-1)^n) \mathfrak{R}(k, -n; u) \} \frac{(i\theta)^n}{n!} \\ &+ \frac{(-1)^k}{2} \sum_{m=0}^{\infty} \frac{u^{-2m}}{(2m + 1)^{4k+2}}. \end{aligned}$$

On the other hand, by combining (2.6) and

$$(2.15) \quad \sum_{n=0}^{\infty} \frac{(-u)^{-n} e^{i(2n+1)\theta}}{(2n + 1)^{2k+1}} = \sum_{n=0}^{\infty} \rho(2k + 1 - n; u) \frac{(i\theta)^n}{n!},$$

we have

$$\begin{aligned} & \left\{ i^{2k} \sum_{j=0}^{2k+1} \rho(2k + 1 - j; u) \lambda_j \frac{(i\theta)^j}{j!} \right\} \sum_{n=0}^{\infty} \frac{(-u)^{-n} e^{i(2n+1)\theta}}{(2n + 1)^{2k+1}} \\ &= (-1)^k \sum_{n=0}^{\infty} \sum_{\nu=0}^k \binom{n}{2\nu} \rho(2k + 1 - 2\nu; u) \rho(2k + 1 + 2\nu - n; u) \frac{(i\theta)^n}{n!}. \end{aligned}$$

By (2.13), we obtain the proof. □

Since (2.7) and (2.15) are uniformly convergent with respect to $u \in (1, 1 + \delta]$ when $\theta \in (-\pi/2, \pi/2)$ by (2.2), so is (2.14), and

$$(2.16) \quad \liminf_{m \rightarrow \infty} \left(\frac{|\beta_m(k; u)|}{m!} \right)^{-1/m} \geq \frac{\pi}{2},$$

for $k \in \mathbb{N}_0$. Furthermore, by (2.8), we have

$$(2.17) \quad \lim_{u \rightarrow 1} \beta_m(k; u) = 0 \quad (m \in \mathbb{N}),$$

$$(2.18) \quad \lim_{u \rightarrow 1} \beta_0(k; u) = -\frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{(2m + 1)^{4k+2}}.$$

3. EVALUATION FORMULAS

By (2.13), we have

$$(3.1) \quad \begin{aligned} \beta_{2j+1}(k; u) &= \frac{1}{2} \mathfrak{S}(k, -2j - 1; u) \\ &- \sum_{\nu=0}^k \binom{2j + 1}{2\nu} \rho(2k + 1 - 2\nu; u) \rho(2k + 2\nu - 2j; u), \end{aligned}$$

for $j \in \mathbb{N}_0$.

Lemma 3.1. For $k, l \in \mathbb{N}_0$, $p \in \{0, 1\}$, $u \in (1, 1 + \delta]$ and $\theta \in \mathbb{R}$,

$$\begin{aligned}
 (3.2) \quad & \frac{1}{2} \sum_{m,n=0}^{\infty} \frac{(-u)^{-m-n} \sin^{(p)}((2m + 2n + 2)\theta)}{\{(2m + 1)(2n + 1)\}^{2k+1} (2m + 2n + 2)^{2l+p}} \\
 & - \sum_{\nu=0}^k \rho(2k + 1 - 2\nu; u) \sum_{\tau=0}^{2\nu} \binom{2l + p - 1 + 2\nu - \tau}{2l + p - 1} \frac{(-\theta)^\tau}{\tau!} \\
 & \quad \cdot \mathcal{X}_{\tau+p}(\theta; 2k + 2l + 1 + p + 2\nu - \tau; u) \\
 & = i^{p-1} \sum_{j=-l-p}^{\infty} \beta_{2j+1}(k; u) \frac{(i\theta)^{2j+2l+1+p}}{(2j + 2l + 1 + p)!}.
 \end{aligned}$$

Proof. By (2.5) and (2.9), we have

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} \frac{(-u)^{-m-n} \sin^{(p)}((2m + 2n + 2)\theta)}{\{(2m + 1)(2n + 1)\}^{2k+1} (2m + 2n + 2)^{2l+p}} \\
 & = i^{p-1} \sum_{m=-2l-p}^{\infty} \mathfrak{S}(k, -m; u) \lambda_{m+1} \frac{(i\theta)^{m+2l+p}}{(m + 2l + p)!}.
 \end{aligned}$$

On the other hand, we use (2.5) and consider the function $f(x; d, \theta) = \sin^{(p)}(x\theta)x^{-d}$ in the argument of Lemma 6 of [6]. Then we obtain that

$$\begin{aligned}
 & \sum_{\tau=0}^r \binom{d - 1 + r - \tau}{d - 1} \frac{(-\theta)^\tau}{\tau!} \frac{\sin^{(\tau+p)}(\theta x)}{x^{d+r+q-\tau}} \\
 & = i^{p-1} \sum_{m=-d}^{\infty} (-1)^r \binom{m}{r} \frac{(i\theta)^{m+d}}{(m + d)!} \lambda_{p+1+m+d} x^{-q-r+m},
 \end{aligned}$$

by using the well-known relation $\binom{-X}{j} = (-1)^j \binom{X+j-1}{j}$. Putting $r = 2\nu$, $q = 2k + 1$ and $d = 2l + p$, we have

$$\begin{aligned}
 & \sum_{\nu=0}^k \rho(2k + 1 - 2\nu; u) \sum_{\tau=0}^{2\nu} \binom{2l + p - 1 + 2\nu - \tau}{2l + p - 1} \frac{(-\theta)^\tau}{\tau!} \\
 & \quad \cdot \mathcal{X}_{\tau+p}(\theta; 2k + 2l + 1 + p + 2\nu - \tau; u) \\
 & = i^{p-1} \sum_{m=-2l-p}^{\infty} \sum_{\nu=0}^k \binom{m}{2\nu} \rho(2k + 1 - 2\nu; u) \rho(2k + 1 + 2\nu - m; u) \\
 & \quad \cdot \lambda_{m+1} \frac{(i\theta)^{m+2l+p}}{(m + 2l + p)!}.
 \end{aligned}$$

Put $m = 2j + 1$. Then, by (3.1), we obtain the proof. □

By (2.16), we can let $u \rightarrow 1$ in both sides of (3.2) when $l \in \mathbb{N}$ and $\theta \in [-\pi/2, \pi/2]$. By (2.17), we obtain the following.

Proposition 3.2. For $k \in \mathbb{N}_0$, $l \in \mathbb{N}$, $p \in \{0, 1\}$ and $\theta \in [-\pi/2, \pi/2]$,

$$\begin{aligned}
 (3.3) \quad & \frac{1}{2} \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n} \sin^{(p)}((2m+2n+2)\theta)}{\{(2m+1)(2n+1)\}^{2k+1} (2m+2n+2)^{2l+p}} \\
 & - \sum_{\nu=0}^k \rho(2k+1-2\nu) \sum_{\tau=0}^{2\nu} \binom{2l+p-1+2\nu-\tau}{2l+p-1} \frac{(-\theta)^\tau}{\tau!} \\
 & \quad \cdot \mathcal{X}_{\tau+p}(\theta; 2k+2l+1+p+2\nu-\tau; 1) \\
 & = i^{p-1} \sum_{j=-l-p}^{-1} \beta_{2j+1}(k; 1) \frac{(i\theta)^{2j+2l+1+p}}{(2j+2l+1+p)!}.
 \end{aligned}$$

For simplicity, we let $\psi(s) := \sum_{m \geq 0} 1/(2m+1)^s = (1-2^{-s})\zeta(s)$ for $s > 1$, where $\zeta(s)$ is the Riemann zeta function. It is well-known that $\sin^{(2j)}((2m+1)\pi/2) = (-1)^{j+m}$ and $\sin^{(2j+1)}((2m+1)\pi/2) = 0$ for $j, m \in \mathbb{N}_0$. Hence $\mathcal{X}_{2j}(\pi/2, s; 1) = (-1)^j \psi(s)$ and $\mathcal{X}_{2j+1}(\pi/2, s; 1) = 0$. Putting $p = 0$, $\theta = \pi/2$ and $l = m + 1$ for $m \in \mathbb{N}_0$ in (3.3), we have

$$\begin{aligned}
 (3.4) \quad & \sum_{r=0}^m \beta_{2r-2m-1}(k; 1) \frac{(i\pi/2)^{2r+1}}{(2r+1)!} \\
 & = -i \sum_{\nu=0}^k \rho(2k+1-2\nu) \sum_{\eta=0}^{\nu} \binom{2m+1+2\nu-2\eta}{2m+1} \\
 & \quad \cdot \frac{(i\pi/2)^{2\eta}}{(2\eta)!} \psi(2k+2m+3+2\nu-2\eta),
 \end{aligned}$$

for $k \in \mathbb{N}_0$. We recall the following lemma which can be obtained by replacing π with $\pi/2$ in Lemma 8 of [6].

Lemma 3.3. Suppose $\{P_m\}$ and $\{Q_m\}$ are sequences which satisfy the relation

$$\sum_{j=0}^m P_{m-j} \frac{(i\pi/2)^{2j+1}}{(2j+1)!} = Q_m,$$

for any $m \in \mathbb{N}_0$. Then the relation

$$P_m = \frac{2}{i\pi} \sum_{\nu=0}^m (1-2^{2\nu+1-2m}) 2^{2\nu+1-2m} \zeta(2m-2\nu) Q_\nu$$

holds for any $m \in \mathbb{N}_0$. Note that $\zeta(0) = -\frac{1}{2}$.

By (3.4), we can apply Lemma 3.3 with $P_m = \beta_{-2m-1}(k; 1)$ and $Q_m = -i\mathcal{Z}_0(k, m)$ for $m \in \mathbb{N}_0$, where

$$\begin{aligned}
 (3.5) \quad \mathcal{Z}_p(k, m) & := \sum_{\nu=p}^k \rho(2k+1-2\nu) \sum_{\eta=0}^{\nu-p} \binom{2m+1-2p+2\nu-2\eta}{2m+1-p} \\
 & \quad \cdot \psi(2k+2m-2p+3+2\nu-2\eta) \frac{(-1)^\eta (\pi/2)^{2\eta+p}}{(2\eta+p)!},
 \end{aligned}$$

for $p \in \{0, 1\}$. Then we have

$$(3.6) \quad \beta_{-2m-1}(k; 1) = -\frac{2}{\pi} \sum_{\nu=0}^m (1-2^{2\nu+1-2m}) 2^{2\nu+1-2m} \zeta(2m-2\nu) \mathcal{Z}_0(k, \nu),$$

for $m \in \mathbb{N}_0$.

Remark 3.4. It follows from (2.2) that both sides of

$$\sum_{m=0}^{\infty} \frac{(-u)^{-m} \cos((2m + 1)\pi/2)}{(2m + 1)^{2k+1}} = \sum_{n=0}^{\infty} \rho(2k + 1 - 2n; u) \frac{(i\pi/2)^{2n}}{(2n)!}$$

are uniformly convergent with respect to $u \in (1, 1 + \delta]$, when $k \in \mathbb{N}$. Letting $u \rightarrow 1$, we have

$$\sum_{j=0}^k \rho(2k + 1 - 2j) \frac{(i\pi/2)^{2j}}{(2j)!} = 0,$$

because $\rho(-2m - 1; u) \rightarrow 0$ ($u \rightarrow 1; m \in \mathbb{N}_0$). Hence we can see that if $k \in \mathbb{N}$, then

$$(3.7) \quad \begin{aligned} \mathcal{Z}_p(k, m) &= \sum_{\nu=1}^k \rho(2k + 1 - 2\nu) \sum_{\eta=0}^{\nu-1} \binom{2m + 1 - 2p + 2\nu - 2\eta}{2m + 1 - p} \\ &\quad \cdot \psi(2k + 2m - 2p + 3 + 2\nu - 2\eta) \frac{(-1)^\eta (\pi/2)^{2\eta+p}}{(2\eta + p)!}, \end{aligned}$$

for $p \in \{0, 1\}$.

Now we can prove the following result on $\mathfrak{T}_{1,1}(2k + 1, 2k + 1, 2l + 1)$ defined by (1.4).

Proposition 3.5. For $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$,

$$(3.8) \quad \begin{aligned} \mathfrak{T}_{1,1}(2k + 1, 2k + 1, 2l + 1) &= -2\mathcal{Z}_1(k, l) + \frac{4}{\pi} \sum_{m=0}^l \sum_{\nu=0}^m (1 - 2^{2\nu+1-2m}) 2^{2\nu+1-2m} \zeta(2m - 2\nu) \\ &\quad \cdot \mathcal{Z}_0(k, \nu) \frac{(i\pi/2)^{2l-2m}}{(2l - 2m)!}. \end{aligned}$$

Proof. We put $p = 1$ and $\theta = \pi/2$ in (3.3). Since $\cos((2m + 2n + 2)\pi/2) = (-1)^{m+n+1}$, we have

$$-\frac{1}{2} \mathfrak{T}_{1,1}(2k + 1, 2k + 1, 2l + 1) = \mathcal{Z}_1(k, l) + \sum_{m=0}^l \beta_{-2m-1}(k; 1) \frac{(i\pi/2)^{2l-2m}}{(2l - 2m)!}.$$

By (3.6), we obtain the proof. □

Finally we prove an evaluation formula for $R(2k + 1, 2k + 1, 2k + 1)$ for any $k \in \mathbb{N}$. By (1.1), (1.2) and (1.4), we can see that

$$(3.9) \quad \begin{aligned} R(2k + 1, 2k + 1, 2k + 1) &= 2^{-6k-3} T(2k + 1, 2k + 1, 2k + 1) - \mathfrak{T}_{1,1}(2k + 1, 2k + 1, 2k + 1), \end{aligned}$$

for $k \in \mathbb{N}_0$. It was proved that

$$(3.10) \quad T(2k + 1, 2k + 1, 2k + 1) = -4 \sum_{j=0}^k \binom{4k - 2j + 1}{2k} \zeta(2j) \zeta(6k - 2j + 3)$$

(see [2], Eq. (1.14)). By combining (3.9), (3.10) and Proposition 3.5, we obtain the following.

Theorem 3.6. For $k \in \mathbb{N}$,

$$\begin{aligned} R(2k+1, 2k+1, 2k+1) &= -2^{-6k-1} \sum_{j=0}^k \binom{4k-2j+1}{2k} \zeta(2j) \zeta(6k-2j+3) + 2\mathcal{Z}_1(k, k) \\ &\quad - \frac{4}{\pi} \sum_{m=0}^k \sum_{\nu=0}^m (1-2^{2\nu+1-2m}) 2^{2\nu+1-2m} \zeta(2m-2\nu) \mathcal{Z}_0(k, \nu) \frac{(i\pi/2)^{2k-2m}}{(2k-2m)!}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{Z}_p(k, m) &= \sum_{\nu=1}^k \rho(2k+1-2\nu) \sum_{\eta=0}^{\nu-1} \binom{2m+1-2p+2\nu-2\eta}{2m+1-p} \\ &\quad \cdot \psi(2k+2m-2p+3+2\nu-2\eta) \frac{(-1)^\eta (\pi/2)^{2\eta+p}}{(2\eta+p)!}, \end{aligned}$$

for $p \in \{0, 1\}$. Note that $\rho(s) = \sum_{m \geq 0} (-1)^m / (2m+1)^s$ and $\psi(s) = (1-2^{-s})\zeta(s)$.

Example 3.7. We list several evaluation formulas for $R(2k+1, 2k+1, 2k+1)$ deduced from Theorem 3.6. Note that we use the relations

$$\rho(2j+1) = \frac{(-1)^j E_{2j}}{2(2j)!} \left(\frac{\pi}{2}\right)^{2j+1} \quad (j \in \mathbb{N}_0),$$

where $\{E_n\}$ are the Euler numbers (see, e.g., [1]).

$$\begin{aligned} R(3, 3, 3) &= \frac{253}{256} \pi^2 \zeta(7) - \frac{2545}{256} \zeta(9) \\ R(5, 5, 5) &= \frac{2039}{18432} \pi^4 \zeta(11) + \frac{285565}{24576} \pi^2 \zeta(13) - \frac{2056257}{16384} \zeta(15) \\ R(7, 7, 7) &= \frac{32639}{2211840} \pi^6 \zeta(15) + \frac{913913}{491520} \pi^4 \zeta(17) + \frac{40212403}{262144} \pi^2 \zeta(19) \\ &\quad - \frac{896163411}{524288} \zeta(21) \\ R(9, 9, 9) &= \frac{522239}{275251200} \pi^8 \zeta(19) + \frac{2978549}{66060288} \pi^6 \zeta(21) + \frac{1194884977}{41943040} \pi^4 \zeta(23) \\ &\quad + \frac{71693105055}{33554432} \pi^2 \zeta(25) - \frac{1625043751045}{67108864} \zeta(27). \end{aligned}$$

Remark 3.8. More general results on *partial* Tornheim's double series $\mathfrak{T}_{b_1, b_2}(p, q, r)$ defined by (1.4) will be given in [7]. Indeed, we will be able to give more general relation formulas for $\mathfrak{T}_{b_1, b_2}(p, q, r)$.

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