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A NOTE ON INVERTIBILITY PRESERVERS ON BANACH ALGEBRAS

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Abstract. Let $\mathcal{A}$ be $\mathcal{B}$ be semisimple Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a unital bijective linear operator that preserves invertibility. If the socle of $\mathcal{A}$ is an essential ideal of $\mathcal{A}$, then ϕ is a Jordan isomorphism.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras with identity elements, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map. We say that ϕ is unital if it maps the identity element of $\mathcal{A}$ into the identity element of $\mathcal{B}$, and we say that ϕ preserves invertibility if $\phi(x)$ is invertible in $\mathcal{B}$ whenever x is invertible in $\mathcal{A}$. It turns out that, under rather mild assumptions, Jordan homomorphisms are unital invertibility preserving maps (see e.g. [10, Proposition 1.3]). Motivated by various relevant results (such as the Gleason-Kahane-Żelazko theorem) Kaplansky [9] asked when the converse is true, that is, under which assumptions a unital invertibility preserving map must be a Jordan homomorphism. There has been a lot of activity concerning this question; we refer the reader to some rather recent papers ([3], [4], [6], [10]) for historical accounts. We shall now only briefly discuss those results that are closely connected with the present paper.

By $\mathcal{B}(X)$ we denote the algebra of all bounded linear operators on a Banach space X. In [8] Jafarian and Sourour proved that Jordan isomorphisms are the only bijective unital linear operators between $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ that preserve invertibility in both directions (i.e., x is invertible if and only if $\phi(x)$ is invertible). Aupetit and du Mouton [4] extended this result to semisimple Banach algebras whose socle is an essential ideal (actually, they considered a slightly more general problem on maps preserving the full spectrum of each element). Finally, Sourour [10] showed that the result from [8] is true for maps that preserve invertibility (in only one direction).

The goal of this note is to obtain results similar to those in [4], however, under the assumption that the invertibility is preserved in one direction only. In particular, we shall thereby obtain a brief proof of Sourour's result [10]. It should be mentioned, however, that several ideas from both [4] and [10] will be used in our proof.

By a Banach algebra we shall mean a complex Banach algebra with an identity element. The socle of the algebra $\mathcal{A}$ will be denoted by $\operatorname{soc}(\mathcal{A})$. Recall that an ideal $\mathcal{I}$ of $\mathcal{A}$ is said to be *essential* if it has a nonzero intersection with every nonzero

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ideal of $\mathcal{A}$; in semisimple algebras this is equivalent to the condition that $a \cdot \mathcal{I} = 0$, where $a \in \mathcal{A}$, implies $a = 0$.

We are now in a position to state our main result, which extends [4, Theorem 3.2 and Corollary 3.3].

Theorem 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a unital bijective linear operator that preserves invertibility. Then*

$$\phi^{-1}(\phi(a^2) - \phi(a)^2) \cdot \operatorname{soc}(\mathcal{A}) = 0 \quad \text{for every } a \in \mathcal{A}.$$

In particular, if $\operatorname{soc}(\mathcal{A})$ *is an essential ideal of $\mathcal{A}$, then ϕ is a Jordan isomorphism.*

In primitive algebras every nonzero ideal is essential, and from a well-known theorem of Herstein [7] on Jordan homomorphisms onto prime rings it follows easily that every Jordan isomorphism ϕ from a primitive algebra onto another algebra is either an isomorphism or an anti-isomorphism (just consider ϕ^{-1}). Hence we have the following corollary, which generalizes [10, Main theorem].

Corollary 1.2. *Let $\mathcal{A}$ be a primitive Banach algebra with nonzero socle, and let $\mathcal{B}$ be a semisimple Banach algebra. If $\phi : \mathcal{A} \to \mathcal{B}$ is a unital bijective linear operator that preserves invertibility, then ϕ is either an isomorphism or an anti-isomorphism.*

2. Proof

We first fix the notation and terminology. By $\mathcal{A}$ and $\mathcal{B}$ we denote semisimple Banach algebras, and by ϕ a bijective unital linear operator from $\mathcal{A}$ onto $\mathcal{B}$ that preserves invertibility. Note that the latter assumption can be equivalently formulated as $\sigma(\phi(a)) \subseteq \sigma(a)$ for every $a \in \mathcal{A}$, where $\sigma(x)$ denotes the spectrum of the element x.

Recall that every minimal left ideal of $\mathcal{A}$ is of the form $\mathcal{A}e$ where e is a minimal idempotent, i.e., $e^2 = e \neq 0$ and $e\mathcal{A}e = \mathbb{C}e$. In this case $e\mathcal{A}$ is a minimal right ideal of $\mathcal{A}$. The sum of all minimal left ideals of $\mathcal{A}$ is called the *socle* of $\mathcal{A}$ and it coincides with the sum of all minimal right ideals of $\mathcal{A}$. For example, for any Banach space X, $\operatorname{soc}(\mathcal{B}(X))$ is equal to the ideal of all finite rank operators in $\mathcal{B}(X)$. If $\mathcal{A}$ has no minimal one-sided ideals, then we define $\operatorname{soc}(\mathcal{A}) = 0$. We say that a nonzero element $u \in \mathcal{A}$ has rank one if u belongs to some minimal left ideal of $\mathcal{A}$ (equivalently, $u = ue$ for some minimal idempotent e in $\mathcal{A}$). By $\mathcal{F}_1(\mathcal{A})$ we denote the set of all elements of rank one in $\mathcal{A}$. It is easy to see (see [5] for details) that $u \in \mathcal{F}_1(\mathcal{A})$ if and only if $u \neq 0$ and u lies in some minimal right ideal of $\mathcal{A}$, and furthermore, this is equivalent to the condition that $u\mathcal{A}u = \mathbb{C}u \neq 0$. Another, less obvious characterization is that $u \in \mathcal{F}_1(\mathcal{A})$ if and only if $u \neq 0$ and $|\sigma(zu) \setminus \{0\}| \leq 1$ for every $z \in \mathcal{A}$ or, equivalently, $|\sigma(uz) \setminus \{0\}| \leq 1$ for every $z \in \mathcal{A}$ (see [4] or [5]).

Lemma 2.1. $\phi(\mathcal{F}_1(\mathcal{A})) \subseteq \mathcal{F}_1(\mathcal{B})$.

Proof. Pick $u \in \mathcal{F}_1(\mathcal{A})$. We have to show that $v = \phi(u)$ lies in $\mathcal{F}_1(\mathcal{B})$, that is, that $|\sigma(zv) \setminus \{0\}| \leq 1$ for every $z \in \mathcal{B}$.

From a well-known result of Aupetit [1] (see also [2, Theorem 5.5.2]) it follows that ϕ is continuous, and therefore of course ϕ^{-1} is also continuous. Set $M = (2\|\phi^{-1}\| + 1)^{-1}$ and pick $z \in \mathcal{B}$ such that $\|z\| \leq M$. Since $M < 1$, $1 + z$ is invertible, and we have $y = (1+z)^{-1} - 1 = -z + z^2 - z^3 + z^4 - \cdots$. Set $x = \phi^{-1}(y)$ and note that

$$\|x\| \leq \|\phi^{-1}\|\|y\| \leq \|\phi^{-1}\|\frac{\|z\|}{1 - \|z\|} \leq \|\phi^{-1}\|\frac{M}{1-M} = \frac{1}{2},$$

so that $1+x$ is invertible, whence it follows that $1+x-\lambda u = (1+x)(1-\lambda(1+x)^{-1}u)$ is invertible for all but possibly one $\lambda \in \mathbb{C}$. Since ϕ preserves invertibility, the same is true for

$$\phi(1+x-\lambda u) = 1+y-\lambda v = (1+y)(1-\lambda(1+z)v),$$

which means that the spectrum of $(1+z)v$ contains at most one nonzero point. Thus we proved that $|\sigma((1+z)v) \setminus \{0\}| \le 1$ whenever $\|z\| \le M$, and similarly we see that in this case also $|\sigma(v(1+z)) \setminus \{0\}| \le 1$.

Now let $z \in \mathcal{B}$ be any element. Define the analytic function $f : \mathbb{C} \to \mathcal{B}$ by $f(\lambda) = (\lambda + z)v$ and note that $|\sigma(f(\lambda)) \setminus \{0\}| \le 1$ whenever $|\lambda| > \frac{\|z\|}{M}$.

Suppose that v does not have a left inverse. Since, in particular, $|\sigma(f(\lambda))| \le 2$ whenever $|\lambda| > \frac{\|z\|}{M}$, it follows from [2, Theorem 3.4.25] that $|\sigma(f(\lambda))| \le 2$ for every $\lambda \in \mathbb{C}$. Taking $\lambda = 0$ we thus get $|\sigma(zv)| \le 2$. However, since $0 \in \sigma(zv)$, it follows that $|\sigma(zv) \setminus \{0\}| \le 1$, as desired. The case when v does not have a right inverse can be treated similarly, by considering the function $\lambda \mapsto v(\lambda + z)$. So we may assume that v is invertible. In this case $|\sigma(f(\lambda))| = 1$ whenever $|\lambda| > \frac{\|z\|}{M}$, and so applying [2, Theorem 3.4.25] again we see that this holds true for any $\lambda \in \mathbb{C}$. Accordingly, $|\sigma(zv)| = 1$, and so, in particular, $|\sigma(zv) \setminus \{0\}| \le 1$ (incidentally we mention that in the case when v is invertible we actually have $\mathcal{A} \cong \mathcal{B} \cong \mathbb{C}$). ☐

Given $u \in \mathcal{F}_1(\mathcal{A})$, there is $\tau(u) \in \mathbb{C}$ such that $u^2 = \tau(u)u$. Clearly $\tau(u) \in \sigma(u)$, and moreover, either $\tau(u) = 0$ or $\tau(u)$ is the only nonzero point in $\sigma(u)$. Since $\tau(u)$ is unique, we may consider τ as a function from $\mathcal{F}_1(\mathcal{A})$ to $\mathbb{C}$, and we extend it by defining $\tau(0) = 0$. Using $u\mathcal{A}u = \mathbb{C}u$, $u \in F_1(\mathcal{A})$, and considering $(xu)^2$ and $(ux)^2$ it follows easily that $\tau(xu)u = uxu = \tau(ux)u$ for any $x \in \mathcal{A}$. Furthermore, we claim that $\tau(x_1u + x_2u) = \tau(x_1u) + \tau(x_2u)$ for all $x_1, x_2 \in \mathcal{A}$ and $u \in \mathcal{F}_1(\mathcal{A})$. This follows from [4, Lemma 2.3], but it can also be proved using only elementary tools. Indeed, examining $(x_1u + x_2u)^2 = x_1ux_1u + x_1ux_2u + x_2ux_1u + x_2ux_2u$ and applying $uxu = \tau(xu)u$ we get

$$(\tau(x_1u + x_2u) - \tau(x_1u) - \tau(x_2u))(x_1u + x_2u) = 0,$$

from which our assertion can be easily inferred. Also, it is straightforward to check that $\tau(\lambda u) = \lambda\tau(u)$ for all $\lambda \in \mathbb{C}$ and $u \in \mathcal{F}_1(\mathcal{A})$. Therefore, the restriction of τ to any minimal left ideal $\mathcal{A}u$ is a linear functional. Moreover, from $u^2 = \tau(u)u$, $u \in \mathcal{F}_1(\mathcal{A})$, we conclude that $|\tau(u)| \le \|u\|$ and so τ is bounded on $\mathcal{A}u$.

Lemma 2.2. *$\tau(xu) = \tau(\phi(x)\phi(u))$ and $\tau(x^2u) = \tau(\phi(x)^2\phi(u))$ for all $x \in \mathcal{A}$ and $u \in \mathcal{F}_1(\mathcal{A})$.*

Proof. Let $u \in \mathcal{F}_1(\mathcal{A})$ be a fixed element. Pick a nonzero $x \in \mathcal{A}$ and let $D_x = \{\lambda \in \mathbb{C} \,|\, |\lambda| < (\|\phi\|\|x\|)^{-1}\}$. Then $1 - \lambda\phi(x)$ is invertible for every $\lambda \in D_x$; moreover, since $\|\phi\| \ge 1$ (ϕ is unital!), the same is true for $1 - \lambda x$. We have $\phi(u) \in \mathcal{F}_1(\mathcal{B})$ and so we can define $F_x, G_x : D_x \to \mathbb{C}$ by

$$F_x(\lambda) = \tau((1-\lambda x)^{-1}u), \quad G_x(\lambda) = \tau((1-\lambda\phi(x))^{-1}\phi(u)).$$

Since τ is a continuous linear functional on $\mathcal{A}u$ (resp. $\mathcal{B}\phi(u)$), we have

$$F_x(\lambda) = \sum_{k=0}^{\infty} \tau(x^k u)\lambda^k, \quad G_x(\lambda) = \sum_{k=0}^{\infty} \tau(\phi(x)^k\phi(u))\lambda^k.$$

Suppose that $G_x(\lambda) = \alpha \ne 0$ for some $\lambda \in D_x$. Then $(1 - \lambda\phi(x))^{-1}\phi(u) - \alpha$ is not invertible, and hence also $\phi(u) - \alpha(1 - \lambda\phi(x))$ is not invertible. Since ϕ is unital and preserves invertibility, it follows that $u - \alpha(1 - \lambda x)$ is not invertible. Accordingly, $(1 - \lambda x)^{-1}u - \alpha$ is not invertible, which means that $F_x(\lambda) = \alpha$. That is, we showed that $G_x(\lambda) = F_x(\lambda)$ whenever $G_x(\lambda) \neq 0$. Since F_x and G_x are analytic functions, it follows that either $F_x \equiv G_x$ or $G_x \equiv 0$.

Comparing coefficients at the expansions of F_x and G_x we see, in particular, that for any $x \neq 0$ in $\mathcal{A}$ we have either $\tau(\phi(x)\phi(u)) = 0$ or $\tau(xu) = \tau(\phi(x)\phi(u))$ and $\tau(x^2u) = \tau(\phi(x)^2\phi(u))$. Both conditions are trivially satisfied for $x = 0$.

If $\tau(\phi(x)\phi(u)) = 0$ for all $x \in \mathcal{A}$, then we would have $\phi(u)\phi(x)\phi(u) = 0$ for every $x \in \mathcal{A}$. However, since ϕ is onto and $\mathcal{B}$ is semisimple (and so, in particular, semiprime) this would yield $\phi(u) = 0$, a contradiction. Thus $\tau(\phi(x_1)\phi(u)) \neq 0$ for some $x_1 \in \mathcal{A}$. Then of course $\tau(x_1^2u) = \tau(\phi(x_1)^2\phi(u))$. Now suppose there exists $x_2 \in \mathcal{A}$ such that $\tau(x_2^2u) \neq \tau(\phi(x_2)^2\phi(u))$. Then $\tau(\phi(x_2)\phi(u)) = 0$; hence $\tau(\phi(x_1 + \mu x_2)\phi(u)) \neq 0$ for any $\mu \in \mathbb{C}$, which in turn implies that $\tau((x_1 + \mu x_2)^2u) = \tau(\phi(x_1 + \mu x_2)^2\phi(u))$. That is,

$$\mu\Big(\tau(x_1x_2u + x_2x_1u) - \tau\big(\phi(x_1)\phi(x_2)\phi(u) + \phi(x_2)\phi(x_1)\phi(u)\big)\Big) \\ + \mu^2\Big(\tau(x_2^2u) - \tau\big(\phi(x_2)^2\phi(u)\big)\Big) = 0$$

for every $\mu \in \mathbb{C}$, which clearly contradicts our assumption that $\tau(x_2^2u) \neq \tau(\phi(x_2)^2\phi(u))$. This means that $\tau(x^2u) = \tau(\phi(x)^2\phi(u))$ for every $x \in \mathcal{A}$. In a similar (but of course shorter) fashion one shows that also $\tau(xu) = \tau(\phi(x)\phi(u))$ for every $x \in \mathcal{A}$. □

Proof of Theorem 1.1. Let $x \in \mathcal{A}$ and let $u \in \mathcal{F}_1(\mathcal{A})$. From the first identity in Lemma 2.2 we see that $\tau(x^2u) = \tau(\phi(x^2)\phi(u))$ and from the second one we see that $\tau(x^2u) = \tau(\phi(x)^2\phi(u))$. Comparing we get $\tau((\phi(x^2) - \phi(x)^2)\phi(u)) = 0$. Set $x_0 = \phi^{-1}(\phi(x^2) - \phi(x)^2)$, and note that $\tau(x_0u) = \tau(\phi(x_0)\phi(u)) = 0$ for every $u \in \mathcal{F}_1(\mathcal{A})$. But this yields that $x_0u = 0$ for every $u \in \mathcal{F}_1(\mathcal{A})$. Indeed, if x_0u_0 was not 0 for some $u_0 \in \mathcal{F}_1(\mathcal{A})$, then, by the semisimplicity of $\mathcal{A}$, there would be $x \in \mathcal{A}$ such that $\sigma(x_0u_0 \cdot x) \neq \{0\}$, meaning that $\tau(x_0 \cdot u_0x) \neq 0$, a contradiction. Accordingly, $x_0 \cdot \operatorname{soc}(\mathcal{A}) = 0$. □

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